9 Games of Incomplete Information

9.1 Cournot Duopoly With Unknown Costs

Assume that two firms produce a homogenous good with inverse demand $P(Q) = \max\{2 - Q, 0\}$, where Q is interpreted as the total quantity of the good on the market. Also, assume that the firms produce the good at constant unit cost $c_1, c_2 \geq 0$.

The payoff function for firm i is thus

$$u_i(q_i, q_j) = P(q_1 + q_2) q_i - c_i q_i = [2 - q_1 - q_2 - c_i] q_i.$$

If costs would be known we would just have a simple (possibly asymmetric) static Cournot duopoly. However, assume that:

1. Firm 1 has cost $c_1 = 1$, which is common knowledge.

2. Firm 2, has cost $c_2 = \frac{1}{2}$ with probability $\frac{1}{2}$ and $c_2 = \frac{3}{2}$ with probability $\frac{1}{2}$.

Assume that q_1 is the quantity chosen by firm 1, then the optimal quantity for firm 2 is

$$q_2^L(q_1) = \arg\max \left[2 - q_1 - q_2 - \frac{1}{2}\right] q_2 = \arg\max \left[\frac{3}{2} - q_1 - q_2\right] q_2$$

if $c_2 = \frac{1}{2}$ and

$$q_2^H(q_1) = \arg\max \left[2 - q_1 - q_2 - \frac{3}{2}\right] q_2 = \arg\max \left[\frac{1}{2} - q_1 - q_2\right] q_2$$

if $c_2 = \frac{3}{2}$. From the first order conditions we have that the cost specific best replies are

$$\begin{aligned} q_2^L(q_1) &= \frac{\frac{3}{2} - q_1}{2} \\ q_2^H(q_1) &= \frac{\frac{1}{2} - q_1}{2} \end{aligned}$$

Firm 1 maximizes its expected profit given q_2^L, q_2^H

$$\begin{aligned} q_1\left(q_2^L, q_2^H\right) &= \arg\max_{q_1} \frac{1}{2}\left[2 - q_1 - q_2^L - 1\right] q_1 + \frac{1}{2}\left[2 - q_1 - q_2^H - 1\right] q_1 \\ &= \arg\max_{q_1} \left[1 - q_1 - \left(\frac{1}{2} q_2^L + \frac{1}{2} q_2^H\right)\right] q_1 \Rightarrow \\ q_1\left(q_2^L, q_2^H\right) &= \frac{1 - \left(\frac{1}{2} q_2^L + \frac{1}{2} q_2^H\right)}{2} \end{aligned}$$

In a Nash equilibrium we have that q_1 satisfies

$$\begin{aligned} q_1 &= q_1\left(q_2^L(q_1), q_2^H(q_1)\right) = \frac{2 - q_2^L - q_2^H}{4} = \frac{2 - \frac{\frac{3}{2}-q_1}{2} - \frac{\frac{1}{2}-q_1}{2}}{4} \\ &= \frac{2 - \frac{2-2q_1}{2}}{4} = \frac{2-1+q_1}{4} \Leftrightarrow q_1\frac{3}{4} = \frac{1}{4} \Leftrightarrow q_1^* = \frac{1}{3}. \end{aligned}$$

This is plausible since the expected cost is 1 for both firms, and $\left(\frac{1}{3}, \frac{1}{3}\right)$ is the equilibrium if $c_1 = c_2 = 1$. We then have that

$$\begin{aligned} q_2^{*L} &= q_2^L(q_1^*) = \frac{\frac{3}{2} - q_1^*}{2} = \frac{\frac{3}{2} - \frac{1}{3}}{2} = \frac{7}{12} \\ q_2^{*H} &= q_2^H(q_1^*) = \frac{\frac{1}{2} - q_1^*}{2} = \frac{\frac{1}{2} - \frac{1}{3}}{2} = \frac{1}{12} \\ &\Rightarrow \frac{1}{2}q_2^{*L} + \frac{1}{2}q_2^{*H} = \frac{7}{24} + \frac{1}{24} = \frac{8}{24} = \frac{1}{3}, \end{aligned}$$

which confirms the intuition on the relationship with the complete information game.

9.2 Formal Treatment

Definition 1 *A game of incomplete information is an object $G = (n, A, \Theta, u, p)$ where $A = \times_{i=1}^n A_i$ is the action space, $\Theta = \times_{i=1}^n \Theta_i$ is the type space, $u = (u_1, ..., u_n)$, where*

$$u_i : A \times \Theta \to R$$

is the payoff function and $p \in \Delta(\Theta)$ is the prior type distribution.

