1 Set Theory and Functions

1.1 Basic Definitions and Notation

- A **set** A is a collection of objects of any kind.
- We write $a \in A$ to indicate that a is an element of A. We express this as "a is contained in A".
- We write $A \subset B$ if every element of A also is an element in B . We express this as "A is a subset of B"
- We write $A = B$ if A and B consist of exactly the same elements. Otherwise we write $A \neq B$.
- If $A \subset B$ and $A \neq B$ we say that A is a proper subset of B.
- We write $\varnothing$ for the **empty set**, the set that contains no elements at all.

1.2 Operations on Sets

1. **The union**: Let A and B be two sets. Then, the union, denoted $A \cup B$ is the set of all elements belonging to at least one of the two sets. That is,

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

Given an arbitrary collection of sets A_i, where i is some parameter such that $i \in I$ (allows for finite or infinite numbers) we write $\cup_i A_i$ for the union, which is the set of all elements that belongs to at least one of the sets in the collection,

$$\cup_i A_i = \{x | \ x \in A_i \text{ for some } i \in I \ \}.$$

2. **The Intersection:** Given A and B we write $A \cap B$ for its intersection- the set of all elements belonging to both A and B.

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The generalization for a collection of sets A_i is obviously

$$\cap_i A_i = \{x| \ x \in A_i \text{ for every } i \in I \ \}$$

3. **Disjoint Sets:** A and B are said to be disjoint if they have no elements in common. That is, if

$$A \cap B = \varnothing.$$

4. **The Complement (sometimes called difference)**: We (KF uses different notation) write $A \backslash B$ for the set of all elements in A that don't belong to B,

$$A \backslash B = \{x | x \in A \text{ and } x \notin B\}$$

Remark 1 *It follows immediately from the definitions that $\cup$ and $\cap$ satisfy communicative and associative laws,*

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \\ (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

Hence, we can simply write $A \cup B \cup C$.

In addition:

Proposition 1 *$\cup$ and $\cap$ satisfy the distributive law*

$$\begin{aligned} (A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C) \end{aligned}$$

Proof. Exercise on Problem Set 1 ■

Sometimes we will also use the Cartesian product:

Definition 1 *Given non empty sets A and B, the Cartesian product, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$*

Example 1 *If* $A = \{a_1, a_2, a_3, a_4\}$ *and* $B = \{b_1, b_2\}$ *then*

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2), (a_4, b_1), (a_4, b_2)\}$$

Example 2 *If* $A = [0, 1]$ *and* $B = [0, 1]$ *we write* $A \times B = [0, 1] \times [0, 1]$ *or* $A \times B = [0, 1]^2$, *where geometrically the Cartesian product of the two (unit) intervals is the (unit) square.*

1.3 Functions

The terminology I'll use is as follows:

Definition 2 *A function (or mapping) from a set* X *to* Y *(denoted* $f : X \to Y$*) is a rule that assigns a unique element* $y = f(x) \in Y$ *to every* $x \in X$.

Usually, X is referred to as the *domain* of f.

Remark 2 *Sometimes functions are defined in terms of the graph* G*, which is a collection of ordered pairs* $\langle x, f(x) \rangle \in X \times Y$*. We say that* G *is the graph of a function if and only if for every* $x \in X$ *there is a unique pair* $\langle x, f(x) \rangle \in G$.

We call the element $f(x) \in Y$ the image of x (under f). This is generalized to subsets as follows:

Definition 3 *If* $A \subset X$ *we say that*

$$f(A) \equiv \{y \in Y | y \in f(a) \text{ for some } a \in A\}$$

is the (direct) image of A.

Also,

Definition 4 *If* $B \subset Y$ *we say that*

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

is the inverse image (or preimage) of B.

The image and the inverse image can be used to define some important types of functions:

Definition 5 *$f : X \to Y$ is said to be surjective (onto) if $f(X) = Y$*

Definition 6 *$f : X \to Y$ is said to be injective (one to one) if $f(x_1) \neq f(x_2)$ for every pair $(x_1, x_2) \in X$ such that $x_1 \neq x_2$.*

Definition 7 *$f : X \to Y$ is said to be bijective if it is surjective and injective.*

It is more or less immediate that:

Proposition 2 *If f is a bijection, then there is a (uniquely defined) function $g : Y \to X$ defined by letting $g(y) \in X$ be given by $g(y) = f^{-1}(y)$ for every $y \in Y$. This function is called the inverse of f (since no confusion can arise we will use f^{-1} also to denote the inverse function in the rest of the course).*

In some sense the conclusion is trivial. However, we'll write down a proof to make sure we understand what the definitions mean

Proof. Since f is injective there are two possibilities:

1. $f^{-1}(y) = \varnothing$

2. there exists a unique element $x \in X$ such that $\{x\} = f^{-1}(y)$

To see this, assume that for contradiction that there exists two distinct elements x_1 and x_2 such that $x_1 \in f^{-1}(y)$ and $x_2 \in f^{-1}(y)$. Then,

$$\{x_1, x_2\} \subset f^{-1}(y) = \{x \in X | f(x) \in \{y\}\} = \{x \in X | f(x) = y\}$$

which contradicts the assumption that f is injective. Moreover, since f is surjective

$$Y = f(X) \equiv \{y \in Y | y \in f(x) \text{ for some } x \in X\},$$

which can be rephrased as saying that there exists some $x \in X$ such that $f(x) = y$ for every $y \in Y$. This rules out $f^{-1}(y) = \varnothing$, so the result follows ■

Definition 8 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ the composition $f \circ g$ is given by the function $h : X \rightarrow Y$ such that $h(x) = g(f(x))$ for every $x \in X$.*

Proposition 3 *Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are bijective, then the composition $f \circ g$ is bijective.*

Proof. Fix any $z' \in Z$. Since $g : Y \rightarrow Z$ is surjective there exists $y' \in Y$ such that $g(y') = z'$. Since f is surjective there exists $x' \in X$ such that $f(x') = y'$. Hence,

$$g(f(x')) = g(y') = z'$$

Since $z' \in Z$ was arbitrary $g(f(X)) = Z$.

