

Problem Set 2

1. Let $l^2 = \{X, \rho\}$ where X is the set of all infinite sequences such that $\sum_{i=1}^{\infty} x_i^2 \leq \infty$ and let $\rho(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$. Carefully demonstrate that l^2 is a metric space.
2. Explicitly construct an open cover of $(0, 1)$ for which no finite subcover exists.
3. Prove that if $a, b \geq 0, p, q > 0$, and $1/p + 1/q = 1$, then we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: one strategy is to compare $\ln(ab)$ and $\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right)$.

4. Use the inequality above to prove that if $p, q > 0$, and $1/p + 1/q = 1$, then

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}},$$

which is Holder's inequality.

5. Use Holders inequality to deduce that if $p > 1$ then,

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{p}}.$$

6. Prove that an arbitrary union of open sets is open and that a finite intersection of open sets is open.
7. Use the result above and de Morgans laws to show that an arbitrary intersection of closed sets is closed and that a finite union of closed sets is closed.
8. Construct an example of an intersection of open sets that is closed.