Problem Set 4 Solutions

Extensive Form

\((0,0)\)

\((1,1)\)

\((2,2)\)

\((3,3)\)
Sequential play is a refinement of Perfect Bayesian Equilibrium. So I begin by solving for all PBE.

Separating: Tough plays B, Wimp plays Q.

\[ \Rightarrow \mu = 1, \quad q = 0 \]

Barnie will play N when B is played and D when Q is played.

Wimp will always have incentive to deviate.

\[ \Rightarrow \text{Not PBE} \]

Separating: Tough plays Q, Wimp plays B.

\[ \Rightarrow \mu = 0, \quad q = 1 \]

Barnie will play N when Q is played and D when B is played.

Wimp will always have incentive to deviate.

\[ \Rightarrow \text{Not PBE} \]

Pooling at Q

When Q is observed, \[ q = \frac{9}{10} \]

Barnie gets: \% from D, \% from N

So will play N after Q.
continued

→ \( W \) never has incentive to deviate,
→ \( T \) will depending on off the equ. path beliefs. Specifically we need \( D \) to be played with a high enough probability

Barnie will play \( D \) when

\[ 1 - \mu > \mu \implies 1 > 2\mu \implies \mu < \frac{1}{2} \]

PBE where both types play \( 0 \), and
Barnie plays \( N \) after \( 0 \).
If \( B \) is observed Barrie believes \( \mu \leq \frac{1}{2} \) and plays \( D \).

Pooling at \( B \)
when \( B \) is observed \( \mu = \frac{9}{110} \)

Barrie gets: \( \frac{9}{110} \) from \( D \), \( \frac{9}{110} \) from \( N \)
so he will play \( N \) after \( B \)

→ \( T \) will never deviate
→ \( W \) will depending on off equ. beliefs.

we want Barrie to play \( D \)

\[ 1 - q > q \implies q < \frac{1}{2} \]

PBE where both types play \( B \), and
Barnie plays \( N \) after \( B \). If \( q \) is observed Barrie believes \( q \leq \frac{1}{2} \) and plays \( D \).
continued

\[ a = \frac{pr(T|B) - pr(B|T) \cdot p(T)}{pr(B|T) \cdot p(T) + pr(B|Q) \cdot p(Q)} \]

\[ \frac{q}{p + d(1-p)} \]

\[ \frac{g}{g+l} \]

\[
\begin{align*}
8 = & \frac{pr(T|Q) - pr(Q|T) \cdot p(T)}{pr(Q|T) \cdot p(T) + pr(Q|Q) \cdot p(Q)} \\
& \frac{g}{g+l} \]
\end{align*}
\]

Semi-separating

Semi-separating
(3) continued

b is chosen to force indifference:

\[
\frac{1}{9b+1} = \frac{9b}{9b+1} \quad b = \frac{1}{9}.
\]

Player 2 is indifferent, how must he mix to prevent player 1 from deviating when 2 is played.
He plays prob. d on D

W type:

\[
d + (1-d)3 \geq 2
\]

\[
d + 1-d > \quad \Rightarrow \quad 1>0 \quad \text{always holds}
\]

T type:

\[
2(1-d) = 3
\]
\[
2 - 2d = 3
\]
\[
-2d = -1 \quad d = \frac{1}{2} \quad \text{not possible}
\]

Cannot make player 1 indifferent, so he will never mix.
We have 2 PBEs, both are pooling.

As discussed in class (on 02/17), we can always find a sequence that rationalizes the necessary off-the-equilibrium path beliefs.

Thus, the following are also sequential equilibria.

(A) Both types play $Q$
   - Barnie plays $N$ after $Q$
     - $q = \frac{9}{10}$, $r = \frac{1}{2}$
   - Barnie plays $D$ after $B$

(B) Both types play $B$
   - Barnie plays $N$ after $B$
     - $u = \frac{9}{10}$, $q = \frac{1}{2}$
   - Barnie plays $D$ after $Q$
### Normal Form

<table>
<thead>
<tr>
<th></th>
<th>DD</th>
<th>DN</th>
<th>ND</th>
<th>NN</th>
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</thead>
<tbody>
<tr>
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<td>0.29/10, 0.9/10</td>
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<td>0.1/10, 0.1/10</td>
<td>0.5/10, 0.9/10</td>
</tr>
</tbody>
</table>

Where BB means: Bif T, Bif N

DD means: Dif B, Dif Q

Nash: \((QQ, DN), (BB, ND)\)

and a mixed.
2.4) I am going to apply the intuitive criterion to part 2.2.

