

B Appendix B: Proofs for Results in Section 2.

Proof of Proposition 1.

Claim B1 For any incentive feasible mechanism \mathcal{G} of the form (3), there exist an incentive feasible mechanism

$$G = \left(\left(\rho^j, \eta_1^j, \dots, \eta_n^j \right)_{j \in \mathcal{J}}, (t_i)_{i \in \mathcal{I}} \right), \quad (\text{B35})$$

that generates the same social surplus, where $\rho^j : \Theta^n \rightarrow [0, 1]$ is the provision rule for good j , $\eta_i^j : \Theta \rightarrow [0, 1]$ is the inclusion rule for agent i and good j , and $t_i : \Theta \rightarrow R$ is the transfer rule for agent i .

Proof. Consider an incentive feasible mechanism \mathcal{G} . Pick $k \in [0, 1]$ arbitrarily and define,

$$\begin{aligned} \rho^j(\theta) &= \mathbb{E}_{\Xi} \zeta^j(\theta, \vartheta) = \int_0^1 \zeta^j(\theta, \vartheta) d\vartheta \\ \eta_i^j(\theta_i) &= \begin{cases} \frac{\mathbb{E}_{-i} \zeta^j(\theta, \vartheta) \omega_i^j(\theta, \vartheta)}{\mathbb{E}_{-i} \zeta^j(\theta, \vartheta)} = \frac{\int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\theta, \vartheta) \omega_i^j(\theta, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i})}{\int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\theta, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i})} & \text{if } \int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\theta, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i}) > 0 \\ k & \text{if } \int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\theta, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i}) = 0 \end{cases} \\ t_i(\theta_i) &= \mathbb{E}_{-i} \tau(\theta) = \int_{\Theta_{-i}^n} \tau(\theta) d\mathbf{F}(\theta_{-i}), \end{aligned} \quad (\text{B36})$$

for each $\theta \in \Theta^n$, $j \in \mathcal{J}$ and $i \in \mathcal{I}$. This is a mechanism of the form in (B35), and we will call it G . Use of the law of iterated expectations on $\rho^j(\theta)$ and $t_i(\theta_i)$ shows that (BB) is unaffected when switching from \mathcal{G} to G . It remains to show that the surplus is unchanged, and that (IC) and (IR) continue to hold under G . The utility of agent i of type $\theta_i \in \Theta$ who announces $\hat{\theta}_i \in \Theta$ is

$$\mathbb{E}_{-i} \left[\sum_{j \in \mathcal{J}} \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \omega_i^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \theta_i - \tau(\hat{\theta}_i, \theta_{-i}) \right] \text{ in mechanism } \mathcal{G} \quad (\text{B37})$$

$$\mathbb{E}_{-i} \left[\sum_{j \in \mathcal{J}} \rho^j(\hat{\theta}_i, \theta_{-i}) \eta_i^j(\hat{\theta}_i) \theta_i - t_i(\hat{\theta}_i) \right] \text{ in mechanism } G. \quad (\text{B38})$$

If $\int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i}) = 0$, we trivially have that the payoffs in (B37) and (B38) are identical, whereas if $\int_{\Theta_{-i}^n} \int_0^1 \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) d\vartheta d\mathbf{F}(\theta_{-i}) > 0$, we have that

$$\begin{aligned} \mathbb{E}_{-i} \rho^j(\hat{\theta}_i, \theta_{-i}) \eta_i^j(\hat{\theta}_i) \theta_i &= \mathbb{E}_{-i} \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \frac{\mathbb{E}_{-i} \omega_i^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta)}{\mathbb{E}_{-i} \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta)} \\ &= \mathbb{E}_{-i} \omega_i^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \theta_i. \end{aligned} \quad (\text{B39})$$

Trivially, $\mathbb{E}_{-i} t_i(\theta_i) = t_i(\theta_i) = \mathbb{E}_{-i} \tau(\theta)$, which combined with (B39) implies that the payoffs in (B37) and (B38) are identical. Since the equality between (B37) and (B38) were established for any i , θ_i and $\hat{\theta}_i$, it follows that all incentive and participation constraints (IC) and (IR) hold for mechanism G given that they are satisfied in mechanism \mathcal{G} . Moreover, [again by (B39)]

$$\mathbb{E}_{-i} \left[\sum_{j \in \mathcal{J}} \rho^j(\theta) \eta_i^j(\theta_i) \theta_i \right] = \mathbb{E}_{-i} \left[\sum_{j \in \mathcal{J}} \omega_i^j(\theta, \vartheta) \zeta^j(\theta, \vartheta) \theta_i \right], \quad (\text{B40})$$

so it follows by integration over Θ and summation over i that

$$\mathbb{E} \left[\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \rho^j(\theta) \eta_i^j(\theta_i) \theta_i \right] = \mathbb{E} \left[\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \zeta^j(\theta, \vartheta) \omega_i^j(\theta, \vartheta) \theta_i \right], \quad (\text{B41})$$

