Econ 890-005
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Problem Set 4

1. Let f be a linear functional defined on a linear space L. Recall that the nullspace N is given by the set

$$\{x \in L | f(x) = 0\}.$$

   Assume that $f(x) \neq 0$ for some $x \in L$:

   1. Prove that N is a subspace of L.
   2. Let $z \in L \backslash N$. Prove that for every $x \in L$ there is a unique pair (α, y) where $y \in N$ and $\alpha \in R$ such that
$$x = \alpha x + y.$$

2. Prove that the closure of a convex set is convex.

3. Recall that a *convex cone* is a set $K \subset R^n$ satisfying 1) $x, y \in K \Rightarrow x + y \in K$, and, 2) $x \in K$ and $\alpha \geq 0s \Rightarrow \alpha x \in K$. Prove that a convex cone is convex.

4. Let $A = \{a_1, ..., a_k\}$ be a set of vectors $a_i \in R^n$ and let

$$K(A) = \left\{ x \in R^n | \text{there exists } \lambda \in R_+^k \text{ such that } x = \sum_1^k \lambda_i a_i \right\}.$$

   Show that $K(A)$ is a convex cone.

5. Let $H = \left\{ x = (x_1, ..., x_k, ..) \in l^2 | \, |x_k| \leq \frac{1}{k} \text{ for each } k \in N \right\}.$

   1. Is H convex?
   2. Is H a *convex body* (=convex set with a nonempty interior)?
   3. Is H totally bounded?

6. Give an example of two convex bodies with an intersection that is not a convex body.

7. Consider the program

$$\begin{aligned} & \max_x f(x) \\ \text{s.t. } g_j(x) & \geq 0 \text{ for } j = 1, .., k \end{aligned}$$

   1. Show that the constraint set is convex if $g_j(\cdot)$ is quasi-concave for every j.
   2. If, in addition, f is strictly quasi-concave, prove that there exists at most one solution to the problem.