The idea behind this is that we can eliminate a PBE if there is some type \( \Theta \) who has a deviation that is assured of yielding him a higher payoff as long as the other players do not assign a positive probability to the deviation having been made by any type \( \Theta \) to whom the action isagu.
dominated.

Look at PBE A (pooling at \( \Theta \))

\( W \) never has incentive to deviate so player \( \Theta \) should not place positive prob on \( W \) type if \( B \) is observed

\[ \Rightarrow 1 - \mu = 0 \Rightarrow \mu = 1 \]

However, if \( \mu = 1 \), \( T \) type will now have incentive to deviate.

Thus this equ. does not pass the intuitive criterion.

Look at PBE B (pooling at B)

\( T \) never has incentive to deviate

\[ \Rightarrow q = 0 \]

\( W \) will also not be able to profitably dev. This passes intuitive criterion.
Re-do assuming \( p = \frac{1}{2} \)

2.2

- Separating, the equations have the same analysis as when \( p = \frac{9}{10} \)

Pooling at 0

When 0 is observed, \( q = \frac{1}{3} \)

Barnie gets \( \frac{1}{2} \) from D

\( \frac{1}{3} \) from N

\[ \Rightarrow \] he will mix placing probabilistic on D.

Will W deviate?

- Depends on beliefs

If \( \mu > \frac{1}{2} \), Barnie plays N

\[ 2(1-d) \geq 3 \] not possible, always deviates.

If \( \mu < \frac{1}{2} \), Barnie plays D

\[ 2(1-d) \geq 1 \Rightarrow d \leq \frac{1}{2} \]

Will W deviate?

If \( \mu < \frac{1}{2} \), Barnie plays D

\[ d + (1-d)^3 \geq 0 \Rightarrow d \leq \frac{3}{2} \]
PRE:
Both types play $\mathcal{Q}$, Barnie mixes, placing prob. $d \leq \frac{1}{2}$ on $\mathcal{D}$ when $\mathcal{Q}$ is observed. Barnie plays $\mathcal{D}$ when $\mathcal{B}$ is observed.

$\mu \leq \frac{1}{2}, \quad q = \frac{1}{2}$

Pooling at $\mathcal{B}$ when $\mathcal{B}$ is observed $\mu = \frac{1}{2}$

Barnie again will mix, placing prob. $d$ on $\mathcal{D}$

- Suppose $q > \frac{1}{2}$, such that if $\mathcal{Q}$ is observed Barnie plays $\mathcal{N}$

will T deviate

\[ d + 3(1-d) \geq 2 \implies d \leq \frac{1}{2} \]

will W deviate

\[ q(1-d) \geq 3, \quad \text{no } d \text{ can prevent deviation.} \]

- Suppose $q < \frac{1}{2}$, such that if $\mathcal{Q}$ is observed Barnie plays $\mathcal{D}$

will T deviate

\[ d + 3(1-d) \geq 0 \implies d \leq \frac{3}{2} \]
Both Types play B. Barrie plays D when B is observed on D when D is observed.

Both Types play B. Barrie plays D when B is observed on D when D is observed.

\[ \alpha(c, d) \geq 1 \Rightarrow d \leq 1/2. \]

Semi-separating.

T plays B w/ prob. \( b \) and

\[ x = \frac{1}{1 + b}. \]

l is chosen to force indifference in players 2.

\[ l = \frac{1}{1 + l}. \]

L = 1 \text{ not mixing back to pooling eqn.}
\[ g = \frac{b}{1+b} \]

- \( g \) is chosen to force indifference.
- \( b \) is chosen sequentially.
- Just as discussed previously, both PBE and NE are always sequentially rationalized.

<table>
<thead>
<tr>
<th>Normal Form</th>
<th>( P )</th>
<th>( D )</th>
<th>( N )</th>
<th>( D )</th>
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</tbody>
</table>

- \( N = \{ (QQ, DN), (BB, ND) \} \)
Notice that player 2 will never play \( P_2 \) knowing this move as \( P_1 \) is played.

We use this information to work our way upwards from the following game.
Rolling at R

- \( q = \frac{1}{2} \)
- \( T \) yields \( \frac{1}{2} \)
- \( B \) yields \( \frac{1}{2} \)

Rolling at L

- \( 0 \leq t < 2/3 \)

indifferent

how must mixing occur to make R dominant for both?