By construction, we also have that $\rho^j(\theta) = \mathbb{E}_{\Xi} \zeta^j(\theta, \vartheta)$ for every θ . Thus $\mathbb{E} [\rho^j(\theta) C^j(n)] = \mathbb{E} [\zeta^j(\theta, \vartheta) C^j(n)]$, implying that

$$\sum_{j \in \mathcal{J}} \mathbb{E} \rho^j(\theta) \left[\sum_{i \in \mathcal{I}} \eta_i^j(\theta_i) \theta_i - C^j(n) \right] = \sum_{j \in \mathcal{J}} \mathbb{E} \zeta^j(\theta, \vartheta) \left[\sum_{i \in \mathcal{I}} \omega_i^j(\theta, \vartheta) \theta_i - C^j(n) \right]. \quad (\text{B42})$$

Hence, \mathcal{G} and G generate the same social surplus. \blacksquare

Claim B2 *For every incentive feasible mechanism of the form (B35), there exists an anonymous simple incentive feasible mechanism g of the form (5) that generates the same surplus.*

Proof. Consider an incentive feasible simple mechanism G on form (B35). For $k \in \{1, \dots, n!\}$, let $P_k : \mathcal{I} \rightarrow \mathcal{I}$ denote the k -th permutation of the set of agents \mathcal{I} . Note that $P_k^{-1}(i)$ gives the index of the agent who takes agent i 's position in permutation P_k . Moreover, for any given $\theta \in \Theta^n$, let $\theta^{P_k} = (\theta_{P_k^{-1}(1)}, \dots, \theta_{P_k^{-1}(n)}) \in \Theta^n$ denote the corresponding k -th permutation of θ .²² For each $k \in \{1, \dots, n!\}$, let $G_k = \left((\rho_k^j, \eta_{k1}^j, \dots, \eta_{kn}^j)_{j=1,2}, t_{k1}, \dots, t_{kn} \right)$ be given by

$$\begin{aligned} \rho_k^j(\theta) &= \rho^j(\theta^{P_k}) \quad \forall \theta \in \Theta^n, j \in \mathcal{J}, \\ \eta_{ki}^j(\theta_i) &= \eta_{P_k^{-1}(i)}^j(\theta_i) \quad \forall \theta_i \in \Theta, j \in \mathcal{J}, i \in \mathcal{I}, \\ t_{ki}(\theta_i) &= t_{P_k^{-1}(i)}(\theta_i) \quad \forall \theta_i \in \Theta, i \in \mathcal{I}, \end{aligned} \quad (\text{B43})$$

and let $g = \left((\tilde{\rho}^j, \tilde{\eta}_1^j, \dots, \tilde{\eta}_n^j)_{j=1,2}, \tilde{t}_1, \dots, \tilde{t}_n \right)$ be given by

$$\begin{aligned} \tilde{\rho}^j(\theta) &= \frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\theta) \quad \forall \theta \in \Theta^n, j \in \mathcal{J} \\ \tilde{\eta}_i^j(\theta_i) &= \frac{\sum_{k=1}^{n!} \mathbb{E}_{-i} [\rho_k^j(\theta)] \eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!} \mathbb{E}_{-i} [\rho_k^j(\theta)]} \quad \forall \theta_i \in \Theta, i \in \mathcal{I}, j \in \mathcal{J} \\ \tilde{t}_i(\theta_i) &= \frac{1}{n!} \sum_{k=1}^{n!} t_{ki}(\theta_i) \quad \forall \theta_i \in \Theta, i \in \mathcal{I}. \end{aligned} \quad (\text{B44})$$

We now note that: (1) for each $j \in \mathcal{J}$, $\tilde{\rho}^j(\theta) = \tilde{\rho}^j(\theta')$ if θ' is a permutation of θ . This is immediate since the sets $\left\{ \rho_k^j(\theta) \right\}_{k=1}^{n!} = \left\{ \rho^j(P_k(\theta)) \right\}_{k=1}^{n!}$ and $\left\{ \rho_k^j(\theta') \right\}_{k=1}^{n!} = \left\{ \rho^j(P_k(\theta')) \right\}_{k=1}^{n!}$ are the same; (2) for $j \in \mathcal{J}$ and each pair $i, i' \in \mathcal{I}$, $\tilde{\eta}_i^j(\cdot) = \tilde{\eta}_{i'}^j(\cdot)$. That is, the inclusion rules are the same for all agents. To see this, consider agent i and i' , and suppose that $\theta_i = \theta_{i'}$. We then have that $\left\{ \mathbb{E}_{-i} [\rho_k^j(\theta)] \eta_{ki}^j(\theta_i) \right\}_{k=1}^{n!}$ and $\left\{ \mathbb{E}_{-i'} [\rho_k^j(\theta)] \eta_{ki'}^j(\theta_{i'}) \right\}_{k=1}^{n!}$ are identical and that $\mathbb{E}_{-i} [\tilde{\rho}^j(\theta)] =$

²²To illustrate, suppose $n = 3, m = 2, \theta = (\theta_1, \theta_2, \theta_3) = ((1, 2), (3, 2), (2, 1))$. Consider, for example, permutation k given by $P_k(1) = 2, P_k(2) = 1, P_k(3) = 3$. Then $P_k^{-1}(1) = 2, P_k^{-1}(2) = 1, P_k^{-1}(3) = 3$ and $\theta^{P_k} = (\theta_{P_k^{-1}(1)}, \theta_{P_k^{-1}(2)}, \theta_{P_k^{-1}(3)}) = (\theta_2, \theta_1, \theta_3) = ((3, 2), (1, 2), (2, 1))$.

