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Efficient Mechanisms for Public Goods with Use Exclusions

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Constrained efficient provision of an excludable public good is studied in a model where preferences are private information. The provision level is asymptotically deterministic, making it possible to approximate the optimal mechanism with a mechanism that provides a fixed quantity of the good and charges fixed user fees for access. In general, the fixed fees involve third degree price discrimination, but, if names are uninformative about preferences, the analysis provides a justification for average cost pricing.

> Being able to limit a public goods' consumption does not make it a turn-blue private good. For what, after all, are the true marginal costs of having one extra family tune in on the program. They are literally zero. Why then limit any family which would receive positive pleasure from tuning in on the program from doing so? (Samuelson, 1958, p. 335)

1. INTRODUCTION

Virtually all theory on collective goods considers a "pure" public good, which is non-excludable and non-rival in consumption. However, for many non-rival goods it is feasible to exclude consumers from usage. To the extent that copying can be prevented, electronic libraries, computer programs, and other goods that can be stored in digital format are almost perfect examples of such *excludable public goods*. Other examples include cable TV, parks, gyms, zoos, museums, trains (as long as there is excess capacity), innovations, and protection by a police or fire department. These examples may also be thought of as natural monopolies, and an excludable public good may in general be considered as a special case of a natural monopoly, with zero marginal cost.

No consumers can be excluded from usage of a non-rival good in a first best allocation. The relevance and desirability of use exclusions in this paper therefore comes from constraints, which rule out first best. Consumers are assumed to be privately informed about their preferences, so provision mechanisms are required to be consistent with incentive compatibility. In addition, I require the provision mechanism to be self-financing, and assume that individuals who are left with a negative net benefit cannot be compelled to participate. The problem analysed is to decide how much of the good to provide, which individuals should be given access, and how to share the costs, subject to these constraints. The essence of this problem is to find a cost sharing arrangement that does not create incentives to misrepresent preferences, or to opt out from the mechanism.

If preferences are public information, the participation and self-financing constraints have no bite. Pareto efficient allocations, which exclude no consumers from usage, may be implemented through personalized Lindahl prices, which recover the costs and make no agent worse off. It is therefore necessary to restrict the ability to price discriminate for exclusions to matter in a complete information environment. Given that pricing must be uniform and that the budget constraint binds, Drèze (1980) shows that use exclusions are active in the constrained optimum. Drèze defends the constraint on the ability to price discriminate by arguing that individualized Lindahl prices are never observed, and suggests that the explanation may be the lack of incentives for truthful revelation of preferences.

Asymmetric information in itself is not sufficient to rationalize uniform prices. "Pivot mechanisms" (Clarke (1971), Groves (1973), d'Aspremont and Gerald-Varet (1979)) can implement first best for a pure public good. Excluding consumers is then again a pure waste of resources.

In this paper, the preference revelation problem is made harder by the participation and self-financing constraints. For reasons familiar from Myerson and Satterthwaite (1983), the *ex post* efficient rule is impossible to implement, and exclusions are active in the constrained optimum. The basic reason is that excluding low types ameliorates the free riding problem by making it less appealing for high types to mimic lower types (Moulin (1994), Cornelli (1996), Dearden (1997)).

The main contribution of this paper is to demonstrate that simple pricing schemes, third degree price discrimination and average cost pricing, can be justified in a large economy, without any exogenous restrictions on the ability to price discriminate. A fixed fee mechanism, that provides a fixed quantity of the public good, is asymptotically optimal. In such a mechanism, the (potential) provision level is determined using knowledge of the distributions only, and may thus be thought of as an *ex ante* decision.[1] Each agent faces a fixed user fee, which is individualized if identity is informative about the preference parameter. The mechanism provides the good if and only if revenues cover the costs, and gives access to those consumers that are willing to pay the fee. Besides its simplicity and approximate optimality, truth-telling is a dominant strategy, budget balance holds *ex post*, and participation is *ex post* individually rational.

Fees collapse to average cost pricing if all preference parameters are drawn from the same distribution. The analysis thus provides a justification for the standard textbook treatment of natural monopolies, as well as the approach in Drèze (1980) and Brito and Oakland (1980) and other studies of excludable public goods in complete information models.