Remark 1 *Definition builds in common prior assumption.*

A pure strategy is a map $s_i : \Theta_i \to A_i$ and a mixed strategy would be a map $\sigma_i : \Theta \to \Delta(A_i)$. We also note that the beliefs over Θ_{-i} are given by

$$\Pr[\theta_{-i}|\theta_i] = \frac{\Pr[\theta_{-i}, \theta_i]}{\sum_{\theta'_{-i} \in \Theta_{-i}} \Pr\left[\theta'_{-i}, \theta_i\right]} = \frac{\Pr[\theta_{-i}, \theta_i]}{\Pr[\theta_i]}$$

Definition 2 *A pure strategy Bayesian Nash equilibrium is a strategy profile* $s^* = (s_1^*, ..., s_n^*)$ *such that*

$$\sum_{\theta_i \in \Theta_i} \left[\sum_{\theta_{-i} \in \Theta_{-i}} u_i \left(s_i^* (\theta_i), s_{-i}^* (\theta_{-i}), \theta \right) \Pr [\theta_{-i} | \theta_i] \right] \Pr [\theta_i]$$
$$\geq \sum_{\theta_i \in \Theta_i} \left[\sum_{\theta_{-i} \in \Theta_{-i}} u_i \left(s_i (\theta_i), s_{-i}^* (\theta_{-i}), \theta \right) \Pr [\theta_{-i} | \theta_i] \right] \Pr [\theta_i]$$

for every $i \in I$ *and every* $s_i : \Theta_i \to A_i$.

An (almost) equivalent formulation is

Definition 3 *A pure strategy Bayesian Nash equilibrium is a strategy profile* $s^* = (s_1^*, ..., s_n^*)$ *such that*

$$\sum_{\theta_{-i} \in \Theta_{-i}} u_i \left(s_i^* (\theta_i), s_{-i}^* (\theta_{-i}), \theta \right) \Pr [\theta_{-i} | \theta_i] \Pr [\theta_i] \geq \sum_{\theta_{-i} \in \Theta_{-i}} u_i \left(s_i (\theta_i), s_{-i}^* (\theta_{-i}), \theta \right) \Pr [\theta_{-i} | \theta_i] \Pr [\theta_i]$$

for every $i \in I, \theta_i \in \Theta_i$ *and* $a_i \in A_i$.

9.3 Example: Double Auction

Assume that there is a seller and a buyer, where:

- Seller has valuation $\theta_S \sim U[0,1]$
- Buyer has valuation $\theta_B \sim U[0,1]$
- Buyer announces bid price $b \in R_+$
- Seller announces ask price $a \in R_+$
- If $b \geq a$, then the object is transacted at price

$$\frac{b+a}{2}.$$

- If $b < a$, the seller keeps object.

Note that in terms of generic notation we have that $\Theta_B = \Theta_S = [0,1]$ and $A_B = A_S = R_+$

The payoff for the buyer is

$$u_B(b, a, \theta_B) = \begin{cases} \theta_B - \frac{b+a}{2} & \text{if } b \geq a \\ 0 & \text{if } b < a \end{cases}$$

and the payoff for the seller is

$$u_S(b, a, \theta_S) = \begin{cases} \frac{b+a}{2} - \theta_S & \text{if } b \geq a \\ 0 & \text{if } b < a \end{cases}$$

Fix θ_B and assume that $a : \Theta_S \to R_+$ is strictly increasing. Then a^{-1} exists and the fact that we get trade if and only if $b \geq a(\theta_S)$ is equivalent with $a^{-1}(b) \geq \theta_S$. Hence, the expected payoff for the buyer is

$$\mathrm{E}\left[u_B\left(b, a(\theta_S), \theta_B\right)\right] = \int_0^{a^{-1}(b)} \left[\theta_B - \frac{b + a(\theta_S)}{2}\right] f(\theta_S)\, d\theta_S$$

Symmetrically

$$\mathrm{E}\left[u_S\left(b(\theta_S), a, \theta_B\right)\right] = \int_{b^{-1}(a)}^{1} \left[\frac{b(\theta_B) + a}{2} - \theta_S\right] f(\theta_B)\, d\theta_B.$$