$E_{-i'} [\tilde{\rho}^j(\theta)]$; and (3) for each pair $i, i' \in \mathcal{I}, \tilde{t}_i(\cdot) = \tilde{t}_{i'}(\cdot)$, which is obvious since the sets $\{t_{ki}(\theta_i)\}_{k=1}^{n!}$ and $\{t_{ki}(\theta'_i)\}_{k=1}^{n!}$ are identical. Together, (1), (2) and (3) establishes that g is anonymous and simple.

Now we show that g is incentive feasible and generates the same expected surplus as G . First, since G and G_k are identical except for the permutation of the agents, we have, for $k = 1, \dots, n!$,

$$\sum_{j \in \mathcal{J}} E \left\{ \rho_k^j(\theta) \left[\sum_{i \in \mathcal{I}} \eta_{ki}^j(\theta_i) \theta_i^j - C^j(n) \right] \right\} = \sum_{j \in \mathcal{J}} E \left\{ \rho^j(\theta) \left[\sum_{i \in \mathcal{I}} \eta_i^j(\theta_i) \theta_i^j - C^j(n) \right] \right\}. \quad (\text{B45})$$

Hence,

$$\begin{aligned} & \sum_{j \in \mathcal{J}} E \left\{ \tilde{\rho}^j(\theta) \left[\sum_{i \in \mathcal{I}} \tilde{\eta}^j(\theta_i) \theta_i^j - C^j(n) \right] \right\} = \sum_{j \in \mathcal{J}} E \left\{ \frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\theta) \left[\sum_{i \in \mathcal{I}} \frac{\sum_{k=1}^{n!} E_{-i} \rho_k^j(\theta) \eta_{ki}^j(\theta_i) \theta_i^j}{\sum_{k=1}^{n!} E_{-i} \rho_k^j(\theta)} - C^j(n) \right] \right\} \\ &= \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} E_{\theta_i} \left\{ \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \rho_k^j(\theta) \eta_{ki}^j(\theta_i) \theta_i^j \right\} - E \left[\frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\theta) C^j(n) \right] \\ &= \frac{1}{n!} \sum_{k=1}^{n!} \sum_{j \in \mathcal{J}} E \left\{ \rho_k^j(\theta) \left[\sum_{i \in \mathcal{I}} \eta_{ki}^j(\theta_i) \theta_i^j - C^j(n) \right] \right\} = \sum_{j \in \mathcal{J}} E \left\{ \rho^j(\theta) \left[\sum_{i \in \mathcal{I}} \eta_i^j(\theta_i) \theta_i^j - C^j(n) \right] \right\}, \end{aligned} \quad (\text{B46})$$

where the last equality follows from (B45). Hence the surplus generated by g is identical to that by original mechanism G . To show that g is incentive feasible we first note that $E \rho_k^j(\theta) = E \rho^j(\theta)$ and $E \sum_{i \in \mathcal{I}} t_{ki}(\theta_i) = E \sum_{i \in \mathcal{I}} t_i(\theta_i)$ for all k , since the agents' valuations are drawn from identical distributions and G_k and G only differ in the index of the agents. Thus

$$\begin{aligned} E \sum_{i \in \mathcal{I}} \tilde{t}_i(\theta_i) - \sum_{j \in \mathcal{J}} E \tilde{\rho}^j(\theta) C^j(n) &= E \sum_{i \in \mathcal{I}} \frac{1}{n!} \sum_{k=1}^{n!} t_{ki}(\theta_i) - \sum_{j \in \mathcal{J}} E \frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\theta) C^j(n) \\ &= E \sum_{i \in \mathcal{I}} t_i(\theta_i) - \sum_{j \in \mathcal{J}} E \rho^j(\theta) C^j(n), \end{aligned} \quad (\text{B47})$$

so g satisfies (BB) if G does. Second, (IC) holds for any permuted mechanism, that is,

$$E_{-i} \sum_{j \in \mathcal{J}} \rho_k^j(\theta) \eta_{ki}^j(\theta_i) \theta_i^j - t_{ki}(\theta_i) \geq E_{-i} \sum_{j \in \mathcal{J}} \rho_k^j(\hat{\theta}_i, \theta_{-i}) \eta_{ki}^j(\hat{\theta}_i, \theta_{-i}) \theta_i^j - t_{ki}(\hat{\theta}_i, \theta_{-i}) \quad (\text{B48})$$