Suppose that $f \circ g$ is not injective. Then there exists $x_1 \neq x_2$ such that $g(f(x_1)) = g(f(x_2))$. Since f is injective $f(x_1) = y_1 \neq y_2 = f(x_2)$. Hence, there exists $y_1 \neq y_2$ such that $g(y_1) = g(y_2)$ contradicting that g is an injection. ■

1.4 Sequences I

Just like functions can be defined in terms of ordered pairs (defining the function from its graph) we may define an ordered pair as a function

$$f : \{1, 2\} \rightarrow X.$$

Similarly, a finite sequence can be defined as a mapping

$$f : \{1, ..., n\} \rightarrow X.$$

Now, letting N be the set of natural numbers we can define. an infinite sequence as a mapping

$$f : N \rightarrow X.$$

It is customary to depart somewhat from standard functional notation when dealing with sequences and,

1. Write $\langle x_i \rangle_{i=1}^{n}$ instead of $(f(1), f(2), ..., f(n))$ or $\langle f(i) \rangle_{i=1}^{n}$

2. Write $\langle x_i \rangle_{i=1}^{\infty}$ instead of $\langle f(i) \rangle_{i=1}^{\infty}$

1.5 Finite, Infinite, Countable and Uncountable Sets

1.5.1 Finite vs Infinite Sets

A finite set is a set which is either empty or a set for which there exists some natural number n such that the set has exactly n elements. That is:

Definition 9 *The set A is finite if it is empty or if there exists some $n \in N$ and a bijection $f : \{1, ..., n\} \to A$ (this defines a set with exactly n elements too). A set A is infinite if it is not finite.*

Remark 3 *Since the bijection $f : \{1, ..., n\} \to A$ defines an inverse $f^{-1} : A \to \{1, ..., n\}$ finiteness could be defined as a bijection from A onto $\{1, ..., n\}$. Both conventions are used to define finiteness (and similarly for countability).*

The remark above also proves part of the following useful claim:

Proposition 4 *A set A is finite if and only if there exists a finite set B and a bijection $g : A \to B$*

Proof. ($\Longrightarrow$) If A is finite there exists some finite n and a bijection $f : \{1, ..., n\} \to A$, implying that $g = f^{-1} : A \to \{1, ..., n\}$ is well-defined. Letting $B = \{1, ..., n\}$ proves the first part.

($\Longleftarrow$) Suppose that there exists finite set B and a bijection $g : A \to B$. Since B is finite there exists some $n \in N$ and (using the remark) a bijection $h : B \to \{1, ..., n\}$. Since g and h are bijective $g \circ h : A \to \{1, ..., n\}$ is bijective. Hence, there is a bijection $f : \{1, ..., n\} \to A$ where

$$f(i) = g \circ h(i) \text{ for } i = 1, ..., n$$

■

We now prove a result sometimes used in combinatorics, which is referred to as the "Pigeonhole principle". If there are more pigeons than pigeonholes at least two pigeons need to share a hole.

Proposition 5 *Suppose that $m, n \in N$. Then:*

1. *If $m \leq n$ there exists an injection (onto map) $f : \{1, ..., m\} \rightarrow \{1, .., n\}$*

2. *If $m > n$ there is no injection $f : \{1, ..., m\} \rightarrow \{1, .., n\}$.*

Proof. For part 1, let $f : \{1, ..., m\} \rightarrow \{1, .., n\}$ be given by

$$f(j) = j \text{ for every } j \in \{1, ..., m\} \subseteq \{1, .., n\}.$$

Obviously, f is injective

For the part 2, we begin by noting that if $n = 1$, then the only map from $f : \{1, ..., m\} \rightarrow \{1, .., n\}$ is $f(j) = 1$ for every $j \in \{1, ..., m\}$, which is not an injection for any $m > 1$.

Next, assume that there is no injection $f : \{1, ..., m\} \rightarrow \{1, .., k\}$ for any $m > k$. We need to demonstrate that it follows that there is no injection $f : \{1, ..., m\} \rightarrow \{1, .., k+1\}$ for any $m > k + 1$, which will prove the claim by induction.

For contradiction, suppose such an injection would exist. Then, there must be some $j \in \{1, ..., m\}$ such that $f(j) = k+1$ since otherwise $h : \{1, ..., m\} \rightarrow \{1, .., k\}$ is an injection despite $m > k + 1 > k$, which violates the induction hypothesis. Moreover, it must be the case that exactly one element $j^* \in \{1, ..., m\}$ is such that $f(j) = k+1$ since obviously there is no injection more than one element is mapped into $k + 1$. It follows that the function

$$\widetilde{h} : \{1, ..., m\} \setminus \{j^*\} \rightarrow \{1, .., k\}$$

defined as

$$\widetilde{h}(j) = h(j) \text{ for every } j \in \{1, ..., m\} \setminus \{j^*\}$$

must be an injection. At this point we may either just assert that $\{1, ..., m\} \setminus \{j^*\}$ and $\{1, ..., m-1\}$ are equivalent sets or relabel by constructing the map $g : \{1, ..., m-1\} \rightarrow \{1, .., k\}$

$$g(j) = \begin{cases} \widetilde{h}(j) = h(j) & \text{if } j < j^* \\ \widetilde{h}(j-1) = h(j-1) & \text{if } j > j^* \end{cases}.$$

It is obvious that g is injective if and only if $\widetilde{h}$ is injective, but g is not injective according to the induction hypothesis. The result follows ■

We can extend this result to

Proposition 6 *Suppose that $m \in N$. Then:*

1. *There exists an injection (onto map) $f : \{1, ..., m\} \to N$*

2. *There is no injection $f : N \to \{1, .., m\}$*

Proof. For part 1, let $f : \{1, ..., m\} \to N$ be given by

$$f(j) = j \text{ for every } j \in \{1, ..., m\} \subseteq N.$$

Obviously, f is injective

For part 2, if $f : N \to \{1, .., m\}$ is an injection it follows that $f(j) \neq f(i)$ for every pair $i, j \in N$, which implies that $f(j) \neq f(i)$ for every pair $i, j \in \{1, ..., m+1\}$, which contradicts the previous proposition ■

I will take the following for granted (see Bartle and Sherbert pages 61-62)

1.5.2 Countable vs. Uncountable Sets

Countably infinite sets can be defined in similar fashion to the definition of finite sets in Definition 9. However, it is more convenient to use the following definition:

Definition 10 *A set A is said to be countable if there exists an injective function $f : A \to N$.*

Bartle and Sherbert use a different definition and establish existence of an injection $f : A \to N$ as a proposition.

Example 3 *Let $A = N$. Consider the identity function $f : N \to N$ given by*

$$f(i) = i \text{ for every } i \in N.$$

Obviously, this is injective, so N is countable.

Example 4 *Let* $Z = \{... - 1, 0, 1, ...\}$ *be the set of positive and negative integers. Let* $f : Z \to N$ *be given by*

$$f(i) = \begin{cases} -2i & \text{if } i < 0 \\ 2i + 1 & \text{if } i \geq 0 \end{cases},$$

which clearly is injective (as negatives map into even and positives map into odd numbers).

Example 5 *Let* A *be finite. Then there is some* $n \in N$ *and a bijection* $f : A \to \{1, ..., n\}$. *Defining* $f_1 : A \to N$ *as*

$$f_1(a) = f(a)$$

for each $a \in A$ *is obvious that* f_1 *is injective (but not surjective as there exists no* $a \in A$ *such that* $f_1(a) = j$ *when* $j > n$*). Hence, any finite set is countable*

Combining with previous results we conclude that:

Proposition 7 *Let* A *be a countable set. Then,* A *is (countably) infinite if and only if there exists some bijective* $f : A \to N$.