Place t prob on T, want to prevent deviation

Player 1: \( t(3) > 1 \)

\( 1-t \) \( \geq 1 \)

\( 3-3t \geq 1 \)

\( t \leq \frac{2}{3} \)

\( t \geq \frac{1}{3} \)

\( t \leq \frac{1}{3} \)

no t can prevent deviation.
Separating \( \Theta_2 \) goes R, \( \Theta_3 \) goes L

\[ \Rightarrow \quad \Phi = 1 \]

Player 2 goes T after R

No one has incentive to deviate

PBE

Separating \( \Theta_2 \) goes L, \( \Theta_3 \) goes R

\[ \Rightarrow \quad \Phi = 0 \]

B played after R

No one has incentive to deviate

PBE

Semi-Pooling

\( \Theta_2 \) always plays R

\( \Theta_3 \) plays R with prob \( R \) and L with prob \( 1 - R \)

\[ P = \frac{\Pr(\Theta_2 | R) \cdot \Pr(\Theta_3 | R)}{\Pr(\Theta_2 | R) \cdot \Pr(\Theta_3 | R) + \Pr(\Theta_1 | R) \cdot \Pr(\Theta_3 | R)} \]

\[ = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{1 + r}} \]
r is chosen to make player 2 indifferent when they see R

\[ \frac{1}{1+r} = \frac{r}{1+r} \implies r = 1 \]

\[ = \text{ not mixed any more!} \]

\[ \text{Semi-Pooling} \]

\[ \Theta_3 \text{ always plays } R \]

\[ \Theta_2 \text{ plays } R \text{ with prob } r \]

\[ f = \Pr(\Theta_2 | R) = \frac{r^{1/2}}{r^{1/2} + 1/2} = \frac{r}{r+1} \]

r chosen to make player 2 indifferent

\[ \frac{r}{r+1} = \frac{1}{1+r} \implies r = 1 \]

\[ = \text{ not mixed any more!} \]
Sequential Equilibrium

Just as before, all our PBE are sequential eqv. They are:

(A) \( \Theta_1 \) plays L
\( \Theta_2 \) and \( \Theta_3 \) play R
\text{Player 2 mixes, placing prob on Top, where}
\[ 1/3 < t < 1/3 \]
\text{when } R \text{ is observed}
\[ \eta = 1/2, \quad \Pr(\Theta_1|R) = 0 \]

(B) \( \Theta_1 \) plays L
\( \Theta_2 \) plays R, \( \Theta_3 \) plays L
\text{Player 2 plays T when } R \text{ is obs.}
\[ \eta = 1, \quad \Pr(\Theta_1|R) = 0 \]

(C) \( \Theta_1 \) plays L
\( \Theta_2 \) plays L, \( \Theta_3 \) plays R
\text{Player 2 plays B when } R \text{ is obs.}
\[ \eta = 0, \quad \Pr(\Theta_1|R) = 0 \]

\text{Note: See Rubenberg + Tirole pg 344}
Thm 8.4.
Equilibrium Dominance

$m$ is equi. dominated if

$$U_i^*(\theta) > \max_{a \in A} U_i(m, a, \theta)$$

Equi. dominance occurs when receiver assigns zero prob. to $\theta$ after observing $m$ when,

(i) $m$ is equi. dominated for $\theta$
(ii) $\exists \theta'$ s.t. $m$ is not equi. dominated.

$m \in \mathbb{L}, R^2$ \hspace{1cm} $A = \mathbb{E}, L, M, B^3$

$\text{Equ}(A)$

$$U_i^*(\theta_1) = 1$$

$$U_i^*(\theta_2) \in (1, 2)$$

$$U_i^*(\theta_3) \in (1, 2)$$

$L$ is equi. dominated for both $\theta_2$ and $\theta_3$ but not $\theta_1 =>$ zero prob. must be placed on $\theta_2$ and $\theta_3$ when $L$ is observed

This holds trivially since player 2 doesn't act when $L$ is played.

$R$ is equi. dominated for $\theta_1$, but not $\theta_2$ or $\theta_3$ => zero prob. must be placed on $\theta_1$ when $R$ is observed. This holds

Equi. passes...
Equ. (B)

$U_i^*(\Theta_1) = 1$
$U_i^*(\Theta_2) = 3$
$U_i^*(\Theta_3) = 1$

R is Equ. dominated for $\Theta_1$
L is Equ. dominated for $\Theta_2$

$\Rightarrow \Pr(\Theta_1 | R) = 0$
$\Pr(\Theta_2 | L) = 0$

These both hold, the Equ. passed

Equ. (C)

$U_i^*(\Theta_1) = 1$
$U_i^*(\Theta_2) = 1$
$U_i^*(\Theta_3) = 3$

R Equ. dominated for $\Theta_1$
L Equ. dominated for $\Theta_3$

$\Rightarrow \Pr(\Theta_1 | R) = 0$
$\Pr(\Theta_3 | L) = 0$

These both hold, the Equ. passed
3.3 Intuitive Criterion.

