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1.1 Sufficient Conditions for Strict Concavity

\[ \pi_1(q_1 + q_2) = q_1 p(q_1 + q_2) - c(q_1) \]
\[ \pi_2(q_1 + q_2) = q_2 p(q_1 + q_2) - c(q_2) \]

For \( \pi_i \) to be strictly concave the following must hold:

1. \( p(q_1 + q_2) \) must be \( C^2 \) in \( q_1 \) and \( q_2 \)
2. \( c(q_i) \) must be \( C^2 \)
3. \( \frac{\partial^2}{\partial q_1 \partial q_1} = p''(q_1 + q_2)q_1 + 2p'(q_1 + q_2) - c''(q_1) < 0 \)

1.2 Proof of Uniqueness: \( \pi_i(q_i + q_j) \) is strictly concave this should imply that there is a unique \( q_i^* \) that solves the optimization problem.

Suppose not:

\[ \Rightarrow q_i^* \text{ and } q_i^{**} \text{ both solve the optimization problem.} \]

Define \( q_i^\lambda = (1 - \lambda)q_i^* + \lambda q_i^{**} \)

By concavity we know:

\[ \pi_i(q_i^\lambda, q_j) > (1 - \lambda)\pi_i(q_i^*, q_j) + \lambda \pi_i(q_i^{**}, q_j) \]

Given that both \( q_i^* \) and \( q_i^{**} \) are maximizers: \( \pi_i(q_i^*, q_j) = \lambda \pi_i(q_i^{**}, q_j) \)

\[ \Rightarrow \pi_i(q_i^\lambda, q_j) > \pi_i(q_i^*, q_j) = \pi_i(q_i^{**}, q_j) \]

This contradicts the assumption that \( q_i^* \) and \( q_i^{**} \) are both profit maximizing.

\[ \Rightarrow \text{there must be a unique solution to the optimization problem.} \]

1.3 Proof that \( B_1(q) = B_2(q) \)

Assume \( B_1(q) \) is player 1’s best response to \( q \) and \( B_2(q) \) is player 1’s best response to \( q \).

Then by definition of best response:

\[ B_1(q)p(B_1(q) + q) - c(B_1(q)) = \pi_1(B_1(q), q) \geq \pi_1(B_2(q), q) = B_2(q)p(B_2(q) + q) - c(B_2(q)) \]

and

\[ B_1(q)p(B_1(q) + q) - c(B_1(q)) = \pi_2(B_1(q), q) \leq \pi_2(B_2(q), q) = B_2(q)p(B_2(q) + q) - c(B_2(q)) \]
\[ \Rightarrow \pi_1(B_1(q), q) = \pi_1(B_2(q), q) \text{ and } \pi_2(B_1(q), q) = \pi_2(B_2(q), q) \]

From before we know that there is a unique solution to each optimization, thus it must be that \( B_1(q) = B_2(q) \)

1.4 Proof that \( B_i \) is continuous

The theorem of the maximum says that given sets \( X \subseteq \mathbb{R}^n \) and \( \Omega \subseteq \mathbb{R}^p \), let \( f : X \times \Omega \rightarrow \mathbb{R} \) be a continuous function, and \( C : \Omega \rightarrow X \) a compact-valued and continuous correspondence, and consider the parameterized maximization problem:

\[ \max_{x \in C(\alpha)} f(x, \alpha) \]

Then the value function

\[ V(\alpha) = \max_{x \in C(\alpha)} f(x, \alpha) \]

is continuous, and the solution correspondence \( S : \Omega \rightarrow X, \)

\[ S(\alpha) = \operatorname{argmax}_{x \in C(\alpha)} \{ x(\alpha; f(x, \alpha) = V(\alpha) \} \]

is nonempty, compact-valued, and upper-hemicontinuous.

It is clear that our maximization problem meets the above conditions. Thus we can conclude that the Best Response functions are nonempty, compact-valued, and upper-hemicontinuous. Upper-hemicontinuity, in addition to the fact that the best responses are unique (as shown above) implies that they are also continuous.

**Proof** that \( B_i(0) > 0 \)

\[ \pi_i(0, 0) = 0 \]

\[ \pi_i(q_i, 0) = p(q_i) - \frac{c(q_i)}{q_i} \]

If the limit of this is positive this implies that for a small enough \( q_i \), \( \pi_i \) will be positive.

Thus \( B(0) > 0 \iff \lim_{q_i \to 0} p(q_i) - \frac{c(q_i)}{q_i} > 0 \)

1.5 Proof of a Symmetric equilibrium

\( B(0) - 0 > 0 \) By the above proof.

\( B(\bar{q}) - \bar{q} < 0 \) \( \bar{q} \) is large and the best response to it is zero.

By the intermediate value theorem this implies that there is some \( q' \) such that:

\[ B(q') - q' = 0 \quad \Rightarrow \quad B(q') = q' \quad \Rightarrow \text{there is at least one symmetric equilibrium.} \]
1.6 The basic argument here is that the Best Response functions need not be well behaved. Specifically, we can imagine a situation, as depicted in the following image, in which there are both symmetric and asymmetric equilibria. Note: There is a more formal proof, Peter may or may not supply it to me, if he does I will relay it to you.