In terms of the provision decision, I show that the optimal provision level is asymptotically deterministic, and that the good is provided if and only if a profit maximizer would provide. Monopolistic provision is distorting due to over-exclusions and, if the quantity is variable, under-provision, but conditions for when to provide coincide with the optimal rule in a large economy.

The intuition behind these results is closely related to the logic driving the "asymptotic impossibility" results for non-excludable public goods (Güth and Hellwig (1986), Rob (1989), Mailath and Postlewaite (1990)). Agents correctly perceive to have a negligible influence on the quantity provided (see Ledyard and Palfrey (1994), Al-Najjar and Smorodinsky (2000) for discussions of influence), so the scope to use the provision rule to price discriminate is vanishingly small in large economies.

Adjusting the quantity based on reports makes it possible to provide more when the surplus from provision is high. Providing at higher levels when many agents have high valuations also discourages high types from misreporting. But, both these considerations vanish in a large economy. Agents also prefer the expectation for sure to a lottery. The latter effect dominates in a large economy, and the provision level therefore converges in probability.

The asymptotic optimality of fixed fees is driven by the same force. Efficient inclusion rules include agents with valuations above a certain threshold. The threshold type pays her exact valuation, and higher types are willing to spend more only to the extent that expected consumption increases in the announcement. In a large economy, the average agent has little influence on the quantity provided. Transfers from participating agents are thus almost independent of type, so the efficiency loss of fixed fees is negligible with many consumers.

1. It is also possible to approximate the optimal mechanism by providing at a constant level for sure. The only difference is that such a mechanism fail *ex post* budget balance (with a vanishing probability).

The fixed fee result is somewhat akin to the asymptotic efficiency of simple voting schemes in Ledyard and Palfrey (1994, 2002). The difference is that they assume equal cost shares. Voluntary participation is then not an issue, so there is no difficulty in raising sufficient revenues.

Cornelli (1996) and Schmitz (1997) consider the same environment as the binary version of the model in Section 3 of this paper, but study profit maximizing mechanisms, rather than constrained efficiency. The focus in Cornelli (1996) is the opposite from this paper, the main insight being that the threat of non-production is a useful tool to discriminate between high and low valuation customers. However, she also shows that excludability may lead to provision also in a large economy. In Schmitz (1997), the result most directly related to this paper is that a simple mechanism, where the provision decision is undertaken before collecting the reports and prices are ordinary monopoly prices, is asymptotically profit maximizing.

Constrained efficient provision of an excludable public good is studied in Dearden (1997) and Hellwig (2003). The relationships and differences with Hellwig (2003) are discussed in some detail in Section 4.4. Dearden (1997) considers a version of the binary model that allows for crowding effects where, also without crowding, exclusions mitigate the free-riding problem. Exclusions are irrelevant asymptotically in this model, because the provision cost is kept constant as additional participants are added. Finally, Moulin (1994) studies exclusions in the context of strategy-proof implementation where a class of "serial cost sharing" mechanisms fully characterizes the set of mechanisms satisfying voluntary participation, anonymity and strategy proofness (see also Dearden, 1998). Within this class, exclusions mitigate the free-rider problem for reasons similar to those in this paper.

2. THE MODEL

Consider an economy with a set of agents $I = \{1, \dots, n\}$ bargaining over the provision of an excludable public good. Agents differ in their valuations for the public good and preferences are private information to the agents. Specifically, I let y denote the quantity of the public good and assume that, if agent $i \in I$ is given access to the public good, her pay-off is

$$\theta_i v(y) - t_i, \tag{1}$$

where $v(\cdot)$ is a continuously differentiable, strictly increasing and concave function satisfying $v(0) = 0$, t_i is a transfer paid by agent i, and θ_i, the type of the agent, is an unobservable taste parameter. If the agent is excluded from usage her utility is $-t_i$. Preferences over lotteries are of expected utility form.