for all $i \in \mathcal{I}$, and $\theta_i, \hat{\theta}_i \in \Theta$. Hence,

$$\begin{aligned} & E_{-i} \sum_{j \in \mathcal{J}} \tilde{\rho}^j(\theta) \tilde{\eta}^j(\theta_i) \theta_i^j - \tilde{t}_i(\theta_i) = E_{-i} \sum_{j \in \mathcal{J}} \left[\frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\theta) \right] \frac{\sum_{k=1}^{n!} E_{-i} [\rho_k^j(\theta)] \eta_{ki}^j(\theta_i) \theta_i^j}{\sum_{k=1}^{n!} E_{-i} [\rho_k^j(\theta)]} - \frac{1}{n!} \sum_{k=1}^{n!} t_{ki}(\theta_i) \\ &= \frac{1}{n!} \sum_{k=1}^{n!} \left[E_{-i} \sum_{j \in \mathcal{J}} \rho_k^j(\theta) \eta_{ki}^j(\theta_i) \theta_i^j - t_{ki}(\theta_i) \right] \geq \frac{1}{n!} \sum_{k=1}^{n!} \left[E_{-i} \sum_{j \in \mathcal{J}} \rho_k^j(\hat{\theta}_i, \theta_{-i}) \eta_{ki}^j(\hat{\theta}_i, \theta_{-i}) \theta_i^j - t_{ki}(\hat{\theta}_i, \theta_{-i}) \right] \\ &= E_{-i} \sum_{j \in \mathcal{J}} \frac{1}{n!} \sum_{k=1}^{n!} \rho_k^j(\hat{\theta}_i, \theta_{-i}) \eta_{ki}^j(\hat{\theta}_i, \theta_{-i}) \theta_i^j - \frac{1}{n!} \sum_{k=1}^{n!} t_{ki}(\hat{\theta}_i, \theta_{-i}) = \sum_{j \in \mathcal{J}} E_{-i} \tilde{\rho}^j(\hat{\theta}_i, \theta_{-i}) \tilde{\eta}_i^j(\hat{\theta}_i) \theta_i^j - \tilde{t}_i(\hat{\theta}_i), \end{aligned} \quad (\text{B49})$$

where the inequality follows from (B48). Hence g satisfies (IC). Finally, g also satisfies the (IR) because (see the second line in (B49)) all the permuted mechanisms satisfy participation constraints. Proposition 1 follows by combining Claims B1 and B2. \blacksquare

Proof of Proposition 2.

Notation: This proof requires us to be explicit about the coordinates of the vector θ when permuting \mathcal{J} . We therefore need some extra notation for this proof (only). We write $\theta_i^{-j} = (\theta_i^1, \dots, \theta_i^{j-1}, \theta_i^{j+1}, \dots, \theta_i^m)$ for a type vector where good j has been removed. Analogously, $\theta^{-j} = (\theta_1^{-j}, \dots, \theta_n^{-j})$ stands for the type profile with good j coordinate removed for all agents and $\theta^j = (\theta_1^j, \dots, \theta_n^j)$ is the vector collecting the valuations for good j for all agents. Furthermore, $\theta_{-i}^{-j} = (\theta_1^{-j}, \dots, \theta_{i-1}^{-j}, \theta_{i+1}^{-j}, \dots, \theta_n^{-j})$ and $\theta_{-i}^j = (\theta_1^j, \dots, \theta_{i-1}^j, \theta_{i+1}^j, \dots, \theta_n^j)$ are used for the vectors obtained respectively from θ^{-j} and θ^j by removing agent i . These conventions are used also on the distributions, so, for example, \mathbf{F}_{-i}^{-j} denotes the cumulative distribution of θ_{-i}^{-j} . Conditional distributions are denoted in the natural way: for example $\mathbf{F}_{-i}^{-j}(\cdot | \theta_i^j)$ denotes the joint distribution of θ_{-i}^{-j} conditional on θ_i^j . Since no integrals are taken over subsets of the range of integration, we also conserve space and write $\int_{\theta} h(\theta) d\mathbf{F}(\theta)$ rather than $\int_{\theta \in \Theta^n} h(\theta) d\mathbf{F}(\theta)$ when integrating a function h over θ and similarly for integrals over various components of θ .

Proof. Consider a simple anonymous incentive feasible mechanism g . For $k \in \{1, \dots, m!\}$ let $P_k : \mathcal{J} \rightarrow \mathcal{J}$ be the k -th permutation of \mathcal{J} and $\theta_i^{P_k} = (\theta_i^{P_k^{-1}(1)}, \dots, \theta_i^{P_k^{-1}(m)}) \in \Theta$ be the permutation of θ_i when the goods are permuted according to P_k . Let $\theta^{P_k} = (\theta_1^{P_k}, \dots, \theta_n^{P_k}) \in \Theta^n$ denote the corresponding permutation of θ .²³ For each $k \in \{1, \dots, m!\}$ define mechanism $g_k = (\{\rho_k^j\}_{j \in \mathcal{J}}, \{\eta_k^j\}_{j \in \mathcal{J}}, t_k)$, where for every $\theta \in \Theta^n$;

1. $\rho_k^j(\theta) = \rho^{P_k^{-1}(j)}(\theta^{P_k})$ for every $j \in \mathcal{J}$;²⁴
2. $\eta_k^j(\theta_i) = \eta^{P_k^{-1}(j)}(\theta_i^{P_k})$ for every $j \in \mathcal{J}$;²⁵
3. $t_k(\theta_i) = t(\theta_i^{P_k})$.