Proof. ($\Rightarrow$) For contradiction, suppose f is surjective and that A is finite. If f is bijective $f^{-1} : N \to A$ exists and is injective (and surjective). If A is finite there exists $n \in N$ and bijection $g : A \to \{1, ..., n\}$. Hence $f^{-1} \circ g : N \to \{1, ..., n\}$ is injective (and surjective), which contradicts Proposition 6. Hence, A cannot be finite, implying that it is infinite

($\Leftarrow$)Suppose that A is finite and that there exists a bijection $f : A \to N$. Then

$$f^{-1} : N \to A$$

is bijective. Moreover, since A is finite there exists a bijection $g : A \to \{1, ...m\}$, where $m \in N$. Hence,

$$f^{-1} \circ g : N \to \{1, ..., n\}$$

is bijective, which is impossible due to Proposition 6. ■

Theorem 1 *If* A *is countable and* $B \subset A$, *then* B *is countable (if* A *is finite, then* B *is finite).*

Proof. First, consider the case with A finite. Without loss, suppose B is non-empty (empty set if finite and therefore countable by default). If B is infinite there exists bijection

$$f : B \to N$$

Let $b_0 \in B \subset A$ and consider map $g : A :\to B$

$$g(a) = \begin{cases} a & \text{if } a \in B \\ b_0 & \text{if } a \notin B \end{cases}.$$

As this map restricted on B is the identity map, this is obviously surjective. Moreover, since A is finite there exists n and bijective map $h : \{1, ..., n\} \to A$. Hence

$$h \circ g \circ f : \{1, ..., n\} \to N$$

is surjective, which which is impossible due to Proposition 6.

Now, suppose A is countably infinite so that there exists bijection $f : A \to N$. Let $g : B \to A$ be given by

$$g(b) = b$$

for every $b \in B$. This is obviously injective, so $g \circ f : B \to N$ is injective. ■

Theorem 2 *The union of a countable number of countable sets is countable.*

Proof. Proof is very similar to proof of countability of rationals.

Without loss, suppose that $\{A_i\}_i$ is a collection of disjoint sets (otherwise look at

$$A_1, A_1 \backslash A_2, A_3 \backslash (A_1 \cup A_2) .....).$$

. Write $\{a_{ij}\}_j$ for the elements. Let $f : \cup_i A_i \to N$ be given by

$$f(a_{ij}) = 2^i 3^{f_i(a_{ij})}.$$

Which is well defined and injective.

[If not there is some $a_{ij} \neq a_{kl}$ with

$$\begin{aligned} 2^i 3^{f_i(a_{ij})} &= 2^k 3^{f_k(a_{kl})} \Leftrightarrow \\ 2^{i-k} &= 3^{f_k(a_{kl}) - f_i(a_{ij})} \end{aligned}$$

LHS values

$$....\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16....$$

RHS values

$$\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81, ...$$

■

Theorem 3 *The set of rationals is countable.*

Proof is very similar.

Theorem 4 *The set of real numbers in the closed interval* $[0, 1]$ *is uncountable.*

Proof. If $[0, 1]$ is countable we could find an injection $f : N \to [0, 1]$. Let $\{x_i\}_{i \in N}$ denote the implied sequence of elements in $[0, 1]$ and write them in decimal form as

$$\begin{aligned} x_1 &= f(1) = 0.x_{11}x_{12}......x_{1n}........ \\ x_2 &= f(2) = 0.x_{21}x_{22}......x_{2n}........ \\ &.... \\ x_n &= f(n) = 0.x_{n1}x_{n2}......x_{nn}........ \\ &..... \end{aligned}$$

Consider

$$y = 0.y_1 y_2 ....y_n......$$

where every $y_i \notin \{0, x_{ii}, 9\}$ [ruling out 0 and 9 is to avoid dealing with 0.5 versus 0.499999....] Hence, y does not belong to $\{x_i\}_{i \in N}$. Since the sequence was arbitrary, no sequence contains all the reals, which completes the proof. ■

2 Sequences and Limits in $\mathbb{R}$

2.1 Suprema and Infima

We will not be careful about the axioms of the real number system, but rather proceed based on our intuitive understanding of ordering properties and arithmetic. However, we will be explicit about ONE crucial property, which relates to a more general completeness property for metric spaces.

Definition 11 *Given a nonempty set $A \subset R$ we say that u is an upper bound for A if $x \leq u$ for every $x \in A$.*

Definition 12 *If an upper bound for A exists we say that A is bounded above.*

Lower bounds/bounded below are defined the same way.

Definition 13 *Given a nonempty set $A \subset R$ we say that u is a least upper bound for A if:*

1. *u is an upper bound for A*
2. *$x \geq u$ for any upper bound x for A*

Example 6 *The set of reals, R, has no upper or lower bound.*

Example 7 *Any $x \geq 0$ is an upper bound for $(-\infty, 0|$. The unique least upper bound for $(-\infty, 0|$ is 0.*

Example 8 *Any $x \geq 0$ is an upper bound for $(-\infty, 0)$. The unique least upper bound for $(-\infty, 0|$ is 0.*

Proposition 8 *If a least upper bound to A exists, then it is unique.*

Proof. Suppose that u and u' are both least upper bounds. Then, $u \leq u'$ since otherwise u wouldn't be a least upper bound and $u' \leq u$ for the same reason. Hence $u = u'$ ■

In what follows we will call the least upper bound to a subset of R the **supremum** of the set. The most crucial assumption about the reals is:

Axiom 1 (Supremum property of R) *For any non empty subset A of R which is bounded above the supremum exists.*

Idea is that there are no gaps among the set of reals.

To appreciate that the axiom is a bit more subtle than one way first think consider the example:

Example 9 *Let Q be the set of rational numbers and $A = \{x \in Q | x^2 < 2\}$. Intuitively we then notice that*

1. *if u is an upper bound for A, then $u^2 = 2$. (proof to follow below)*
2. *there is no rational number such that $u^2 = 2$ (proof to follow below)*

A consequence of the supremum property of the reals is that the **infinum** exists for sets that are bounded below.

Proposition 9 *If a nonempty subset of the reals A is bounded below, then it has a greatest lower bound (infinum).*

Proof. Problem set. Idea is that l is a lower bound of A if and only if $-l$ is an upper bound for

$$\{x \in R | -x \in A\}$$

■

We can then prove "Archimedes axiom" which can be thought of as either of the following two statements:

(1) Given any number, you can always pick an integer that is larger than the original number.

(2) Given any positive number, you can always pick an integer whose reciprocal is less than the original number.