Type θ_i is distributed over $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$ in accordance with distribution F_i, which has a continuous and strictly positive density f_i. Types are stochastically independent, so F_i is the prior about θ_i for all other agents as well as for the mechanism designer.[2] For brevity, Θ denotes $\times_i \Theta_i$ and, to avoid trivialities, I assume that $\overline{\theta}_i > 0$ for all $i \in I$.

The level of the public good is bounded from above by $\overline{y}$ and the *per unit* cost of provision is $C(n) > 0$. Quantity $y \in [0, \overline{y}]$ thus costs $yC(n)$ to provide. Note here that n is the number of *agents* and not the number of *users*. The good is thus fully non-rival.

The rationale for indexing costs of provision by n is that I consider sequences of economies where the number of participants approaches infinity. I then assume that $C(n)/n$ has limit $c^* > 0$. The simplest example of such a cost sequence is if $C(n) = c^* n$ for each n, implying that the cost of providing quantity y is $yc^* n$. The cost of providing even the tiniest amount thus tends to infinity as $n \to \infty$, but the *per capita* cost tends to zero as y tends to zero. There is no need to

2. Independence is important. With correlated values there are circumstances where the *ex post* efficient rule can be implemented in the pure public goods case (Pesendorfer, 1998), which eliminates any role for exclusions.

give these cost sequences any particular economic interpretation. Rather, it may be viewed as a normalization of the necessary *per capita* contributions (see Roberts, 1976 for a more extensive discussion).

Results are sensitive to this assumption. Hellwig (2003) studies more or less the same model, except that $C(n)$ is constant in n. If the level of the public good is bounded above and $\underline{\theta}_i \geq 0$ for all i it is then possible to get arbitrarily close to the first best, where no consumer is excluded from usage (see Section 4.3 and Hellwig, 2003).

A fixed cost in production of a private good is an excludable public good. However, the analysis of this paper does not generally apply if there is also a positive marginal cost (see Section 4.3). Marginal costs can be "netted out" if demands are binary, so the results apply in the special case where $y \in [0, 1]$ is interpreted as a probability of setting up a facility producing a binary good. Another extension that can be mapped into the framework of this paper, also when marginal costs are positive, is if y is the *quality* level and $yC(n)$ the cost of setting up a plant producing quality y (of a binary good). Interpreting y as *quantity* is only possible if the marginal cost is zero.

2.1. The design problem

No matter what mechanism or bargaining institution is set up in the economy, the outcome of this process should determine:

(1) the level of the public good,
(2) which agents should be allowed to use the public good,
(3) how the costs of the public good should be shared.

By appeal to the revelation principle I restrict attention to direct mechanisms for which truth-telling is a Bayesian Nash equilibrium. Allowing randomizations in the discrete inclusion decision, a direct mechanism can be represented as a triple (y, η, ξ), where $y : \Theta \to [0, \overline{y}]$ is the *provision rule*, $\eta : \Theta \to [0, 1]^n$ is the *inclusion rule*, and $\xi : \Theta \to R^n_+$ is the *cost sharing rule*. I adopt the convention that $\eta_i(\theta)$ is the probability that agent i is given access to the public good, given announcement θ. For convenience, agents may be included (to consume nothing) even if the good is not produced, and it is possible to provide a positive quantity and exclude everybody. The exposition below also assumes that agent i contributes $\xi_i(\theta)$ when θ is announced no matter whether she consumes the public good or not.[3]

The expected utility for agent i of type θ_i is $\eta_i(\hat{\theta})\theta_i v(y(\hat{\theta})) - \xi_i(\hat{\theta})$, where $\hat{\theta}$ denotes the vector of reported types. For truth-telling to be an equilibrium in the revelation game it must be *incentive compatible* to announce the true type for every agent i and any possible type realization. I let E_{-i} denote the expectation operator with respect to $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$ and express this as

$$\begin{aligned} &E_{-i}[\eta_i(\theta)\theta_i v(y(\theta)) - \xi_i(\theta)] \geq E_{-i}[\eta_i(\hat{\theta}_i, \theta_{-i})\theta_i v(y(\hat{\theta}_i, \theta_{-i})) - \xi_i(\hat{\theta}_i, \theta_{-i})] \\ &\quad \forall i \in I, \theta_i, \hat{\theta}_i \in \Theta_i. \end{aligned} \tag{2}$$