By construction, each g_k is simple. Each g_k is also anonymous by the anonymity of g . Using the definition of g_k and manipulating the result by observing that the labeling of the variables is irrelevant, we get:²⁶

$$\begin{aligned}
E\rho_k^j(\theta) \eta_k^j(\theta_i) \theta_i^j &= \int_{\theta} \rho_k^j(\theta) \eta_k^j(\theta_i) \theta_i^j d\mathbf{F}(\theta) / \text{def of } g_k / = \int_{\theta \in \Theta^n} \rho^{P_k^{-1}(j)}(\theta^{P_k}) \eta^{P_k^{-1}(j)}(\theta_i^{P_k}) \theta_i^j d\mathbf{F}(\theta) \\
&= \int_{\theta^j} \left[\int_{\theta^{-j}} \rho^{P_k^{-1}(j)}(\theta^{P_k}) \eta^{P_k^{-1}(j)}(\theta_i^{P_k}) \theta_i^j d\mathbf{F}^{-j}(\theta^{-j} | \theta^j) \right] d\mathbf{F}^j(\theta^j) \\
/\text{relabel}/ &= \int_{\theta^{P_k^{-1}(j)}} \left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j} \left((\theta^{-j})^{P_k} \middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}} \right) \right] d\mathbf{F}^j(\theta^{P_k^{-1}(j)})
\end{aligned} \tag{B50}$$

²³To illustrate, suppose $n = 2, m = 3$, and $\theta = (\theta_1, \theta_2) = ((1, 2, 0), (3, 2, 1))$. Consider, for example, permutation k given by $P_k(1) = 2, P_k(2) = 1, P_k(3) = 3$. Then $P_k^{-1}(1) = 2, P_k^{-1}(2) = 1, P_k^{-1}(3) = 3$ and $\theta_1^{P_k} = (\theta_1^{P_k^{-1}(1)}, \theta_1^{P_k^{-1}(2)}, \theta_1^{P_k^{-1}(3)}) = (2, 1, 0), \theta_2^{P_k} = (\theta_2^{P_k^{-1}(1)}, \theta_2^{P_k^{-1}(2)}, \theta_2^{P_k^{-1}(3)}) = (2, 3, 1), \theta^{P_k} = (\theta_1^{P_k}, \theta_2^{P_k}) = ((2, 1, 0), (2, 3, 1))$.

²⁴This implies that $\rho_k^{P_k^{-1}(j)}(\theta^{P_k}) = \rho^j(\theta)$ for every $j \in \mathcal{J}$.

²⁵This implies that $\eta_k^{P_k^{-1}(j)}(\theta_i^{P_k}) = \eta^j(\theta_i)$ for every $j \in \mathcal{J}$.

²⁶It is important to point out that, in reaching the fourth equality in (B50), we can relabel the integrating variables (since they are dummies) but not the integrating functions.

where we recall,

$$(\theta^{-j})^{P_k} \equiv \left(\theta^{P_k^{-1}(1)}, \dots, \theta^{P_k^{-1}(j-1)}, \theta^{P_k^{-1}(j+1)}, \dots, \theta^{P_k^{-1}(n)} \right). \quad (\text{B51})$$

By exchangeability, we have

$$\begin{aligned} & d\mathbf{F}^{-j} \left((\theta^{-j})^{P_k} \middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th (vector) argument}} \right) \\ &= d\mathbf{F}^{-j} \left(\theta^{P_k^{-1}(1)}, \dots, \theta^{P_k^{-1}(j-1)}, \theta^{P_k^{-1}(j+1)}, \dots, \theta^{P_k^{-1}(n)} \middle| j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)} \right) \\ &= d\mathbf{F}^{-j} \left(\theta^{-j} \middle| j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)} \right) \\ &= d\mathbf{F}^{-P_k^{-1}(j)} \left(\theta^{-P_k^{-1}(j)} \middle| P_k^{-1}(j)\text{-th (vector) argument} = \theta^{P_k^{-1}(j)} \right); \end{aligned} \quad (\text{B52})$$

and

$$d\mathbf{F}^j \left(\theta^{P_k^{-1}(j)} \right) = d\mathbf{F}^{\theta^{P_k^{-1}(j)}} \left(\theta^{P_k^{-1}(j)} \right). \quad (\text{B53})$$