Proposition 10 *The set of natural numbers N is not bounded above.*

Proof. If N is bounded above, then $u = \sup N$ exists by the completeness axiom. Let $n \in N$ be picked arbitrary. We have that $n \leq u$ as N is bounded above. But, $n+1$ is also bounded by u, so

$$n+1 \leq u \Leftrightarrow n \leq u-1.$$

Since the argument holds for an arbitrary $n \in N$ it follows that $u-1$ is also an upper bound for N, which contradicts the leastness of u. ■

Corollary 1 *For any pair $(x, y) \in R$ such that $x \neq y$ there exists a rational number in between.*

Proof. Without loss, assume that $x < y$. Suppose that $x \geq 0$. By Proposition 10 there exists n such that

$$n > \frac{1}{y-x},$$

implying that

$$\frac{1}{n} < y-x$$

Moreover, there exists $v \in N$ such that (here we use fact that $x \geq 0$)

$$v-1 \leq nx < v$$

since:

1. Set $\{m \in N | m \geq nx\}$ is nonempty (otherwise nx upper bound for N)

2. Set $\{m \in N | m \geq nx\}$ is bounded below by nx $\Rightarrow$exists a least element $v \in \{m \in N | m \geq nx\}$

$$x < \frac{v}{n} \leq x + \frac{1}{n} < x + y - x = y.$$

Now, let $x < 0$. Suppose that $y > 0$. Then, the previous argument establishes that there exists $n, v \in N$ such that

$$x < 0 < \frac{v}{n} < y.$$

Finally, let $x < y < 0$. Then, the previous argument establishes that there exists n, v such that

$$-y < \frac{v}{n} < -x,$$

implying that

$$x < -\frac{v}{n} < y.$$

■

We can now get back to the example:

Example 10 (continued) *CLAIM: if u is a least upper bound for $A = \{x \in Q | x^2 < 2\}$ then $u^2 = 2$.*

Proof. *Obviously an upper bound exists (20 works...and is in Q).*

CASE 1: Suppose the least upper bound is such that $u^2 < 2$. By the Archimedan property just proved we may pick n such that

$$\begin{aligned} 0 &< \frac{1}{n} < \frac{2-u^2}{4u} \Rightarrow \frac{4u}{n} < 2-u^2 \\ \text{and } \frac{1}{n} &< 2u, \end{aligned}$$

But then (observing that we can trivially rule out the possibility that u is negative)

$$\begin{aligned} u^2 &< \left(u+\frac{1}{n}\right)^2 = u^2 + \frac{2u}{n} + \frac{1}{n^2} \\ \bigg/ \frac{1}{n} < 2u \bigg/ &< u^2 + \frac{2u}{n} + \frac{2u}{n} = u^2 + \frac{4u}{n} \\ \bigg/ \frac{4u}{n} < 2-u^2 \bigg/ &< u^2 + 2 - u^2 = 2, \end{aligned}$$

implying that u is not a lower bound.

CASE 2: Suppose that $u^2 > 2$. By the Archimedan property we can pick n such that

$$0 < \frac{1}{n} < \frac{u^2-2}{2u} \Rightarrow$$

$$\left(u - \frac{1}{n}\right)^2 = u^2 - \frac{2u}{n} + \frac{1}{n^2} > u^2 - \frac{2u}{n} > u^2 - \left(u^2 - 2\right) = 2,$$

implying that $u - \frac{1}{n}$ is a smaller upper bound of A, contradicting the leastness of u. ■

Remark 4 *CHECK: SUM OF RATIONAL S IS RATIONAL. HENCE THE EXAMPLE SHOWS (ACCEPTING THAT $\sqrt{2}$ IS IRRATIONAL) THAT THE SUPREMUM PROPERTY DOES NOT APPLY TO THE SET OF RATIONAL NUMBERS.*

Remark 5 *By the supremum property of R the example also shows that $\sqrt{2}$ exists in the real number system.*

2.2 Real Sequences

While formally a real sequence is a map $f : N \to R$, but we will usually write $\langle x_1, x_2, ..., x_i, ...\rangle$ or $\langle x_n \rangle_{n=1}^{\infty}$ when this cannot cause any confusion.

A crucial concept is that of convergence:

Definition 14 $\langle x_i \rangle_{i=1}^{\infty}$ *converges to the (real number) x^* if given any (real number) $\varepsilon > 0$ there exists some finite K (possibly dependent on ε) such that*

$$x^* - \varepsilon < x_n < x^* + \varepsilon$$

for every $n \geq K$.

Example 11 *Let $\langle x_n \rangle_{n=1}^{\infty}$ be given by $x_n = \frac{n-1}{n}$ for every $n \in N$. By the Archimedean property of the naturals there exists K such that*

$$0 < \frac{1}{K} < \varepsilon$$

for every $\varepsilon > 0$. Hence, if $n \geq K$

$$|x_n - 1| = \left| \frac{n-1}{n} - 1 \right| = \left| \frac{-1}{n} \right| = \frac{1}{n} \leq \frac{1}{K}.$$

We conclude that $x_n = \frac{n-1}{n}$ converges to 1.

Example 12 *Sometimes limit not so easy to guess, as with*

$$x_n = \left(1 - \frac{1}{n}\right)^n$$

This can be motivated from probability theory: The chance of of k successes in n Bernoulli trials is

$$p_{k|n} = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

So, if the probability of success is $\frac{1}{n}$, then the probability of no success out of n draws is

$$\frac{n!}{1 \times n!!} \left(\frac{1}{n}\right)^0 \left(1 - \frac{1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n.$$

Can show that $\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}$.

Proposition 11 *If $\langle x_n \rangle_{n=1}^{\infty}$ converges there is a unique limit.*

Proof. Suppose that x^* and x^{**} are limits of $\langle x_n \rangle_{n=1}^{\infty}$ and let $x^* < x^{**}$ without loss of generality. Let $\varepsilon = \frac{x^{**} - x^*}{3}$. Because x^* is a limit of $\langle x_n \rangle_{n=1}^{\infty}$ there exists K^* such that

$$x_n < x^* + \varepsilon = x^* + \frac{x^{**} - x^*}{3} = \frac{2}{3}x^* + \frac{1}{3}x^{**}$$

for all $n \geq K^*$. Symmetrically, as x^{**} is a limit of $\langle x_n \rangle_{n=1}^{\infty}$ there exists K^{**} such that

$$x_n > x^{**} - \varepsilon = x^{**} - \frac{x^{**} - x^*}{3} = \frac{2}{3}x^{**} + \frac{1}{3}x^*$$

Hence,

$$\begin{aligned} \frac{2}{3}x^{**} + \frac{1}{3}x^* &< x_n < \frac{2}{3}x^* + \frac{1}{3}x^{**} \Rightarrow \\ \frac{2}{3}\left(x^{**} - x^*\right) &< \frac{1}{3}\left(x^{**} - x^*\right). \end{aligned}$$

$\Rightarrow 2 < 1$.

