Question 1

1. Static Nash:

\[ \Pi_1 = q_1 (1 - q_1 - q_2) \]

s.t. \[ 1 - 2q_1 - q_2 = 0 \quad \Rightarrow \quad q_1 = \frac{1 - q_2}{2} \]

by symmetry: \[ q_2 = \frac{1 - q_1}{2} \]

\[ \Rightarrow \text{Nash:} \quad q_1 = q_2 = \frac{1}{3} \]

\[ \Pi_i^N = \frac{1}{9} \quad \forall \ i = 1, 2 \]

Cartel

\[ \Pi_T = Q(1-Q) \]

s.t. \[ 1 - 2Q = 0 \quad \Rightarrow \quad Q = \frac{1}{2} \]

Assuming they split the market:

Cartel: \[ q_1 = q_2 = \frac{1}{4} \]

\[ \Pi_i^C = \frac{1}{8} \quad \forall \ i = 1, 2 \]
Question 1

3) Nash Reversion Strategy.

1st we must determine what a player will deviate to if he knows the other player is choosing \( q = \frac{1}{4} \)

\[
\text{BR}_i(q) = \frac{1 - q}{2q} = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}
\]

such that \( \Pi_i^p = \frac{3}{8} (1 - \frac{1}{4} - \frac{3}{8}) = \frac{9}{64} \)

Let's state the strategy explicitly:

\( S_i^t = \frac{1}{4} \)

\[
S_i^t = \begin{cases} 
\frac{1}{4} \text{ if } h_t = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots) \\
\frac{1}{3} \text{ otherwise}
\end{cases}
\]

All we have to do is check whether there is a profitable single deviation strategy payoff:

\[
\frac{1}{8} \left( \frac{1}{1 - \frac{3}{8}} \right)
\]

deviation payoff:

\[
\frac{9}{64} + \frac{1}{9} \left( \frac{8}{1 - \frac{3}{8}} \right)
\]
Question 1 continued

\[
\frac{1}{8} \left( \frac{1}{1-8} \right) \geq \frac{9}{89} + \frac{1}{9} \left( \frac{8}{1-8} \right)
\]

\[= \Rightarrow 1 \geq \frac{9}{8} (1-8) + \frac{8 \cdot 8}{9}
\]

\[\Rightarrow \delta \geq \frac{9}{1/7} \text { will support this strategy.}
\]

3) Assume \(q_1 = q_2 = x\) rather than \(= \frac{1}{4}\)

we need

\[
\frac{x (1-2x)}{1-8} \geq \left( \frac{1-x}{\delta} \right)^2 + \frac{8}{9 (1-8)}
\]

to hold such that \(\delta\) has a large range of support.

\[\Rightarrow \delta \geq \left[ \left( \frac{1-x}{\delta} \right)^2 - \frac{1}{9} \right]^{-1} \left[ \left( \frac{1-x}{\delta} \right)^2 - x (1-2x) \right]
\]

\[\text{Want this } \leq \frac{9}{1/7}
\]

Calculation yields: \(x \in \left[ \frac{1}{4}, \frac{1}{3} \right]\)

will be supportable by \(\delta\) that is less than \(\frac{9}{1/7}\).
Question 1

4. Suppose the following strategy: (known as one-period carrot and stick)

\[ S_i = \frac{1}{4} \]

\[ S_i^* = \begin{cases} \frac{1}{4} & \text{if } (a_{1i}^{*}, a_{2i}^{*}) = (\frac{1}{4}, \frac{1}{4}) \text{ or } (x, x) \\ x & \text{otherwise} \end{cases} \]

There are 2 possible states of this game:

The cooperation stage, and punishment stage.

Cooperation

Strategy payoff: \( \frac{1}{8(1-x)} \)

Deviation payoff: \( \frac{9}{8x} + 8x(1-3x) + \frac{8x^2}{8(1-x)} \)

Punishment

Strategy payoff: \( x(1-2x) + \frac{8}{8(1-x)} \)

Deviation payoff: \( \frac{(1-x)^2}{2} + 8x(1-3x) + \frac{8x^2}{8(1-x)} \)
(4) continued

Goal: find an $x$ such that a smaller $\frac{8}{8}$ will support $S\Pi$.