Using (B50), (B52) and (B53), we have that

$$\begin{aligned} & \mathbb{E} \rho_k^j(\theta) \eta_k^j(\theta_i) \theta_i^j \\ &= \int_{\theta^{P_k^{-1}(j)}} \left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j} \left((\theta^{-j})^{P_k} \middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}} \right) \right] d\mathbf{F}^j \left(\theta^{P_k^{-1}(j)} \right) \\ &= \int_{\theta^{P_k^{-1}(j)}} \left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-P_k^{-1}(j)} \left(\theta^{-P_k^{-1}(j)} \middle| \underbrace{\theta^{P_k^{-1}(j)}}_{P_k^{-1}(j)\text{-th argument}} \right) \right] d\mathbf{F}^{P_k^{-1}(j)} \left(\theta^{P_k^{-1}(j)} \right) \\ &= \int_{\theta} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)} d\mathbf{F}(\theta) = \mathbb{E} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)}. \end{aligned} \quad (\text{B54})$$

Moreover, exchangeability implies that $\mathbb{E} t_k(\theta_i) = \mathbb{E} t \left(\theta_i^{P_k} \right) = \mathbb{E} t(\theta_i)$. The ex ante utility,

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^m \rho_k^j(\theta) \eta_k^j(\theta_i) \theta_i^j - t_k(\theta_i) \right] &= \left[\sum_{j=1}^m \mathbb{E} \rho^{P_k^{-1}(j)}(\theta) \eta^{P_k^{-1}(j)}(\theta_i) \theta_i^{P_k^{-1}(j)} \right] - \mathbb{E} t(\theta_i) \\ \left/ \begin{array}{l} \text{Same elements in } \mathcal{J} \text{ and} \\ \{P_k^{-1}(1), \dots, P_k^{-1}(m)\} \end{array} \right/ &= \left[\sum_{j=1}^m \mathbb{E} \rho^j(\theta) \eta^j(\theta_i) \theta_i^j \right] - \mathbb{E} t(\theta_i), \end{aligned} \quad (\text{B55})$$

is thus unchanged when changing from g to g_k . The same steps as in (B50) through (B54) (only somewhat simpler) establishes that $\mathbb{E} \rho_k^j(\theta) = \mathbb{E} \rho^{P_k^{-1}(j)}$ for every j , implying that

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^m \rho_k^j(\theta) C^j(n) - \sum_i t_k(\theta_i) \right] &= \left[C(n) \mathbb{E} \sum_{j=1}^m \rho_k^j(\theta) - \sum_i \mathbb{E} t_k(\theta_i) \right] \\ &= \left[C(n) \mathbb{E} \sum_{j=1}^m \rho^j(\theta) - \sum_i \mathbb{E} t(\theta_i) \right] = \mathbb{E} \left[\sum_{j=1}^m \rho^j(\theta) C(n) - \sum_i t(\theta_i) \right], \end{aligned} \quad (\text{B56})$$

so the feasibility constraint is unaffected when changing from g to g_k . Next, write $U(\theta_i, \theta'_i; g)$ and $U(\theta_i, \theta'_i; g_k)$ for the expected utility from announcing θ'_i when the true type is θ_i in mechanisms g and g_k respectively. Next, by a calculation in the same spirit as (B50) through (B54):

$$\begin{aligned}
\mathbb{E}_{-i} \rho_k^j(\theta_{-i}, \theta'_i) &= \int_{\theta_{-i}} \rho_k^j(\theta_{-i}, \theta'_i) d\mathbf{F}_{-i}(\theta_{-i}) / \text{def of } g_k / = \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}((\theta_{-i}, \theta'_i)^{P_k}) d\mathbf{F}_{-i}(\theta_{-i}) \\
&= \int_{\theta_{-i}^j} \left[\int_{\theta_{-i}^{-j}} \rho^{P_k^{-1}(j)}((\theta_{-i}, \theta'_i)^{P_k}) d\mathbf{F}_{-i}^{-j}(\theta_{-i}^{-j} | \theta_{-i}^j) \right] d\mathbf{F}_{-i}^j(\theta_{-i}^j) \tag{B57} \\
/\text{relabel}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}} \left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k}) d\mathbf{F}_{-i}^{-j}((\theta_{-i}^{-j})^{P_k} | \theta_{-i}^{P_k^{-1}(j)}) \right] d\mathbf{F}_{-i}^j(\theta_{-i}^{P_k^{-1}(j)}) \\
/\text{exchangeability}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}} \left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k}) d\mathbf{F}_{-i}^{-P_k^{-1}(j)}(\theta_{-i}^{-P_k^{-1}(j)} | \theta_{-i}^{P_k^{-1}(j)}) \right] d\mathbf{F}_{-i}^j(\theta_{-i}^{P_k^{-1}(j)}) \\
&= \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k}) d\mathbf{F}_{-i}(\theta_{-i}) = \mathbb{E}_{-i} \rho^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k})
\end{aligned}$$