■

Definition 15 *A sequence of real numbers $\langle x_n \rangle_{n=1}^{\infty}$ is:*

1. *monotonic increasing if $x_{n+1} \geq x_n$ for every $n \in N$*

2. *monotonic decreasing if $x_{n+1} \leq x_n$ for every $n \in N$*

3. *monotonic if either monotonic increasing or decreasing*

Definition 16 *A sequence of real numbers* $\langle x_n \rangle_{n=1}^{\infty}$ *is bounded if there exists* $a, b \in R$ *such that* $a \leq x_n \leq b$ *for every* $n \in N$.

Proposition 12 *Every bounded monotonic sequence is convergent.*

Proof. Let $X = \{x_1, x_2..., x_n, ...\} \subset R$ be the set of elements of the sequence $\langle x_n \rangle_{n=1}^{\infty}$

Remark 6 *Formally, a real valued sequence is defined as a function* $f : N \to R$, *so* $X = f(N)$, *the direct image of* N

Consider the case where the sequence is increasing. Then, X is bounded above by a. By the completeness/supremum property of the real number system there exists a supremum of X. Write $\overline{x} = \sup X$.

Consider an arbitrary $\varepsilon > 0$ and note that $\overline{x} - \varepsilon$ can not be an upper bound of X, since that would contradict that $\overline{x}$ is the LEAST upper bound. Hence, there exists $K < \infty$ such that $x_K > \overline{x} - \varepsilon$, which implies that

$$\overline{x} \geq x_n \geq x_K > \overline{x} - \varepsilon$$

for every $n \geq K$. Hence

$$|x_n - \overline{x}| \leq \varepsilon$$

for every $n \geq K$. Since ε was arbitrary, $\overline{x}$ is the limit of $\langle x_n \rangle_{n=1}^{\infty}$.

Decreasing sequences-same argument.

■

2.3 Subsequences

Definition 17 *Let* $\langle x_n \rangle_{n=1}^{\infty}$ *be a sequence and* $\{k_i\}_{i=1}^{\infty}$ *a strictly monotonic sequence with* $k_i \in N$ *for every* i. *Then, the sequence*

$$\langle x_{k_n} \rangle_{k_n} = \langle x_{k_1}, x_{k_2}, ..., x_{k_n} \rangle$$

(constructed by setting $x_{k_i} = x_n$ *for every* k_i*) is called a subsequence of* $\langle x_n \rangle_{n=1}^{\infty}$

Example 13 *Let* $\langle x_n \rangle_{n=1}^{\infty}$ *be given by*

$$x_n = \begin{cases} a & \text{if } n \text{ is odd} \\ b & \text{if } n \text{ is even} \end{cases},$$

then two obvious subsequences are $(a, a, ..., a, ...)$ *and* $(b, b, ..., b,)$

Theorem 5 (Bolzano & Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let a_0 and b_0 be bounds for $\langle x_n \rangle_{n=1}^{\infty}$, so that

$$x_n \in [a_0, b_0]$$

for every $n \in N$. Now let

$$\begin{aligned} L_1 &= \left[a_0, \frac{a_0 + b_0}{2}\right] \text{ and let } N_1^L = \left\{n \in N | x_n \in \left[a_0, \frac{a_0 + b_0}{2}\right]\right\} \\ H_1 &= \left[\frac{a_0 + b_0}{2}, b_0\right] \text{and let } N_1^H = \left\{n \in N | x_n \in \left[\frac{a_0 + b_0}{2}, b_0\right]\right\} \end{aligned}$$

There are now two possibilities:

1. N_1^L is infinite. If this is the case we let $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$ and let $N_1 = \left\{n \in N | x_n \in \left[a_0, \frac{a_0+b_0}{2}\right]\right\}$

2. N_1^L is finite. If this is the case it follows that N_1^H is infinite. If this is the case we let $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$, and let $N_1 = N_1^H$.

We conclude that there exists an interval $[a_1, b_1] \subset [a_0, b_0]$ and an infinite set $N_1 = \{n \in N | x_n \in [a_1, b_1]\}$, where

$$b_1 - a_1 = \frac{1}{2} [b_0 - a_0]$$

We can then construct $[a_2, b_2], [a_3, b_3]$ .... and N_2, N_3... by continuing to divide the intervals in two halfs. We then have::

CLAIM: For every $k \in N$ there exists an interval $[a_k, b_k]$ with

$$b_k - a_k = \frac{1}{2^k} [b_0 - a_0]$$

and an infinite set

$$N_1 = \{n \in N | x_n \in [a_k, b_k]\}$$

.Suppose claim is true for $k \in N$. Let

$$\begin{aligned} L_{k+1} &= \left[a_k, \frac{a_k + b_k}{2}\right] \text{ and let } N_{k+1}^L = \left\{n \in N | x_n \in \left[a_k, \frac{a_k + b_k}{2}\right]\right\} \\ H_{k+1} &= \left[\frac{a_k + b_k}{2}, b_k\right] \text{ and let } N_{k+1}^H = \left\{n \in N | x_n \in \left[\frac{a_k + b_k}{2}, b_k\right]\right\} \end{aligned}$$

Since

$$\begin{aligned} N_{k+1}^L \cup N_{k+1}^H &= \left\{n \in N | x_n \in \left[a_k, \frac{a_k + b_k}{2}\right]\right\} \cup \left\{n \in N | x_n \in \left[\frac{a_k + b_0}{2}, b_k\right]\right\} \\ &= \{n \in N | x_n \in [a_k, b_k]\} \end{aligned}$$

is infinite by the induction hypothesis we conclude that at either N_{k+1}^L or N_{k+1}^H is infinite. As

$$\begin{aligned} \frac{a_k + b_k}{2} - a_k &= b_k - \frac{a_k + b_k}{2} \\ &= \frac{1}{2}(b_k - a_k) = \frac{1}{2^k}[b_0 - a_0] \end{aligned}$$

by the induction hypothesis this proves the claim. Now:

1. $a_k \in [a_0, b_0]$ for every k and $a_k \geq a_{k-1}$ for every k $\Rightarrow$limit a^* exists (monotone bounded convergence result)

2. $b_k \in [a_0, b_0]$ for every k and $b_k \leq b_{k-1}$ for every k $\Rightarrow$limit b^* exists (monotone bounded convergence result)

3. $c_k = b_k - a_k \in [0, b_0 - a_0]$ for every k and $c_{k+1} = b_{k+1} - a_{k+1} = \frac{1}{2}(b_k - a_k) \leq (b_k - a_k) = c_k$ $\Rightarrow$limit c^*exists (again)

4. $c^* = 0$.

5. $c^* = b^* - a^*$. Hence both $\{a_k\}$ and $\{b_k\}$ has the same limit $a^* = b^*$.