Remark 2 *This kind of game has uncountable number of equilibria. We will only solve for one particular kind of equilibrium.*

9.3.1 Solving for a Linear Equilibrium

We guess that $a : \Theta_S \to R_+$ is given on form

$$a(\theta_S) = a_S + c_S\theta_S.$$

Remark 3 *We will NOT impose that strategies are linear. We will guess linearity and show that the best response to linear strategies are linear.*

Now

$$a^{-1}(x) = \frac{x - a_S}{c_S}.$$

Hence, the problem that defines the best response for a type θ_B buyer is

$$\max_{b \geq 0} \int_0^{\frac{b-a_S}{c_S}} \left[\theta_B - \frac{b + a(\theta_S)}{2}\right] d\theta_S = \max_{b \geq 0} \int_0^{\frac{b-a_S}{c_S}} \left[\theta_B - \frac{b + a_S + c_S\theta_S}{2}\right] d\theta_S$$

Cranking it out Use Leibnitz rule to get FOC

$$\begin{aligned}
0 &= \left[\theta_B - \frac{b + a_S + c_S\frac{b-a_S}{c_S}}{2}\right]\frac{d}{db}\left[\frac{b-a_S}{c_S}\right] + \int_0^{\frac{b-a_S}{c_S}}\left[\frac{d}{db}\left[\theta_B - \frac{b+a}{2}\right]\right]d\theta_S \\
&= \frac{\theta_B - b}{c_S} - \int_0^{\frac{b-a_S}{c_S}}\frac{1}{2}d\theta_S = \frac{\theta_B - b}{c_S} - \frac{1}{2}\frac{b-a_S}{c_S} \Leftrightarrow 0 = \theta_B - b - \frac{1}{2}b + \frac{1}{2}a_S \Leftrightarrow \\
\frac{3}{2}b &= \theta_B + \frac{1}{2}a_S \Leftrightarrow b = \frac{1}{3}a_S + \frac{2}{3}\theta_B,
\end{aligned}$$

WHICH IS A STRATEGY THAT IS LINEAR IN TYPE.

- $b(\theta_B) = a_B + c_B\theta_B$
- $\Rightarrow b^{-1}(x) = \frac{x-a_B}{c_B}$

Given a linear buyer strategy, we have that the seller wants to solve the problem

$$\max_{a\geq 0}\int_{b^{-1}(a)}^{1}\left[\frac{b(\theta_B)+a}{2} - \theta_S\right]d\theta_B = \max_{a\geq 0}\int_{\frac{a-a_B}{c_B}}^{1}\left[\frac{a_B + c_B\theta_B + a}{2} - \theta_S\right]d\theta_B$$

Again using Leibnitz rule, the FOC is

$$\begin{aligned}
0 &= \int_{\frac{a-a_B}{c_B}}^{1}\frac{1}{2}d\theta_B - \left[\frac{a_B + c_B\frac{a-a_B}{c_B} + a}{2} - \theta_S\right]\frac{1}{c_B} \\
&= \left[1 - \frac{a-a_B}{c_B}\right]\frac{1}{2} - [a - \theta_S]\frac{1}{c_B} = \frac{[c_B + a_B - a]}{2}\frac{1}{c_B} - [a-\theta_S]\frac{1}{c_B} \\
&\Leftrightarrow\ 0 = \frac{[c_B + a_B - a]}{2} - [a - \theta_S] \\
\frac{3}{2}a &= \frac{c_B + a_B}{2} + \theta_S \Leftrightarrow \\
a &= \frac{c_B + a_B}{3} + \frac{2}{3}\theta_S
\end{aligned}$$

Hence, we started with hypothesized strategies

$$\begin{aligned}
a(\theta_S) &= a_S + c_S\theta_S \\
b(\theta_B) &= a_B + c_B\theta_B
\end{aligned}$$

and derived best replies

$$\begin{aligned}
a(\theta_S) &= \frac{c_B + a_B}{3} + \frac{2}{3}\theta_S \\
b(\theta_B) &= \frac{1}{3}a_S + \frac{2}{3}\theta_B
\end{aligned}$$

Hence

$$c_S = c_B = \frac{2}{3}$$

Moreover a_B, a_S solves the system

$$\begin{aligned} a_B &= \frac{1}{3} a_S \\ a_S &= \frac{c_B + a_B}{3} = \frac{\frac{2}{3} + a_B}{3} \end{aligned}$$