I also require that a voluntary participation, or *individual rationality*, condition is satisfied. Agents are assumed to know their own type, but not the realized types of the other agents, when deciding on whether to participate. Individual rationality is thus imposed at the interim stage as,

$$E_{-i}[\eta_i(\theta)\theta_i v(y(\theta)) - \xi_i(\theta)] \geq 0 \qquad \forall i \in I, \theta_i \in \Theta_i. \tag{3}$$

3. The alternative is to make transfers conditional on the project being undertaken and the agent being included. Due to risk-neutrality this leads to the same characterization of incentive feasible provision and inclusion rules.

Finally, costs of provision must be covered by the revenue collected. It is unclear whether it is more reasonable to impose such a feasibility constraint *ex post* or *ex ante*. The seemingly weaker condition is the *ex ante feasibility* (*budget balance*) *constraint*,

$$E\left(\sum_{i=1}^{n} \xi_i(\theta) - y(\theta)C(n)\right) \geq 0. \tag{4}$$

I use (4) in my analysis, but, by adapting an argument from Cramton, Gibbons and Klemperer (1987), one shows that the *ex ante* and *ex post* constraints are equivalent: if (2)–(4) holds, then a transfer scheme can be constructed which, together with the original provision and exclusion rules, satisfies (2), (3) and *ex post* feasibility.

Mechanisms satisfying (2)–(4) are called *incentive feasible*. Note here that the constraints (3) and (4) makes it natural to think of the problem as a "voluntary bargaining problem". The constraints in (3) capture the option to walk away from an agreement. The constraint (4) means that private consumption must be sacrificed to enjoy the public good, thus making it a bargaining set-up rather than a "pure" problem of preference revelation.

2.2. Preliminaries: characterization of constrained optimal mechanisms

To get a tractable optimization problem I characterize the incentive feasible mechanisms by combining (2), (3) and (4) into a single integral constraint, which eliminates transfers from the problem. This is a routine exercise, using familiar techniques from Myerson (1981) and others. The purpose of the exposition is therefore to introduce notation and all proofs are omitted.

Fix an arbitrary mechanism (y, η, ξ) and let $U_i(\theta_i)$ be the indirect expected utility for an agent of type θ_i. Define $t_i(\theta_i) \equiv E_{-i}\xi_i(\theta)$, which is the expected transfer for agent i of type θ_i given truthful revelation. Similarly, define $v_i(\theta_i) = E_{-i}\eta_i(\theta)v(y(\theta))$, which may be thought of as the "expected consumption benefit" (reduces to a scaling of the expected consumption when v is linear). In a truth-telling equilibrium it must be the case that

$$\begin{aligned} U_i(\theta_i) &= \max_{\hat{\theta}_i \in \Theta_i} \theta_i E_{-i}(\eta_i(\hat{\theta}_i, \theta_{-i})v(y(\hat{\theta}_i, \theta_{-i}))) - E_{-i}(\xi_i(\hat{\theta}_i, \theta_{-i})) \\ &= \max_{\hat{\theta}_i \in \Theta_i} \theta_i v_i(\hat{\theta}_i) - v_i(\hat{\theta}_i) = \theta_i v_i(\theta_i) - t_i(\theta_i) \qquad \forall i \in I, \theta_i \in \Theta_i, \end{aligned} \tag{5}$$

where the last equality is a consequence of (2). Using routine arguments one first shows that:

Lemma 1. *(y, η, ξ) is incentive compatible if and only if $v_i(\theta_i)$ is increasing in θ_i for all i and*

$$U_i(\theta_i) = U_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} v_i(\theta)d\theta \qquad \forall i \in I, \theta_i, \hat{\theta}_i \in \Theta_i\cdot \tag{6}$$