Now, pick $\varepsilon > 0$, let (very inefficient choice)

$$\begin{aligned} k &\in \inf\left\{n \in N | n \geq \frac{2(b_0 - a_0)}{\varepsilon}\right\} \Rightarrow \\ \frac{1}{k} &\leq \frac{\varepsilon}{2(b_0 - a_0)} \end{aligned}$$

And let

$$K(\varepsilon) = \inf\{n \in N | x_n \in [a_k, b_k]\}$$

Then for every $n \geq K(\varepsilon)$ it follows that

$$\begin{aligned} x_n - a^* &\leq b_k - a^* \leq b_k - a_k = \frac{1}{2^k}[b_0 - a_0] \\ &< \frac{1}{k}[b_0 - a_0] \leq \frac{\varepsilon}{2(b_0 - a_0)}[b_0 - a_0] = \frac{\varepsilon}{2}, \end{aligned}$$

which proves the result.

■

2.4 Open and Closed Sets in $\mathbb{R}$

Definition 18 *Given any $\varepsilon > 0$ we say that an open ball (ε neighborhood) of $x \in R$ is given by the set*

$$B(x, \varepsilon) = \{y \in R | \, |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

Definition 19 *A set $A \subset R$ is open if for every $x \in A$ there exists an open ball $B(x, \varepsilon) \subset A$.*

Definition 20 *A set $A \subset R$ is closed if the complement $B = R \backslash A$ is open.*

Example 14 *Let $A = R$. Fix x and let $\varepsilon = 1$. Then any $y \in (x - 1, x + 1) \in R$, so R is open.*

Example 15 *Consider the open interval $A = (a, b)$ .Pick any $x \in (a, b)$ and let*

$$\varepsilon = \min\{x - a, b - x\}$$

then for any $y \in B(x,\varepsilon)$

$$\begin{aligned} y &> x-\varepsilon = x-\min\{x-a,b-x\} \\ &\geq x+\max\{a-x,x-b\} \geq x+a-x=a \\ y &< x+\varepsilon = x+\min\{x-a,b-x\} \\ &\leq x+b-x=b. \end{aligned}$$

Hence, (a,b) is an open set.

Example 16 *Let $A=(\infty,a)$ and $B=(b,\infty)$. Open, by same argument.*

Example 17 *Consider $[a,b]$. Not open since for every $\varepsilon>0$*

$$B(a,\varepsilon)\cap[a,b]=(a-\varepsilon,a]$$

in nonempty for every $\varepsilon>0$.

Note that

$$R\backslash[a,b]=(\infty,a)\cup(b,\infty).$$

By the previous argument, if $x\in(\infty,a)$ we can find a ball $B(x,\varepsilon)\subset(\infty,a)$ and if x is in (b,∞) we can also find a ball $B(x,\varepsilon)\subset(b,\infty)$. Hence, there an open ball $B(x,\varepsilon)\subset(\infty,a)\cup(b,\infty)$ for every $x\in(\infty,a)\cup(b,\infty)$, so $(\infty,a)\cup(b,\infty)$. we conclude that $[a,b]$ is a closed set.

Example 18 *$\varnothing$ is open (vacuously since there is no point in the set). Hence, R is closed and open. Since R is also open, it follows that $\varnothing$ is open and closed.*

Theorem 6 *The union of any collection of open sets is open.*

Proof. Let I be some indexing set and assume that A_i is open for every $i\in I$. If $x\in A=\cup_{i\in I}A_i$. Then there exists $j\in I$ and $\varepsilon>0$ such that $B(x,\varepsilon)\subset A_j$. Hence,

$$B(x,\varepsilon)\subset A_j\subset\cup_{i\in I}A_i.$$

■

Theorem 7 *The intersection of a finite collection of open sets is open.*

Proof. Let $x \in \cap_{i=1}^{k} A_i$. Let $\varepsilon_i > 0$ be such that $B(x, \varepsilon_i) \subset A_i$, which exists since $x \in \cap_{i=1}^{k} A_i \Rightarrow x \in A_i$ for every i and each A_i is open. Let

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, .., \varepsilon_k\} > 0,$$

and note that

$$B(x, \varepsilon) \subset B(x, \varepsilon_i) \subset A_i \text{ for every } i$$

implying that

$$B(x, \varepsilon) \subset \cap_{i=1}^{k} A_i.$$

■

Example 19 *Let* $\left(0, \frac{1}{k}\right)$ *and consider*

$$\cap_{i=1}^{k} \left(-\frac{1}{k}, 1 + \frac{1}{k}\right) = \left(-\frac{1}{k}, 1 + \frac{1}{k}\right)$$

which is open for every finite k. *However*

$$\cap_{i=1}^{\infty} \left(-\frac{1}{k}, 1 + \frac{1}{k}\right) = [0, 1],$$

which is not open.

Theorem 8 *The intersection of an arbitrary collection of closed sets is closed.*

Proof. Let I be some indexing set and assume that A_i is closed for every $i \in I$. By de Morgans laws

$$R \backslash (\cap_{i \in I} A_i) = \cup_{i \in I} R \backslash A_i.$$

Each set $R \backslash A_i$ is open, so (by previous theorem) $\cup_{i \in I} R \backslash A_i$ is open, proving that $R \backslash (\cap_{i \in I} A_i)$ is open, which by definition means that $\cap_{i \in I} A_i$ is closed. ■

Theorem 9 *The union of a finite collection of closed sets is closed.*

Proof. By de Morgans laws

$$R \backslash \left(\cup_{i=1}^{k} A_i \right) = \cap_{i=1}^{k} R \backslash A_i.$$

Each set $R \backslash A_i$ is open and k is finite, so (by previous theorem) $\cap_{i=1}^{k} R \backslash A_i$ is open, proving that $R \backslash \left(\cup_{i=1}^{k} A_i \right)$ is open, which by definition means that $\cup_{i=1}^{k} A_i$ is closed. ■

Definition 21 (ALT 1) *Given set $A \subset R$, $x \in R$ is called a cluster point of A if for every $\varepsilon > 0$ there exists $y \in B(x, \varepsilon) \cap A \backslash \{x\}$.*

Proposition 13 *A set A is closed if and only if it contains all its cluster points.*

Proof. ($\Rightarrow$) Suppose not. Then, there exists an cluster point $x \in R \backslash A$, which is an open set. Hence, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset R \backslash A$, implying that $B(x, \varepsilon) \cap A = \varnothing$, contradicting that x is a cluster point of A.

($\Leftarrow$) Suppose that A contains all cluster points. Pick $y \in R \backslash A$. Then, y is not a cluster point, so there exists open ball $B(y, \varepsilon) \subset R \backslash A$.True for all $y \in R \backslash A$, so $R \backslash A$ is open. Hence, A is closed. ■

Almost the same result stated in terms of sequences.