That is, the perceived probability of getting j when announcing θ'_i in mechanism g_k is the same as the perceived probability of getting good $P_k^{-1}(j)$ when announcing $(\theta'_i)^{P_k}$, so that

$$\begin{aligned}
U(\theta_i, \theta'_i; g_k) &= \mathbb{E}_{-i} \sum_{j=1}^m \rho_k^j(\theta_{-i}, \theta'_i) \eta_k^j(\theta'_i) \theta_i^j - t_k(\theta'_i) \tag{B58} \\
&= \sum_{j=1}^m \eta_k^{P_k^{-1}(j)}((\theta'_i)^{P_k}) \theta_i^j \mathbb{E}_{-i} \rho^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k}) - t((\theta'_i)^{P_k}),
\end{aligned}$$

whereas

$$\begin{aligned}
U(\theta_i, \theta'_i; g) &= \sum_{j=1}^m \eta_k^j(\theta'_i) \theta_i^j \mathbb{E}_{-i} \rho_k^j(\theta_{-i}, \theta'_i) - t(\theta'_i) \Rightarrow \tag{B59} \\
U(\theta_i, \theta'_i; g) \Big|_{\substack{\theta_i = \theta_i^{P_k} \\ \theta'_i = \theta'_i{}^{P_k}}} &= \sum_{j=1}^m \eta_k^j((\theta'_i)^{P_k}) \theta_i^{P_k^{-1}(j)} \mathbb{E}_{-i} \rho_k^j(\theta_{-i}, \theta'_i{}^{P_k}) - t((\theta'_i)^{P_k}) \\
&= \sum_{j=1}^m \eta_k^{P_k^{-1}(j)}((\theta'_i)^{P_k}) \theta_i^j \mathbb{E}_{-i} \rho_k^{P_k^{-1}(j)}(\theta_{-i}, \theta'_i{}^{P_k}) - t((\theta'_i)^{P_k}) = U(\theta_i, \theta'_i; g_k),
\end{aligned}$$

which establishes that type θ_i who announces θ'_i in mechanism g_k gets the same utility as type $\theta_i^{P_k}$ who announces $(\theta'_i)^{P_k}$ in mechanism g . Hence incentive compatibility and individual rationality of g_k follows from incentive compatibility and individual rationality of g . Now, construct a new mechanism $\tilde{g} = (\{\tilde{\rho}^j\}_{j \in \mathcal{J}}, \{\tilde{\eta}^j\}_{j \in \mathcal{J}}, \tilde{t})$ by letting

$$\begin{aligned}
\tilde{\rho}^j(\theta) &= \frac{1}{m!} \sum_{k=1}^m \rho_k^j(\theta) = \frac{1}{m!} \sum_{k=1}^m \rho^{P_k^{-1}(j)}(\theta^{P_k}) \tag{B60} \\
\tilde{\eta}^j(\theta_i) &= \frac{\sum_{k=1}^m \eta_k^j(\theta_i) \mathbb{E}_{-i} \rho_k^j(\theta)}{\sum_{k=1}^m \mathbb{E}_{-i} \rho_k^j(\theta)} = \frac{\sum_{k=1}^m \eta^{P_k^{-1}(j)}(\theta_i^{P_k}) \mathbb{E}_{-i} \rho^{P_k^{-1}(j)}(\theta^{P_k})}{\sum_{k=1}^m \mathbb{E}_{-i} \rho^{P_k^{-1}(j)}(\theta^{P_k})} \\
\tilde{t}(\theta_i) &= \frac{1}{m!} t_k(\theta_i) = \frac{1}{m!} t(\theta_i^{P_k})
\end{aligned}$$

let $P : \mathcal{J} \rightarrow \mathcal{J}$ be an arbitrary perturbation of the set of goods. Then,

$$\tilde{\rho}^{P^{-1}(j)}(\theta^P) = \frac{1}{m!} \sum_{k=1}^m \rho^{P_k^{-1}(P^{-1}(j))}((\theta^P)^{P_k}) = \frac{1}{m!} \sum_{k=1}^m \rho^{P_k^{-1}(j)}(\theta^{P_k}) = \tilde{\rho}^j(\theta), \tag{B61}$$

since the sets $\left\{ \rho^{P_k^{-1}(P^{-1}(j))} \left((\theta^P)^{P_k} \right) \right\}_{k=1}^{m!}$ and $\left\{ \rho^{P_k^{-1}(j)} (\theta^{P_k}) \right\}_{k=1}^{m!}$ are identical. Furthermore

$$\begin{aligned} \tilde{\eta}^{P^{-1}(j)} (\theta_i^P) &= \frac{\sum_{k=1}^{m!} \eta_k^{P_k^{-1}(P^{-1}(j))} \left((\theta_i^P)^{P_k} \right)_{E_{-i} \rho_k^{P_k^{-1}(P^{-1}(j))}} \left((\theta^P)^{P_k} \right)}{\sum_{k=1}^{m!} E_{-i} \rho_k^{P_k^{-1}(P^{-1}(j))} \left((\theta^P)^{P_k} \right)} \\ &= \frac{\sum_{k=1}^{m!} \eta_k^{P_k^{-1}(j)} \left(\theta_i^{P_k} \right)_{E_{-i} \rho_k^{P_k^{-1}(j)}} \left(\theta^{P_k} \right)}{\sum_{k=1}^{m!} E_{-i} \rho_k^{P_k^{-1}(j)} \left(\theta^{P_k} \right)} = \tilde{\eta}^j (\theta_i) \end{aligned} \quad (\text{B62})$$

