The ex post efficient provision rule is to provide the good if
\[ \theta_1 + \theta_2 \geq 12 \]

Define
\[ P = (P_{0,0}, P_{0,5}, P_{0,10}, P_{5,0}, P_{5,5}, P_{5,10}, P_{10,0}, P_{10,5}, P_{10,10}) \]
\[ = (0, 0, 0, 0, 0, 1, 0, 1, 1) \]
\[ t^A = (t_{0,0}^A, t_{0,5}^A, t_{0,10}^A, t_{5,0}^A, t_{5,5}^A, t_{5,10}^A, t_{10,0}^A, t_{10,5}^A, t_{10,10}^A) \]
and symmetrically for \( B \)

by LR constraints
\[ t^A = (0, 0, 0, 0, 0, t_{5,10}^A, 0, t_{10,5}^A, t_{10,10}^A) \]

\[ 1C - 10^A \]
\[ \frac{8}{10} \left( 10 - t_{10,10}^A \cdot P_{10,10} \right) + \frac{1}{10} \left( 10 - t_{10,5}^A \cdot P_{10,5} \right) \geq \frac{8}{10} \left( 10 - t_{5,10}^A \cdot P_{5,0} \right) \]
\[ \Rightarrow \quad 10 \geq 8 t_{10,10}^A + t_{10,5}^A - 8 t_{5,10}^A \]

\[ 1R - 10^A \]
\[ 10 - t_{10,10}^A \geq 0 \]
\[ 10 - t_{10,5}^A \geq 0 \]

\[ 1R - 5^A \]
\[ 5 - t_{5,10}^A \geq 0 \]
This will bind, notice can increase \( t_{5,5}^A \) until it does and all the other relevant constraints still hold.
Budget Balance given

\[ t_{s,10}^a + t_{s,10}^b = 12 \]

\[ \Rightarrow t_{s,10}^a = 7, \quad t_{s,10}^b = 5 \]

by symmetry \( t_{10,10}^a = 7, \quad t_{10,10}^b = 5 \)

Plugging into \( 10 - 10^a \)

\[ 10 > 8 t_{10,10}^a + 4 = 8(5) \]

\[ \frac{50 - 7}{8} \geq t_{10,10}^a \quad \Rightarrow \quad t_{10,10}^a \leq 5.3 \]

Budget Balance Required

\[ t_{10,10}^a + t_{10,10}^b = 12 \]

this cannot be the case when \( t_{10,10}^a = t_{10,10}^b \leq 5.3 \)
A proportional tax mechanism

\[ t = a \theta \]

Budget Balance

\[ \begin{align*}
0 & - P_{0,12} \geq 0 \\
5a & - P_{5,12} \geq 0 \\
5a & - P_{5,15} \geq 0 \\
10a & - P_{10,12} \geq 0 \\
10a & - P_{10,110} \geq 0 \\
15a & - P_{15,12} \geq 0 \\
15a & - P_{15,110} \geq 0 \\
20a & - P_{20,12} \geq 0 \\
\end{align*} \]

IR constraints

\[ \begin{align*}
P_{0,10} (0 - 0a) & \geq 0 \\
P_{0,5} (0 - 0a) & \geq 0 \\
P_{0,110} (0 - 0a) & \geq 0 \\
P_{5,0} (5 - 5a) & \geq 0 \\
P_{10,10} (10 - 10a) & \geq 0 \\
P_{5,10} (5 - 5a) & \geq 0 \\
P_{10,110} (10 - 10a) & \geq 0 \\
\end{align*} \]