Define $BR(T, a_i)$ set of all pure strategy best responses for player 2 to action $a_i$ for beliefs $\mu(.|a_i)$ such that $\mu(T|a_i) = 1$

$$BR(\Theta, R) = \{T, H, B \}$$

$$J(a_i) = \{ \Theta : u^*(\Theta) > \max_{a_j \in BR(\Theta|J_i(a_i), a_i)} u_i(a_i, a_j, \Theta') \}$$

Then if for some $a_i, \exists \Theta' \in \Theta$ s.t.

$$u^*(\Theta') < \min_{a_j \in BR(\Theta|J_i(a_i), a_i)} u_i(a_i, a_j, \Theta')$$

the equ. fails intuitive criterion.

Equ (A)

$$J(R) = \{ \Theta \}$$

$$J(L) = \{ \Theta_0, \Theta_3 \}$$

Passes, no need to check and condition in this case.

Equ (B)

$$J(R) = \{ \Theta_1 \}$$

$$J(L) = \{ \Theta_0, \Theta_3 \}$$

R: $u^*(\Theta_3) = 1 > \min_{a_j \in BR(\Theta_0, \Theta_3, R)} u_i(R, a_2, \Theta_3) = 0$

So this passes.
Equ. (c)

\[ J(R) = 3 \Theta_1 \gamma \]
\[ J(L) = 3 \Theta_3 \gamma \]

R: \( U''_1(\Theta_2) = 1 \) \( \geq \min_{a_0 \in BR(\Theta_4, \Theta_5, R)} U_1(R, a_0, a_0) = 0 \)

**this also passes.**
\[ \theta \in \{1, 2\} \]
\[ q = \Pr(\theta = 2), \quad m(t) - \text{posterior} \quad \Pr(\theta = 2) \]

Utility:
\[ U(w, t, \theta) = w - \frac{t}{q} \]
\[ \Pi = \theta - w = m(t) \cdot q + (1 - m(t)) \cdot (1 - w) - w \]

Assume:
\[ \Pi = 0 \implies w = m(t) + 1 \]

Suppose:
\[ \theta = 1 \text{ type mixes between } t^{\text{**}} \text{ (prob } 1 - q) \text{ and } t^{\text{**}} \text{ (prob } q) \]
\[ \theta = 2 \text{ type only plays } t^{\text{**}} \]

Consequently,
\[ m(2|t) = 0 \text{ where } t \neq t^{\text{**}} \]

\[ m(2|t^{\text{**}}) = \frac{\Pr(t^{\text{**}}|2) \cdot \Pr(2)}{\Pr(t^{\text{**}}|12) \cdot \Pr(2) + \Pr(t^{\text{**}}|11) \cdot \Pr(1)} \]

\[ = \frac{q}{q + p(1 - q)} \]
4) continued

If the $\Theta = 1$ type is mixing it must mean that he is indifferent between the 2 outcomes.

$$1 - t^{**} = \frac{g}{\text{wage}} + 1 - t^{ss}$$

Note: $t^{**}$ must be equal to 0, the logic is if by playing $t^{**}$ the firm assumes the worker is type $\Theta = 1$, and provides wage $1 + \mu(2t) = 1$, then the worker's best response is to choose $t^{**} = 0$ and maximize payoff.

Thus,

$$t^{ss} = \frac{g}{g + f(1-g)}$$

For the $\Theta = 2$ type the following must hold, or he will want to deviate.

$$1 + \frac{g}{g + f(1-g)} - \frac{g}{g + f(1-g)}(\frac{1}{2}) \geq 1 + \mu(\Theta | t^0) - \frac{t^0}{2}$$
4. continued

This clearly will always hold

Equilibrium

\[ t = \frac{t}{t - \frac{g + g_1 - g_2}{8} \cdot \theta} \]

\[ t = \frac{g}{g + g_1 - g_2} \]

\[ \text{Equilibrium:} \quad \mu = \frac{1}{2} \] when \( \theta = 0 \)

\[ \mu = \frac{g}{g + g_1 - g_2} \]

\[ \text{Eq}(\theta = 0) = \frac{g}{g + g_1 - g_2} \]

\[ \text{where } \theta = \mu \cdot \mu \]