This is a standard result which has nothing to do with the collective nature of the good. Equally routine procedures, using Lemma 1, show that the expected transfer from agent i satisfies

$$E\xi_i(\theta) = \int_{\theta_1} \cdots \int_{\theta_n} \eta_i(\theta)\left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)}\right) v(y(\theta))\Pi_k f_k(\theta_k)d\theta_k - U_i(\underline{\theta}_i) \tag{7}$$

in any incentive compatible mechanism. Combining (7) with the feasibility constraint (4), and using that $U_i(\underline{\theta}_i) \geq 0$ for all i is necessary and sufficient for all participation constraints to hold (U_i is increasing in θ_i by (6), since $v_i(\theta_i) \geq 0$ for all θ_i), one concludes that a necessary condition for a mechanism (y, η, ξ) to be incentive feasible is that,

$$\int_{\theta_1} \cdots \int_{\theta_n} \left[\sum_i \eta_i(\theta)\left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)}\right) v(y(\theta)) - y(\theta)C(n)\right] \Pi_k f_k(\theta_k)d\theta_k \geq 0. \tag{8}$$

Conversely, if (y, η) satisfies (8), then there exists a transfer scheme ξ such that (y, η, ξ) is incentive feasible. Hence:

Proposition 1. *There exists a contribution scheme ξ such that (y, η, ξ) satisfies* (2)–(4) *if and only if v_i is increasing for each i and* (8) *holds.*

An immediate consequence of Proposition 1 is that a mechanism that maximizes social surplus subject to incentive feasibility solves

$$\max_{\{y(\cdot),\{\eta(\cdot)\}_{i=1}^n\}} \int_{\theta_1} \cdots \int_{\theta_n} \left[\sum\nolimits_i \eta_i(\theta) v(y(\theta))\theta_i - y(\theta)C(n)\right] \Pi_k f_k(\theta_k) d\theta_k \tag{9}$$

$$\text{s.t. } \int_{\theta_1} \cdots \int_{\theta_n} \left[\sum\nolimits_i \eta_i(\theta) v(y(\theta)) \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)}\right) - y(\theta)C(n)\right] \Pi_k f_k(\theta_k) d\theta_k \geq 0$$

$$v_i(\theta_i) = E_{-i} v(y(\theta))\eta_i(\theta) \text{ increasing in } \theta_i \ \ \forall i \in I \qquad \text{(monotonicity)}$$

$$0 \leq y(\theta) \leq \bar{y} \text{ and } 0 \leq \eta_i(\theta) \leq 1 \ \ \forall i \in I \qquad \text{(boundary).}$$

The only difference between the function in the integral constraint and the objective function of (9) is that the term $\theta_i - (1 - F_i(\theta_i))/f_i(\theta_i)$ replaces θ_i in the constraint. This captures that higher types must have no incentives to mimic types with lower valuations for the public good. That is, there are informational rents for higher types that limit the mechanism designer to extract the "virtual surplus" from each agent and type rather than the actual surplus.

The solution to (9) can be characterized by standard Lagrangian techniques. Define

$$S_n(y, \eta) \equiv \int_{\theta_1} \cdots \int_{\theta_n} \left(\sum\nolimits_i \eta_i(\theta) v(y(\theta))\theta_i - y(\theta)C(n)\right) \Pi_k f_k(\theta_k) d\theta_k \tag{10}$$

$$G_n(y, \eta) \equiv \int_{\theta_1} \cdots \int_{\theta_n} \left(\sum\nolimits_i \eta_i(\theta) v(y(\theta)) x_i(\theta_i) - y(\theta)C(n)\right) \Pi_k f_k(\theta_k) d\theta_k, \tag{11}$$

where $x_i(\theta_i) \equiv \theta_i - (1 - F_i(\theta_i))/f_i(\theta_i)$. Ignoring the monotonicity constraint, the Lagrangian for (9) is $L_n(y, \eta, \lambda) \equiv S_n(y, \eta) + \lambda G_n(y, \eta)$. If $\lambda^n G_n(y^n, \eta^n) = 0$ and

$$(y^n, \eta^n) \in \arg\max L_n(y, \eta, \lambda^n), \tag{12}$$

then (y^n, η^n) solves (9), given that the solution to this relaxed problem is monotonic.[4] This makes problem (9) very tractable since (12) is solved by point-wise optimization:

Lemma 2. *Suppose that $x_i(\theta_i)$ is weakly increasing. Then, there exists $\lambda^n \geq 0$ such that (y^n, η^n) is a solution to* (9) *if and only if, for almost all $\theta \in \Theta^n$,*

$$y^n(\theta) = \arg\max_{y \in [0, \bar{y}]} \sum\nolimits_{i=1}^n v(y) \max[0, \theta_i + \lambda^n x_i(\theta_i)] - (1 + \lambda^n) y C(n), \tag{13}$$

and where for all $i \in I$ and all θ such that $y(\theta) > 0$,[5]

$$\eta_i^n(\theta) = \begin{cases} 1 & \text{if } \theta_i + \lambda^n x_i(\theta_i) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The non-trivial part of Lemma 2 is that it establishes *existence* of a value of the multiplier that together with associated maximizer of (12) is a saddle point of the Lagrangian, thereby

4. To see this, suppose (y', η') is such that $S_n(y', \eta') > S_n(y^n, \eta^n)$ and $G_n(y', \eta') \geq 0$. Then $L_n(y', \eta', \lambda^n) = S_n(y', \eta') + \lambda^n G_n(y', \eta') \dot{>} S_n(y^n, \eta^n) = L_n(y^n, \eta^n, \lambda^n)$, contradicting that (y^n, η^n) maximizes the Lagrangian.

5. The inclusion rule is irrelevant when $y(\theta) = 0$, but it is without loss to assume that inclusions are always in accordance to (14).

proving existence of solutions to (9) as well as providing a useful characterization that is used in Sections 3 and 4. The proof follows the proof of Proposition 3.1 in Hellwig (2003) closely and is omitted (details on how to modify the proof are in Norman, 2003). As usual, the role of assuming that $x_i(\theta_i)$ is increasing is to guarantee that solutions to the relaxed problem are monotonic.

2.3. *Remark on the planners' objective function*

I have chosen to include only consumer welfare in the objective function of the planning problem (9). A natural alternative would be to maximize a weighted average of consumer surplus and profits. The main reason for not following this route is that the solution is unchanged, unless profits have a higher weight than consumer welfare.

To realize this, let (y^*, η^*) be a maximizer of $S_n(y, \eta) + G_n(y, \eta)$, subject to the constraint that $G_n(y, \eta) \geq 0$. If the constraint binds, it is easy to see that (y^*, η^*) solves (9), and that any solution of (9) also maximizes the alternative welfare criterion. Assuming instead that the constraint does not bind, we note that the constant $G_n(y^*, \eta^*) > 0$ may be subtracted from the transfers made by, say, player 1. Given provision and inclusion rules (y^*, η^*) and unchanged transfers for all other agents this generates a social surplus (only counting consumer welfare) of $S_n(y^*, \eta^*) + G_n(y^*, \eta^*)$. If (y^n, η^n) solves (9) and (y^*, η^*) does not, this implies that $S_n(y^*, \eta^*) + G_n(y^*, \eta^*) < S_n(y^n, \eta^n)$, contradicting the definition of (y^*, η^*). Since the constraint binds in any solution to (9) we conclude that the two problems are equivalent.

The key observation in the argument above is that a net profit can be refunded to the consumers, which is as good as letting the provider keep the profit if consumer and producer welfare have equal weights. Decreasing the relative weight on profits leaves the conclusion unchanged, but for a sufficiently high relative weight on profits the two problems are no longer equivalent.[6]

3. A BINARY PUBLIC GOOD

In this section I consider the special case when the public good comes as a single indivisible unit. Propositions 2 and 3 establish that the probability of provision in the surplus maximizing mechanism converges to either zero or one, depending on whether "standard monopoly profits" are positive or negative in the limit. Proposition 4 shows that a "fixed fee mechanism" is asymptotically optimal.