Clearly \( a \leq 1 \)

\[ \Rightarrow P_{0,10}, P_{0,5}, P_{5,0}, P_{10,10}, P_{10,110}, P_{5,10} \text{ all equal 0} \]
Max: total Surplus

\[ \begin{align*}
&+ \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( P_{00} (0 - 12) \right) + \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( P_{5,0} (5 - 12) \right) \\
&+ \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( P_{0,5} (5 - 12) \right) + \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( P_{5,5} (10 - 12) \right) \\
&+ \left( \frac{1}{10} \right) \left( \frac{8}{10} \right) \left( P_{3,10} (15 - 12) \right) + \left( \frac{1}{10} \right) \left( \frac{8}{10} \right) \left( P_{1,5} (15 - 12) \right) \\
&+ \left( \frac{1}{10} \right) \left( \frac{8}{10} \right) \left( P_{0,10} (10 - 12) \right) + \left( \frac{1}{10} \right) \left( \frac{8}{10} \right) \left( P_{0,10} (10 - 12) \right) \\
&\left( \frac{8}{10} \right) \left( \frac{8}{10} \right) \left( P_{0,10} (20 - 12) \right) 
\end{align*} \]

We know we cannot implement the optimal allocation rule, so our options are

- \( P_{0,10} = 1 \Rightarrow \) all else zero, or the good is never provided.
- When \( P_{0,10} = 1 \), surplus \( = \left( \frac{8}{10} \right)^2 (20 - 12) > 0 \)
- So it is better than never providing

\[ \begin{align*}
&\left( \frac{1}{10} \right) A \\
&\frac{8}{10} (10 - t_{10,10}^A) \geq 0 \\
&\Rightarrow \quad 10 > t_{10,10}^A \\
\end{align*} \]

Most efficient \( t = t_{10,10}^A = t_{10,110}^B = 10 \)

**Final Result:** Allocation Rule \( P_{0,10} = 1 \), all else equal zero:

- \( t_{10,10} = t_{10,110}^B = 10 \)
Suppose \( P_{x,10} \neq 0 \), \( P_{x,15} \neq 0 \), \( P_{x,10} = 1 \)

\[
10 - 10
\]

\[
\frac{8}{10} \left(10 - 10a\right) + \frac{1}{10} \left(10 - 10a\right) \geq \frac{8}{10} \left(10 - 5a\right)
\]

\[
\Rightarrow 9 - 9a \geq 8 - 4a \Rightarrow a \leq \frac{1}{5}
\]

Budget balance tells us:

\[
a \geq \frac{12}{15} \quad \text{so this assumption does not work.}
\]

Suppose \( P_{x,10} = 1 \) and all others are zero.

\[
10 - 10
\]

\[
\frac{8}{10} \left(10 - 10a\right) \geq 0
\]

Budget balance tells us:

\[
a = \frac{12}{20} \Rightarrow a = \frac{6}{10}
\]

Final result

Allocation rule: Provide public good only if both value it at 10

Each pay transfer of 10
Again, we know the optimal rule cannot be implemented, so let's look at $P_{1,10} = 1$, all else zero.

$$t_{10,10}^A = t_{10,10}^B = x \cdot 20$$

**[BB]**

$$x \cdot 40 - 12 = 0 \Rightarrow x = \frac{12}{40} = \frac{3}{10}$$

**[IR -10]**

$$10 - x \cdot 20 \geq 0 \Rightarrow x \leq \frac{1}{2}$$

**Final Result:**

**Allocation Rule:** $P_{1,10} = 1$, all else are zero.

$$t_{10,10}^A = t_{10,10}^B = x \cdot 20 \text{ where } x = \frac{3}{10}$$
Optimal auctions are all about maximizing the seller's revenue. From the Revenue Equivalence Theorem, we know that all allocation rules lead to the highest seller revenue. So the question here is what allocation rule will lead to the highest seller revenue.

Recall the seller wants to maximize
\[
\max_{\Theta_i \in [0, 1]} \left( \Theta_i - \int (1 - F(\theta)) \pi_i(\theta) f_i(\theta) d\theta \right)
\]

Subject to
\[
\sum_{j=1}^{n} \Theta_j = 1
\]

We know \( \Theta_i \) is Uniformly distributed on [0, 1].