for the same reason. It is obvious that $\tilde{t}(\theta_i^P) = \tilde{t}(\theta_i)$, which together with (B61) and (B62) establishes that \tilde{g} is symmetric. To complete the proof we need to show that \tilde{g} is incentive feasible and generates the same surplus as g . We note that

$$\begin{aligned} E \tilde{\rho}^j (\theta) \tilde{\eta}^j (\theta_i) \theta_i^j &= \frac{1}{m!} \sum_{k=1}^{m!} E \rho_k^j (\theta) \frac{\sum_{k=1}^{m!} \eta_k^j (\theta_i) E_{-i} \rho_k^j (\theta)}{\sum_{k=1}^{m!} E_{-i} \rho_k^j (\theta)} \theta_i^j \\ &= \frac{1}{m!} E_{\theta_i} \sum_{k=1}^{m!} \left[E_{-i} \rho_k^j (\theta) \frac{\sum_{k=1}^{m!} \eta_k^j (\theta_i) E_{-i} \rho_k^j (\theta)}{\sum_{k=1}^{m!} E_{-i} \rho_k^j (\theta)} \theta_i^j \right] = \frac{1}{m!} E \left[\sum_{k=1}^{m!} \eta_k^j (\theta_i) \rho_k^j (\theta) \theta_i^j \right] \\ &\Rightarrow E \sum_{j=1}^n \left[\tilde{\rho}^j (\theta) \tilde{\eta}^j (\theta_i) \theta_i^j - \tilde{t}(\theta_i) \right] = \frac{1}{m!} \sum_{k=1}^{m!} E \left[\sum_{j=1}^m \eta_k^j (\theta_i) \rho_k^j (\theta) \theta_i^j - t_k (\theta_i) \right] \\ &/(\text{B54}) \ \& \ (\text{B55})/ = E \left[\sum_{j=1}^m \eta^j (\theta_i) \rho^j (\theta) \theta_i^j - t (\theta_i) \right], \end{aligned} \quad (\text{B63})$$

which establishes that the ex ante utility from \tilde{g} and g are the same for all agents. Moreover,

$$\begin{aligned} E \left[\sum_{j=1}^m \tilde{\rho}^j (\theta) C^j (n) - \sum_{i=1}^n \tilde{t} (\theta_i) \right] &= E \left[C (n) \sum_{j=1}^m \frac{1}{m!} \sum_{k=1}^{m!} \rho_k^j (\theta) - \sum_{i=1}^n \sum_{k=1}^{m!} \frac{1}{m!} t_k (\theta_i) \right] \\ &= \sum_{k=1}^{m!} E \left[C (n) \sum_{j=1}^m \rho_k^j (\theta) - \sum_{i=1}^n t_k (\theta_i) \right] / (\text{B56})/ \\ &= \frac{1}{m!} \sum_{k=1}^{m!} E \left[\sum_{j=1}^m \rho^j (\theta) C^j (n) - \sum_{i=1}^n t (\theta_i) \right] = E \left[\sum_{j=1}^m \rho^j (\theta) C^j (n) - \sum_{i=1}^n t (\theta_i) \right], \end{aligned} \quad (\text{B64})$$

so the budget balance constraint is unaffected. All incentive compatibility constraints hold since,

$$\begin{aligned} U(\theta_i, \theta'_i; \tilde{g}) &= \sum_{j=1}^m \tilde{\eta}^j (\theta'_i) \theta_i^j E_{-i} \tilde{\rho}^j (\theta_{-i}, \theta'_i) - \tilde{t} (\theta'_i) \\ &= \frac{\sum_{k=1}^{m!} \eta_k^j (\theta'_i) E_{-i} \rho_k^j (\theta_{-i}, \theta'_i)}{\sum_{k=1}^{m!} E_{-i} \rho_k^j (\theta_{-i}, \theta'_i)} E_{-i} \left[\frac{1}{m!} \sum_{k=1}^{m!} \rho_k^j (\theta_{-i}, \theta'_i) \right] - \frac{1}{m!} \sum_{k=1}^{m!} t_k (\theta'_i) \\ &= \frac{1}{m!} \sum_{k=1}^{m!} \left[\eta_k^j (\theta'_i) E_{-i} \rho_k^j (\theta_{-i}, \theta'_i) - t_k (\theta'_i) \right] \\ &/(\text{B58})/ = \frac{1}{m!} \sum_{k=1}^{m!} U(\theta_i, \theta'_i; g_k) \leq / \text{IC for each } k/ \frac{1}{m!} \sum_{k=1}^{m!} U(\theta; g_k) = U(\theta; \tilde{g}). \end{aligned} \quad (\text{B65})$$

By the same calculation, $U(\theta; \tilde{g}) = \frac{1}{m!} \sum_{k=1}^{m!} U(\theta; g_k) \geq 0$, since all participation constraints hold for each k . This completes the proof. \blacksquare