6 Public Choice

6.1 Voting over 2 Alternatives

To start with, consider a problem where there are n citizens labeled $i = 1, 2, .., n$ and a set of alternatives given by $A = \{x, y\}$. You can for example interpret this as a two-candidate presidential election or as a referendum. We will analyze this as a simultaneous move game.

Let $U_i(x)$ and $U_i(y)$ denote the utility of agent i given alternatives x and y and let a strategy for agent i be given by $s_i \in \{-1, 0, 1\}$, where

- $s_i = -1$ means that i votes for alternative x
- $s_i = 0$ means that the voter abstains
- $s_i = 1$ means that the voter votes for y

Majority rule then means that;

- x wins if $\sum_{i=1}^n s_i < 0$
- There is a tie (assume that we flip a coin in this case) if $\sum_{i=1}^n s_i = 0$
- y wins if $\sum_{i=1}^n s_i > 0$

For brevity, let $\pi(s)$ be the probability that x wins given votes $s = (s_1, ..., s_n)$, that is

$$\pi(s) = \begin{cases} 1 & \text{if } \sum_{i=1}^n s_i < 0 \\ \frac{1}{2} & \text{if } \sum_{i=1}^n s_i = 0 \\ 0 & \text{if } \sum_{i=1}^n s_i > 0 \end{cases}$$

We can then write the payoff function for i in terms of the strategic variables as

$$U_i(s) = \pi(s) U_i(x) + (1 - \pi(s)) U_i(y)$$

We now observe the following:

Remark 1 *Suppose that $n \geq 3$. Then (irrespective of preferences) it is always a Nash equilibrium for all players to vote for x. Symmetrically, there it is also a Nash equilibrium for all players to vote for y.*

This is obvious, because if all citizens vote for x then x wins, and if all agents but one vote for x, then x still wins. Hence, no voter is pivotal (can change the outcome), implying that all voters are indifferent between voting for x and voting for y. The argument for y is symmetric.

It follows that, in a vote between party A and party B, all agents may prefer party B, but is is nevertheless consistent with Nash equilibrium for everyone to vote for A for the simple reason that all agents (correctly) believe that their vote cannot change the outcome. Still, in a vote with two alternatives we sort of think that people cast their vote for what they like, and this belief is supported by the fact that voting for the favorite alternative is a *weakly dominant strategy*:

Proposition 1 *Voting for the most preferred alternative is a weakly dominant strategy.*

To see this, suppose without loss of generality that i prefers x. Consider the following cases;

1. x wins even if i votes for y ($\sum_{j \neq i} s_j < -1$). Then the payoff is $U_i(x)$ regarddless of how i votes.

2. x wins if i votes x and there is a tie if i votes for y ($\sum_{j \neq i} s_j = -1$). In this case the payoff is $U_i(x)$ if i votes for x and

$$\frac{1}{2}U_i(x) + \frac{1}{2}U_i(y) < U_i(x)$$

   if i votes for y

3. x wins if i votes for x and y wins if i votes for y ($\sum_{j \neq i} s_j = 0$), in which case the agent gets $U_i(x)$ if voting for x and $U_i(y)$ if voting for y.

4. a tie if i votes for x and y winning if i votes for y ($\sum_{j\neq i} s_j = 1$), in which case voting for x gives payoff

$$\frac{1}{2}U_i(x) + \frac{1}{2}U_i(y) > U_i(y)$$

and voting for y gives $U_i(y)$

5. y wins no matter what ($\sum_{j\neq i} s_j > 1$). Then, voting for x or y both gives $U_i(y)$.

Taken together, voting for the most preferred alternative never hurts the agent as it weakly increases the chance to win for all voting profiles by the others.

6.2 The Median Voter Theorem

We will now use the dominant strategy result in 2 candidate elections to think about how to model political competition in an environment where citizens disagree about the best policy. An example where the framework here is applicable is if a society needs to decide on the level of a public good under the (say constitutional) restriction that all agents have to split the costs equally (any other cost sharing rule is fine as long as it is given to the problem). We will however present the model in a somewhat more context free form.

We consider an economy with;

- n citizens labeled $i = 1, .., n$
- a single dimensional policy variable p
- each agent i preferences over p denoted by $v^i(p)$
- two political parties, A and B.
- Parties compete in an election for office. Parties care *only about winning the election* and are not policy motivated at all. Payoffs for party j are

$$u^j = \begin{cases} w^j > 0 & \text{if in office} \\ 0 & \text{if not in office} \end{cases}$$

Consider the following timing:

Stage 1 Parties simultaneously commit to "platforms" (p_A, p_B)

Stage 2 Voters vote between two parties. When in office the winning party implement its platform.

Formally this gives us an extensive form game where a strategy for a voter is a function $s^i(p_A, p_B)$ that takes on values in $\{-1, 0, 1\}$(or even a probability distribution for each pair of platforms). However, this level of formalism is not really necessary given that we are (which we are) interested in "credible Nash" or subgame perfect Nash equilibria, meaning that we seek equilibria where (1) all voters behave optimally after any pair of platforms that are announced, (2) voters are voting in accordance with their dominant strategy (to eliminate all the silly equilibria where voters vote for something they dislike because they think they cannot change the outcome).

It is then pretty clear that given platforms (p_A, p_B) we have that the unique dominant strategy in the subgame is

- Voter i votes for A whenever $v^i(p_A) > v^i(p_B)$ and votes for B whenever $v^i(p_A) < v^i(p_B)$.