Because the seller's revenue is
\[
\max_{\Theta_i \in [0, 1]} \left( \Theta_i - \int (1 - F(\theta)) \pi_i(\theta) d\theta \right)
\]

They set their expected revenue to exceed zero for them to sell the item.
\[ \sum_{i=1}^{n} (\theta_i - 1) p_i(\theta) \geq 0 \]

\[ \Rightarrow \text{when } p_i(\theta) = 1 \text{ (meaning the good is sold to person } i) \]

\[ \text{it must be the case that} \]

\[ \theta_i - 1 \geq 0 \Rightarrow \theta_i \geq 1/2 \]

So the optimal mechanism is the following:

\[ p_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \max \left\{ \frac{1}{2}, \theta_{i-1} \right\} \\ 0 & \text{otherwise} \end{cases} \]
If $\Theta_2 \in [0, 2/3]$, then $v_2(\Theta_2) = \Theta_2 - \frac{1 - \Theta_2}{3}$.

If $\Theta_2 \in [0, 2/3]$ and $\Theta_2 \in (3-2\varepsilon)(\Theta_2 - 2/3)$, then $v_2(\Theta_2) = \Theta_2 - \frac{1 - 2\varepsilon/3}{3 - 2\varepsilon}$.

Clearly, it must be the case that

$$2\Theta_2 - \frac{1}{2} \leq \frac{2\Theta_2 - \frac{2}{3}}{3 - 2\varepsilon} \leq 1 - \frac{2\varepsilon/3}{3 - 2\varepsilon}.$$
\[ \Rightarrow 0 \leq \frac{\mu}{3} - 5 + \frac{3}{\omega} \]

\[ \Rightarrow 0 \leq 2\varepsilon^2 - 5\varepsilon + 3 \]

\[ \Rightarrow \varepsilon \leq 1 \]
The virtual valuations are

\[ v_1(\theta_1) = \theta_1 - (1 - \theta_1) + \theta_1 \in [0, 1] \]

\[ v_2(\theta_2) = \begin{cases} 
\frac{2}{\varepsilon} \theta_2 - \frac{1}{\varepsilon} (1 - \varepsilon \theta_2) & \text{if } \theta_2 \in [0, 2/3] \\
\frac{2}{3 - 2\varepsilon} (1 - \frac{2\varepsilon}{3} \theta_2 - (3 - 2\varepsilon)(\theta_2 - \frac{2}{3})) & \text{if } \theta_2 \in (2/3, 1] 
\end{cases} \]

Note that:

\[ v_2(\frac{1}{2}) = \frac{1}{2} - \frac{1 - \varepsilon/2}{\varepsilon} = \frac{2(\varepsilon - 1)}{2\varepsilon} \leq 0 \text{ if } \varepsilon \leq 1 \]

whereas \( v_1(\frac{1}{2}) = 0 \)

Both virtual valuations are continuous on \([0, 2/3]\), so if \( \varepsilon \leq 1 \) there is a range \([\frac{1}{2}, 5]\) where:

\[ v_1(x) > 0 \text{ and } v_2(x) < 0 \]

\( \forall \theta_1, \theta_2 \in [\frac{1}{2}, 5] \times [\frac{1}{2}, 5] \)

\underline{Conclusion:} Optimal Allocation rule is to sell the good to the type one bidder if \( \theta_1, \theta_2 \in [\frac{1}{2}, 5] \times [\frac{1}{2}, 5] \)
Intuitively, the gain from discriminating in favor of the "weak bidder" is that it allows the seller to extract more surplus from agents with higher types. This makes sense, because the probability of a valuation in the range $[\frac{2}{3},1]$ is higher for agent 2 than for agent 1.

\[\lim_{s \to 0} \Pr(Ω_t ≤ Ω_t' + s | Ω_t ≥ Ω_t'') = \lim_{s \to 0} \frac{s}{\int_{Ω_t' + s}^{Ω_t} f_i(c_i) \, dc_i \, \delta(1 - F_i(\Omega_t'))} \]

Use l'Hopital's rule

\[\Rightarrow \lim_{s \to 0} \frac{f_i(\Omega_t' + s) \, dc_i}{1 - F_i(\Omega_t'')} = \frac{f_i(\Omega_t'')}{1 - F_i(\Omega_t'')}\]