Proposition 14 *A is closed if and only if the limit of every converging sequence $\langle x_n \rangle$ with $x_n \in A$ for all n belongs to A.*

Proof. We will prove this from the previous result by showing that

CLAIM: *A contains all its cluster points$\Leftrightarrow$the limit of every converging sequence $\langle x_n \rangle$ with $x_n \in A$ for all n belongs to A.*

($\Rightarrow$) Suppose that $\langle x_n \rangle$ converges to x^*.Then, for every $\varepsilon > 0$ there exists K such that $x_n \in B(x^*, \varepsilon)$ for $n \geq K$, and, by hypothesis, $x_n \in A$. Hence, x^* is a cluster point of A. By assumption A contains its cluster points, so $\lim_{n \to \infty} x_n = x^* \in A$.

Suppose that $x^* \notin A$ is a cluster point. Let $\varepsilon_1 > 0$ and pick $x_1 \in B(x^*, \varepsilon_1) \backslash \{x^*\}$ arbitrarily. As $x_1 \neq x^*$ there exists $\varepsilon_2 > 0$ such that $x_1 \notin B(x^*, \varepsilon_1)$, implying that there exists $x_2 \in B(x^*, \varepsilon_2) \subset B(x^*, \varepsilon_1)$ because x^* is a cluster point. Recursively, if $x_k \neq x^*$

and $x_k \in B(x^*, \varepsilon_k)$ there exists $\varepsilon_{k+1} > 0$ such that $x_k \notin B(x^*, \varepsilon_1)$. Hence, there exists $x_{k+1} \in B(x^*, \varepsilon_{k+1}) \subset B(x^*, \varepsilon_1)$. This defines an infinite sequence which obviously converges to x^*, so $x^* \in A$ if the limit of every converging sequence is in A. ■

An alternative definition is.

Remark 7 *The proof shows that given any set $A \subset R$, $x \in R$ is a cluster point of A if for every $\varepsilon > 0$ there exists an infinite set $X \subset B(x, \varepsilon) \cap A$. This is an alternative definition often used for a cluster point.*

2.5 Compact Sets

Definition 22 *An open cover of a set A in R is a collection $\mathcal{C} = \{C_i\}$ of sets such that $A \subset \cup_{i \in I} C_i$*

Definition 23 *$\mathcal{C}'$ is a subcover of $\mathcal{C}$ if each set C_j in $\mathcal{C}'$ also is in $\mathcal{C}$ and $\mathcal{C}'$ is a cover of A.*

Definition 24 *A in R is said to be compact if **every** open cover $\mathcal{C}$ has a finite open subcovering $\mathcal{C}'$.*

Theorem 10 (Heine Borel) *$A \subset R$ is compact if and only if A is closed and bounded.*

Remark 8 *Every analysis book says that result is "closely related to Bolzano Weierstrass", yet proofs tend to not use that result. Here is an argument that makes explicit use of Bolzano Weierstrass.*

Proof. ($\Rightarrow$) Suppose that A is compact. We note that if $C_i = (-i, i)$ then

$$\cup_{i=1}^{\infty} C_i$$

is an open cover of R. Hence,

$$A \subset R \subset \cup_1^{\infty} C_i$$

so $\{(-i, i) \,|\, i \in N\}$is an open cover of A as well. Since it is compact, there exists some finite K such that

$$A \subset \cup_{i=1}^{K} C_i = (-K, K),$$

so A is bounded. Let $y \notin A$ and let

$$C_i = R \backslash \left[y - \frac{1}{i}, y + \frac{1}{i}\right]$$

Obviously, G_i is open ($\left[y - \frac{1}{i}, y + \frac{1}{i}\right]$ is closed) and we have that $\cup_{i=1}^{\infty} C_i = R$ which covers R and therefore also A. By compactness there is some finite K such that

$$\begin{aligned}
A &\subset \cup_{i=1}^{K} C_i = R \backslash \left[y - \frac{1}{K}, y + \frac{1}{K}\right] \Rightarrow \\
A \cap \left[y - \frac{1}{K}, y + \frac{1}{K}\right] &= \varnothing \Rightarrow \\
A \cap \left(y - \frac{1}{K}, y + \frac{1}{K}\right) &= \varnothing
\end{aligned}$$

Hence, for every $y \notin A \Leftrightarrow y \in R \backslash A$ there exists $\varepsilon > 0$ such that $\left(y - \frac{1}{K}, y + \frac{1}{K}\right) \subset R \backslash A$, so $R \backslash A$ is open. Hence, A is closed.

($\Leftarrow$) For contradiction, suppose A is closed and bounded, but that that there is an open cover $\mathcal{C} = \{C_\alpha\}$ for which there is no finite subcover.

For any n, consider the collection of open sets

$$\mathcal{C}_n = \{C_{\alpha,n}\} \text{ where } C_{\alpha,n} = \left\{x \in C_\alpha | \inf_{y \in R \backslash C_\alpha} |x - y| > \frac{1}{n}\right\}$$

CASE 1: Suppose that there exists K such that $\mathcal{C}_K$ is a subcover of $\mathcal{C}$. Since A is bounded by there exists m such that $A \subset [-m, m]$ so let

$$\begin{aligned}
x_1 &= -m \text{ and } B\left(x_1, \frac{1}{K}\right) = \left(-m - \frac{1}{K}, -m + \frac{1}{K}\right) \\
x_2 &= -m + \frac{1}{2K} \text{ and } B\left(x_2, \frac{1}{K}\right) = \left(-m - \frac{1}{2K}, -m + \frac{3}{2K}\right) \\
&\quad .... \\
x_n &= -m + \frac{(n-1)}{2K} \text{ and } B\left(x_n, \frac{1}{K}\right) = \left(-m + \frac{(n-3)}{2K}, -m + \frac{(n+1)}{2K}\right) \\
&\quad .... \\
x_{4mK+1} &= -m + \frac{(4mK + 1 - 1)}{2K} = -m + 2m = m
\end{aligned}$$

Hence, $\left\{B\left(x_n, \frac{1}{K}\right)\right\}_{n=1}^{4mK+1}$ is a finite cover of A. Moreover, each $x_n \in C_{\alpha_n,K}$ by the hypothesis that $\mathcal{C}_K$ is a subcover of $\mathcal{C}$. Pick any $y \in B\left(x_n, \frac{1}{K}\right)$. Then,

$$\inf_{x \in R \backslash x_n \in C_{\alpha_n,K}} |x - y| \geq \inf_{x_n \in C_{\alpha_n,K}} |x - x_n| - |y - x_n| > 0$$

since $|y - x_n| < \frac{1}{K}$ as y is in $B\left(x_n, \frac{1}{K}\right)$ and $\inf_{x_n \in C_{\alpha_n,K}} |x - x_n| \geq \frac{1}{K}$ by construction of $C_{\alpha,K}$.