Hence, let $s_i^*(p_A, p_B) = -1$ if $v^i(p_A) > v^i(p_B)$ and $s_i^*(p_A, p_B) = 1$ if $v^i(p_A) < v^i(p_B)$ and assume that the voter flips a coin when indifferent, then the probability of winning for A given these sequentially rational (and weakly dominant) continuation strategies is

$$\pi(p_A, p_B) = \begin{cases} 1 & \text{if } \sum_i s_i^*(p_A, p_B) < 0 \\ \frac{1}{2} & \text{if } \sum_i s_i^*(p_A, p_B) = 0 \\ 0 & \text{if } \sum_i s_i^*(p_A, p_B) > 0 \end{cases}$$

Expected payoffs for parties are are then

$$\pi(p_A, p_B)\, w_A \text{ for party } A$$

$$(1 - \pi(p_A, p_B))\, w_B \text{ for party } B$$

Define

$$p_i^* = \arg\max_{p \in P} v^i(p)$$

for each agent i (we assume the most favorite policy is uniquely defined). Moreover, we will (the result below does not work otherwise) assume that preferences are single peaked. That is;

Definition 1 *A utility function (of a single-dimensional variable) $v^i(\cdot)$ is said to be single peaked if there exists some p_i^* such that;*

1. $v^i(p') < v^i(p'')$ *for any* $p' < p'' \leq p_i^*$
2. $v^i(p') > v^i(p'')$ *for any* $p_i^* \leq p' < p''$

We define an equilibrium as an equilibrium of the simultaneous move reduced game where parties chooses platforms and citizens decisions are replaced by the rule to vote for the most preferred platform and flip a coin where indifferent (alternatively, we could just define it as a subgame perfect equilibrium where weakly dominated strategies are eliminated-as you will verify in your problem set the choice of tie-breaking rule is immaterial).

Definition 2 *A pair of platforms (p_A^*, p_B^*) is an equilibrium if*

$$\begin{aligned} p_A^* &\in \arg\max_p \pi(p_A, p_B^*) w_A \\ p_B^* &\in (1 - \pi(p_A^*, p_B)) w_B \end{aligned}$$

In order to state the result in a sensible way we begin by labeling the agents in accordance to their most preferred levels of the policy so that $p_1^* \leq p_2^* \leq ... \leq p_n^*$. Assuming n is odd we have:

Proposition 2 *(Median Voter Theorem, Downs (1957), Black (1948)) Suppose that preferences are single-peaked. Then, the unique equilibrium in the Downsian model has $p_A^* = p_B^* = p_m^*$, where $m = \frac{n-1}{2}$ is the voter who have a most preferred outcome with $\frac{n-1}{2}$ voters on each side.*

Proof. ($p_A^* = p_B^* = p_m^*$ is an equilibrium) Suppose $p_B = p_m^*$. If $p_A < p_m^*$ it follows from strict single-peakedness that $v^i(p_A) < v^i(p_B)$ for $i = m, ..., n$. Hence $|S_A(p_A, p_m^*)| < \frac{n-1}{2}$ and $|S_B(p_A, p_m^*)| \geq \frac{n-1}{2}$, so B wins for sure, $\pi(p_A, p_m^*) = 0$. If $p_A > p_m^*$, the same argument applies, so there is no profitable deviation from p_m^* for player A. Since labeling of parties is arbitrary, this proves that (p_m^*, p_m^*) is an equilibrium.

(uniqueness) Suppose without loss of generality that $p_A \neq p_m^*$. Suppose first that A wins with some probability. Then B can't play a best response, since p_m^* wins with probability 1 due to argument above. On the other hand side, suppose A does never win. Then A can't play a best response since if A deviates and play p_m^* the probability of winning is $\frac{1}{2}$ if $p_B = p_m^*$ and 1 otherwise. ■

7 The Condorcet Paradox

Now, consider the case where there are three policies under consideration, call them $P = \{x, y, z\}$. Assume that there are 3 citizens and suppose that they have the following (strict) preference orderings

1s ranking	2s ranking	3s ranking
x	y	z
y	z	x
z	x	y

It was noted already in the eighteenth century that *there is no decision that can win a majority against all other options.* To see this note that;

1. if there is a pairwise vote between x and y, then 1 and 3 vote for x and x will win
2. if there is a pairwise vote between y and z, then 1 and 2 vote for y and y wins
3. if there is a pairwise vote between z and x, then 2 and 3 vote for z and x wins.
4. That is x beats y that beats z that beats x!

This is called a voting cycle, a phenomenon that has implications for several issues (in particular for "preference aggregation" and political competition. In the context of the median voter result we observe the following:

- Suppose that the two parties pick the same policy, then they presumably win with probability $\frac{1}{2}$
- But, regardless of which policy both parties pick, there is an alternative policy that will win for sure.

How do we make sense of this in the light of the median voter result? Since it is possible to associate numbers with each of the alternatives, the only assumption that can fail is the assumption of single-peaked preferences. Indeed, these preferences are not single-peaked (and the argument is instructive as to how to think about single-peakedness).

1. Label the policies by numbers. First, say that $x < y < z$;
2. Then, if u_i is a utility function representing the ranking for $i = 1, 2, 3$ we have that (draw):
   (a) $u_1(x) > u_1(y) > u_1(z)$, which is OK
   (b) $u_2(x) < u_2(y)$ and $u_2(y) > u_2(z)$, which is OK
   (c) $u_3(x) > u_3(y)$ and $u_3(y) < u_3(z)$, which is not OK (since the minimum is in the middle)
3. Now, in general, it is always possible to find some numbers so that single-peakedness is violated for that particular ordering. In this example however, everything is symmetric, so;
   (a) if you try $y < z < x$, that is just like having 2 take on the role of agent 3, 3 take on the role of agent 1, and 1 take on the role of agent 2.
   (b) in fact, you should convince yourself that any way you order x, y and z, there will be an agent for which single-peakedness fails.