Suppose: $x$ is such that $\Pi = 0$ (this is the worse possible punishment)

$\Rightarrow 1 - 2x = 0 \Rightarrow x = \frac{1}{2}$

We can construct the relevant inequalities as follows:

(A) $\frac{1}{8} + \frac{8}{8} \geq \frac{9}{\epsilon y} + 8(x(1-2x))$

and

(B) $x(1-2x) + \frac{8}{8} \geq \left(\frac{1-x}{2}\right)^2 + 8x(1-2x)$

if $x = \frac{1}{2}$

(A) will hold when

$\frac{1}{8} + \frac{8}{8} \geq \frac{9}{\epsilon y} \Rightarrow 8 \geq \frac{9}{8} - \frac{8}{8}$

$\Rightarrow 8 \geq \frac{1}{8}$

(B) will hold when

$\frac{8}{8} \geq \frac{1}{10} \Rightarrow 8 \geq \frac{1}{2}$
4 continued
So we have constructed a SPE strategy that is supported by $\delta \geq \frac{1}{2}$.
This is the lowest $\delta$ a one-period carrot and stick strategy can support because we have used the strongest punishment: $T = 0$. 
Question 2

Statement: If a subgame perfect strategy profile is history independent then it must induce a stage game Nash Equ. in each stage of G^T

Proof: Suppose not, so that we have a strategy, $s^*$ that includes a non-Nash outcome at period t.

$\Rightarrow$ at time t there is a profitable deviation (by definition of Nash Equ.)

$st. \quad \Pi_t^s < \Pi_t^{Deviation}$

If our strategy is SPE then

$\Pi_t^s + \sum_{j=1}^{T} \delta^j \Pi_j^s \geq \Pi_t^{Dev} + \sum_{j=1}^{T} \delta^j \Pi_j^s$

however, this is not the case, so we must conclude that a stage game Nash must be played in every stage of G^T.

Intuition: In a history independent strategy you are unable to punish or reward, consequently a non-Nash outcome cannot be supported in any stage because the lack of credible threat makes deviation inevitable.
Question 2

2. Suppose we have strategy \( S^* \) which is history independent and induces Nash equilibrium in every stage. We can think of \( S^* \) in the following way:

\[
S^* = (s^*_1, s^*_2, \ldots, s^*_T)
\]

no matter what the history of the game.

To show this is SPE, notice that deviation in any period will result in a lower payoff (by definition of Nash).

So,

\[
\Pi^{s^*_t} \geq \Pi^{d^*_t}
\]

Consequently, Total Profit from following the strategy \( S^* \):

\[
\Pi^{s^*_1} + \sum_{t=1}^{T} s^* \Pi^{s^*_t}
\]

will always be larger than the total profit from a one shot deviation:

\[
\Pi^{d^*_1} + \sum_{t=1}^{T} s^* \Pi^{s^*_t}
\]

\( \therefore \) The strategy profile \( S^* \) is SPE.
Question 2

Construct the following stage game:

\[
\begin{array}{c|cc}
 & A & B \\
\hline
A & 1, 1 & 1, 1 \\
B & 1, 1 & 5, 5 \\
\end{array}
\]

There are 2 Nash equil. of this game: 
\((A, A), (B, B)\)

The following strategy will induce a Nash in every period, but is not a SPE:

\[S_i^t = \begin{cases} 
A & \text{if } (a_i^t, a_{-i}^t) = (A, A) \\
B & \text{otherwise} 
\end{cases}\]

There are 2 possible states of the game \((A, A)\) and \((B, B)\) we must check whether a profitable deviation exists for either state.

**State \(A, A\)**

strategy payoff: \[\sum_{t=0}^{T} S^t(1) = V^s\]

deviation payoff: \[1 + \sum_{t=1}^{T} S^t(5) = V^p\]

\[V^p > V^s \forall s \geq 0\]
So we have found a profitable deviation
\[ \rightarrow S \text{ is not SPE. Even though } (A,A) \]
is a Nash eqv.

**Reasoning:** I set up the strategy so that you were basically rewarded for deviating.
Consider the following stage game:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0,0</td>
<td>0,7</td>
<td>-1,1</td>
</tr>
<tr>
<td>B</td>
<td>7,0</td>
<td>3,3</td>
<td>-1,1</td>
</tr>
<tr>
<td>C</td>
<td>-1,1</td>
<td>-1,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

This game has 2 nash equ: (B,B), (C,C)

The following strategy will induce a SPE that does not consist of a nash equ. being played in each stage:

\[ s_i = A \]

\[ s_i^* = \begin{cases} B & \text{if } (s_i, s_j) = (A, A) \text{ or } (B, B) \\ C & \text{otherwise} \end{cases} \]

States of the game: (A,A) (B,B) (C,C)

Stage A,A:

Strategy payoff: \[ 0 + \sum_{t=1}^{T} s_t \cdot 3 = v^s \]

deviation payoff: \[ v + \sum_{t=1}^{T} s_t(0) = v^p \]

clearly \( v^s \geq v^p \) \( \forall s > 0 \)
(4) Continued

There is no need to check for stage \((B,B)\) and \((C,C)\) because they are both Nash Equilibrium, which implies that there can be no profitable deviation.

Hence, we have constructed a SPE with a single period of non-Nash play.

Question 3: See MWG Solutions Guide.