711 TA/BR.jpg
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2.1 Extensive form

![Game Tree](game1.jpg)

2.2 Normal Form Payoff Matrix

As discussed in class there are actually $2^5 = 32$ strategies available to each player. Below is a reduced normal form. Specifically I have eliminated all strategies that are irrelevant as a result of the players own actions.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C(CC)</strong></td>
<td><strong>C(CC)</strong></td>
</tr>
<tr>
<td>2,2</td>
<td>2,2</td>
</tr>
<tr>
<td>2,2</td>
<td>2,2</td>
</tr>
<tr>
<td>3,0</td>
<td>3,0</td>
</tr>
<tr>
<td>3,0</td>
<td>3,0</td>
</tr>
<tr>
<td><strong>D(CC)</strong></td>
<td>3,0</td>
</tr>
<tr>
<td><strong>D(CD)</strong></td>
<td>3,0</td>
</tr>
<tr>
<td><strong>D(DC)</strong></td>
<td>4,-2</td>
</tr>
<tr>
<td><strong>D(DD)</strong></td>
<td>4,-2</td>
</tr>
</tbody>
</table>

Note: **C(CD)** means play C in the first period and play C in the second period if the other player choose C and play D in the second period if the other player choose D.

2.3 The Nash Equilibria of the reduced normal form game are \{ (D(DD), D(DD)), (D(CD), D(DD)) \}
For player 1 C(CD), C(DC), C(DD), D(DD), D(DC), D(CD) all survive iterated elimination of strictly dominated strategies. For player 2 C(DC), C(DD), D(CD), D(DD) all survive iterated elimination of strictly dominated strategies.

Notice that the Nash Equilibrium of the twice repeated game results in a Nash being played in each stage. We can generalize this to the T-stage repeated game, meaning that the equilibrium of this game will also consist of a Nash Equilibrium being played in each stage of the game.

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3.1 Strategy Spaces

\[ S_1 = q_1 \in [0, \infty) \text{ or } \mathbb{R}_+ \]

\[ S_2 : q_1 \rightarrow q_2 \quad \text{where } q_i \in [0, \infty) \text{ or } \mathbb{R}_+^\infty \]

More simply put, \( S_1 \) is a choice of \( q_1 \) and \( S_2 \) is a mapping from \( q_1 \) to \( q_2 \).

3.2 The outcome corresponding to the strategy profile \((s_1, s_2)\) is the output vector \((s_1, s_2(s_1))\), or to be more consistent with the preceding terminology it is the output vector \((q_1, S_2(q_1))\).

3.3 \((q_1 = 5, q_2 = \frac{9 - q_1}{2})\) is not a Nash Equilibrium. Notice that while player 2 is best responding to player 1, player 1 could increase his payoff by choosing 4.5.

3.4 Backwards Induction Equilibrium

Step 1: Begin with firm 2, their goal is to:

\[
\max_{q_2} (9 - q_1 - q_2)q_2 \tag{1}
\]

\[
f.o.c: 9 - q_1 - 2q_2 = 0 \implies q_2 = \frac{9 - q_1}{2} \tag{2}
\]

Step 2: Firm 1 is aware of how firm 2 will react to their choice of \( q_1 \), so their goal is to:

\[
\max_{q_1} (9 - q_1 - \frac{9 - q_1}{2})q_1 \tag{3}
\]

This can be simplified to:
\[ \max_{q_1} \left( \frac{9 - q_1}{2} \right) q_1 \]  

(4)

\[ f.o.c: \left( \frac{9 - 2q_1}{2} \right) = 0 \implies q_2 = 4.5 \]  

(5)

The unique backwards induction equilibrium = \( \left( \frac{9}{7}, \frac{9-q_1}{2} \right) \)

The unique backwards induction outcome = \( \left( \frac{9}{7}, \frac{9}{7} \right) \)

3.5 A nash equilibrium of this game that is not a backwards induction equilibrium is the so-called “cournot” equilibrium. That is we solve the game as if the 2 firms are acting simultaneously. As discussed in class this yields \( q_1 = q_2 = 3 \).

To check that this is a Nash Equilibrium note:

\[ BR_2(q_1 = 3) = \frac{9-q_1}{2} = 3 \]

\[ BR_1(q_2 = 3) = \arg\max_{q_1} (9 - q_1 - 3)q_1 = 3 \]

3.5 \( Q_1 = \{ q_1 : q_1 \in [0,9) \} \)

Argument: Player 2 can threaten Player 1 by claiming that if player 1 does not choose \( \tilde{q}_1 \), then player will play \( 9 - q_1 \), giving player 1 and 2 zero profit. If player 1 believes this threat, his best response is to play the requested \( q_1 \), and player 2 will of course play \( \frac{9-q_2}{2} \), his best response. Using this argument, any \( q_1 \in [0,9) \) can be requested by player 2, and player 1 will comply, thus all these \( q_1 \) belong in \( Q_1 \).

Note: in second stage player 2 is required to play a best response, so the threat is not credible, this is why these Nash Equilibria are not Subgame Perfect Equilibria.