We conclude that $B\left(x_n, \frac{1}{K}\right) \subset C_{\alpha_n,K}$. As $\left\{B\left(x_n, \frac{1}{K}\right)\right\}_{n=1}^{4mK+1}$ is an open cover it follows that $\{C_{\alpha_n,K}\}_{n=1}^{4mK+1}$ is an open cover of A. Obviously, $\{C_{\alpha_n,K}\}_{n=1}^{4mK+1}$ is a finite subcover of $\mathcal{C}_n = \{C_{\alpha,n}\}$, and by assumption $\mathcal{C}_n = \{C_{\alpha,n}\}$ is a subcover of $\mathcal{C}$. Hence, A is compact.

CASE 2: Suppose A is not covered by $\mathcal{C}_n = \{C_{\alpha,n}\}$ for any n. Then there is a sequence $\{x_n\}$ with $x_n \in A$ and $x_n \in R \backslash \cup_\alpha C_{\alpha n}$ for every $n \in N$. Let $\{x_{k_n}\}$ be a convergent subsequence with limit x^*. Then we note that:

1. Since A is closed $x^* \in A$

2. Since $\mathcal{C}$ is a covering of A there exists α^* such that $x^* \in C_{\alpha^*}$

3. C_{α^*} is open so there exists $\varepsilon > 0$ such that $B(x^*, \varepsilon) \subset C_{\alpha^*}$

4. Since there is no n such that $x^* \in \cup_\alpha C_{\alpha,n}$ there is no n such that

$$\begin{aligned} x^* \quad &\in \quad C_{\alpha^* n} = \left\{x \in C_{\alpha^*} | \inf_{y \in R \backslash C_{\alpha^*}} |x - y| > \frac{1}{n}\right\} \\ &\Leftrightarrow \quad \text{there is no } n \text{ such that } B\left(x^*, \frac{1}{n}\right) \subset C_{\alpha^*} \end{aligned}$$

   [if $B\left(x^*, \frac{1}{n}\right) \subset C_{\alpha^*}$ then $|x^* - y| \geq \frac{1}{n} > \frac{1}{n+1}$ for every $y \notin C_{\alpha^*}$ implying that $x^* \in C_{\alpha^* n+1}$]

5. Statements there exists $\varepsilon > 0$ such that $B(x^*, \varepsilon) \subset C_{\alpha^*}$ and there exists no n such that $B\left(x^*, \frac{1}{n}\right) \subset C_{\alpha^*}$ contradict each other.

We conclude that there cannot be any convergent subsequence to $\{x_n\}$, which (since $\{x_n\}$ is bounded) contradicts the Bolzano Weierstrass theorem. It follows that A must be covered by $\mathcal{C}_n = \{C_{\alpha,n}\}$ for some n, which by analysis in CASE 1 means that A is compact.

2.6 Cauchy Convergence

An alternative way of thinking about convergent sequences is as a sequence with terms that get closer and closer together further out in the sequence.

Definition 25 $\langle x_n \rangle_{n=1}^{\infty}$ *is said to be a Cauchy sequence if given any* $\varepsilon > 0$ *there exists* K *such that*

$$|x_n - x_m| < \varepsilon$$

for every $m, n \geq K$.

The set of real numbers is a simple example of a complete metric space. Such spaces have many important properties, one being that the two convergence criteria are equivalent (which in the end gives a nice characterization of compactness).

The easy direction is:

Lemma 1 *Every convergent sequence of real numbers* $\langle x_n \rangle_{n=1}^{\infty}$ *is a Cauchy sequence.*

Proof. Fix $\varepsilon > 0$ and let $\langle x_n \rangle_{n=1}^{\infty}$ have limit x^*. As $\langle x_n \rangle_{n=1}^{\infty}$ converges to x^* there exists $K\left(\frac{\varepsilon}{3}\right) \in N$ such that

$$\begin{aligned} |x_n - x^*| &< \frac{\varepsilon}{2} \\ |x_m - x^*| &< \frac{\varepsilon}{2} \end{aligned}$$

for every $n, m \geq K\left(\frac{\varepsilon}{2}\right)$. Thus (triangle inequality "direct path shorter than any other path")

$$\begin{aligned} |x_n - x_m| &= |(x_n - x^*) + (x_m - x^*)| \\ &\leq |(x_n - x^*)| + |(x_m - x^*)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

■

To establish the other direction is a bit more work, but we may rely on some of the work we have already done. In addition, the following fact is important.

Lemma 2 *If* $\langle x_n \rangle_{n=1}^{\infty}$ *is a Cauchy sequence, then there exists* a, b *such that* $a < x_n < b$ *for every* n.

Proof. Pick some $\varepsilon > 0$ and let K be such that $|x_n - x_m| < \varepsilon$ for every $n, m \geq K$. Then

$$|x_n - x_K| < \varepsilon$$

for every n, implying that

$$\begin{aligned} x_n - x_K &< \varepsilon \text{ and } x_K - x_n < \varepsilon \\ x_K - \varepsilon &< x_n < x_K + \varepsilon \end{aligned}$$

for every $n \geq K$. The set

$$\{x_1, x_2, ..., x_K - \varepsilon, x_K + \varepsilon\}$$

is obviously bounded above and below and any bounds a, b to this set also bounds $x_K + \varepsilon$. ■

Theorem 11 $\langle x_n \rangle_{n=1}^{\infty}$ *converges if and only if it is a Cauchy sequence.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence. By Lemma above it is then bounded above and below. Applying the Bolzano-Weierstrass Theorem we know that there exists a convergent subsequence $\langle x_{k_n} \rangle_{k_n}$. Let the limit of the convergent subsequence be denoted by x^*. Fix $\varepsilon > 0$. Since $\langle x_{k_n} \rangle$ converges to x^* there exists $K_1 \in \{k_i\}_{i=1}^{\infty}$ such that

$$|x_n - x^*| < \frac{\varepsilon}{2}$$

for every $n \geq K_1$ such that there exists an element in $\{k_i\}_{i=1}^{\infty}$ with $k_i = n$. This implies that

$$|x_{K_1} - x^*| < \frac{\varepsilon}{2}.$$

But, since $\langle x_n \rangle_{n=1}^{\infty}$ is Cauchy there exists K_2 such that

$$|x_n - x_{K_1}| < \frac{\varepsilon}{2}$$

holds for every $n \geq K_2$. Therefore

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_{K_1}) + (x_{K_1} - x^*)| \\ &\leq |x_n - x_{K_1}| + |x_{K_1} - x^*| < \varepsilon \end{aligned}$$

holds for every $n \geq K_2$. We conclude that x^* is the limit of $\langle x_n \rangle_{n=1}^{\infty}$. ■