Hence

$$\begin{aligned} a_B &= \frac{1}{3} a_S \Leftrightarrow 3a_B = a_S = \frac{\frac{2}{3} + a_B}{3} \\ &\Rightarrow \\ 9a_B &= \frac{2}{3} + a_B \Leftrightarrow 8a_B = \frac{2}{3} \Leftrightarrow \\ a_B &= \frac{1}{12} \Rightarrow \\ a_S &= 3a_B = \frac{3}{12} = \frac{1}{4} \end{aligned}$$

So, we've verified that

$$\begin{aligned} a(\theta_S) &= \frac{1}{4} + \frac{2}{3}\theta_S \\ b(\theta_B) &= \frac{1}{12} + \frac{2}{3}\theta_B \end{aligned}$$

is an equilibrium in the double auction. DRAW!

9.4 Purification

9.4.1 Example

Consider "perturbed" game

	o	f
O	$2+\varepsilon, 1$	$0, 0$
F	$0, 0$	$1, 2+\delta$

where,

- $\varepsilon \sim U[0, x]$
- $\eta \sim U[0, x]$
- Eg $\Theta_1 = \Theta_2 = [0, x]$ with uniform i.i.d. priors.

Remark 4 *Both pure strategy equilibria (O, o) and (F, f) are equilibria in extended game.*

Remark 5 *Will look for a pure strategy equilibrium in which each player plays both strategies with strictly positive probability.*

Let;

- $q = \Pr[2 \text{ plays } f]$
- $p = \Pr[1 \text{ plays } O]$

We note that O is the best response for player 1 if and only if

$$\begin{aligned} (2+\varepsilon)(1-q) &\geq q \Leftrightarrow \\ \varepsilon &\geq \frac{q-2(1-q)}{1-q} = \frac{3q-2}{1-q} \end{aligned}$$

By the same token f is a best response iff

$$\eta \geq \frac{3p-2}{1-p}$$

We conclude that the equilibrium strategies must have a cutoff property. Hence we'll look for a (Bayesian) Nash equilibrium where

$$\begin{aligned} s_1^*(\varepsilon) &= \begin{cases} O & \text{if } \varepsilon \geq a \\ F & \text{if } \varepsilon < a \end{cases} \\ s_2^*(\eta) &= \begin{cases} o & \text{if } \eta < b \\ f & \text{if } \eta \geq b \end{cases} \end{aligned}$$

It follows that

$$
\begin{aligned}
q &= \Pr\left[\eta \geq b\right] = 1 - \int_0^b \frac{1}{x} d\eta = 1 - \frac{b}{x} = \frac{x-b}{x} \\
p &= \frac{x-a}{x}
\end{aligned}
$$

Simplifying we have that

$$
\begin{aligned}
a &= \frac{x-3b}{b} \\
b &= \frac{x-3a}{b} \\
&\Rightarrow a = b
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
a &= \frac{x-3a}{a} \Leftrightarrow a^2 = x - 3a \Leftrightarrow \\
\left(a + \frac{3}{2}\right)^2 - \frac{9}{4} - x &= 0 \\
a &= -\frac{3}{2} \pm \sqrt{\frac{9}{4} + x}
\end{aligned}
$$

Since only a positive root can solve the problem we have that

$$
a = b = -\frac{3}{2} + \sqrt{\frac{9}{4} + x},
$$

and

$$
p\left(x\right) = q\left(x\right) = \frac{x-b}{x} = \frac{x + \frac{3}{2} - \sqrt{\frac{9}{4} + x}}{x}
$$

Using l'Hopitals rule we have that

$$
\lim_{x \to 0} q\left(x\right) = \frac{1 - \frac{1}{2}\left(\frac{9}{4} + x\right)^{-\frac{1}{2}}}{1} = \lim_{x \to 0}\left[1 - \frac{\frac{1}{2}}{\sqrt{\left(\frac{9}{4} - x\right)}}\right] = 1 - \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{2}{3}
$$

That is, $\left(q\left(x\right), p\left(x\right)\right) \to \left(\frac{2}{3}, \frac{2}{3}\right)$, which is the mixed strategy equilibrium with complete information.

The point of the example is that it illustrates (somewhat sloppily stated):

Theorem 1 *For almost all payoffs there exists a sequence of incomplete information games so that for every equilibrium in the unperturbed game there is a sequence of pure strategy equilibria s.t the distribution over actions converge towards the distribution of the unperturbed game.*