To deal with the indivisibility of the public good I now allow randomizations also in the provision rule. I let $\rho : \Theta \to [0, 1]$ denote a generic random provision rule, where $\rho(\theta)$ is the provision probability given announcements θ. The cost $C(n)$ is now taken to be the cost of the project, and the feasibility constraint (4) simplifies to

$$E\left[\sum_{i=1}^{n} \xi_i(\theta) - \rho(\theta)C(n)\right] \geq 0. \tag{15}$$

The utility is now $\theta_i - t_i$ if agent i consumes the public good and $-t_i$ otherwise, so the expected utility of agent i of type θ_i given reports $\hat{\theta}$ is $E_{-i}[\rho(\hat{\theta})\eta_i(\hat{\theta})\theta_i - \xi_i(\hat{\theta})]$, where $\eta_i(\hat{\theta})$ is interpreted as the probability of inclusion conditional on provision. The only change in preferences is thus that $\rho(\theta)$ replaces $v(y(\theta))$, so the model is equivalent to a model with a quantity decision where $v(y) = y$ and $\overline{y} = 1$, a case covered by the characterization in Section 2.2. We conclude that a surplus maximizing mechanism solves

6. The problem studied in this paper may be viewed as maximizing a weighted average of consumer and producer surplus, where the "profit weight" is endogenously determined as $\lambda^n/(1 + \lambda^n)$. A positive welfare weight on profits can only matter if profits are positive (so that the constraint is non-binding). In this case, the solution is characterized just like in Lemma 2, except that the weight on profit is exogenously given.

$$\max_{\{\rho(\cdot),\{\eta(\cdot)\}_{i=1}^n\}} \int_{\theta_1} \cdots \int_{\theta_n} \left(\sum_i \eta_i(\theta)\theta_i - C(n)\right) \rho(\theta) \Pi_k f_k(\theta_k) d\theta_k \tag{16}$$

$$\text{s.t.} \int_{\theta_1} \cdots \int_{\theta_n} \left(\sum_i \eta_i(\theta) x_i(\theta_i) - C(n)\right) \rho(\theta) \Pi_k f_k(\theta_k) d\theta_k \geq 0 \tag{17}$$

$$\rho_i(\theta_i) = E_{-i}\rho(\theta)\eta_i(\theta) \text{ increasing in } \theta_i \quad \forall i \in I \qquad \text{(monotonicity)}$$

$$0 \leq \rho(\theta) \leq 1 \text{ and } 0 \leq \eta_i(\theta) \leq 1 \quad \forall i \in I \qquad \text{(boundary).}$$

By applying Lemma 2 we know that there is some $\lambda^n \geq 0$ such that (ρ^n, η^n) solves problem (16) if and only if η_i^n is given by (14) for every i and θ, and

$$\rho^n(\theta) = \begin{cases} 1 & \forall\, \theta \text{ s.t. } \frac{1}{1+\lambda^n}\left[\sum_i \eta_i(\theta)\theta_i - C(n)\right] + \frac{\lambda^n}{1+\lambda^n}\left[\sum_i \eta_i(\theta) x_i(\theta_i) - C(n)\right] \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

To interpret the optimization problem it is useful to observe that the L.H.S. of constraint (17) is the objective function for a profit maximizer. Provisions are either zero or one, depending on whether a weighted average of social and virtual surplus is positive or negative. Since $\theta_i > x_i(\theta_i)$ for all $\theta_i < \overline{\theta}_i$, (18) shows that the *ex ante* provision probability is higher in the constrained efficient mechanism than in the profit maximizing mechanism, as one would expect.

In the remainder of the paper I assume that $x_i(\theta_i)$ is strictly increasing for each agent i and define θ_i^*, the "monopoly price", as

$$\theta_i^* = \arg\max_{\theta_i \in \Theta_i} \theta_i(1 - F_i(\theta_i)). \tag{19}$$

This is not only the profit maximizing price when restricted to take-it-or-leave-it offers. If the provision cost is sunk, the mechanism that provides access to all types $\theta_i \geq \theta_i^*$ at price θ_i^* and excludes lower types maximizes profits over all conceivable selling mechanisms.

3.1. *The surplus maximizing mechanism in a large economy*

To construct a sequence of economies I take as primitives an infinite sequence of costs $\{C(n)\}_{n=1}^\infty$ and an infinite sequence of (independent) distributions of types $\{F_i\}_{i=1}^\infty$. The n-th economy of a sequence consists of agents $\{1, \ldots, n\}$ together with the provision cost $C(n)$. For brevity, I let $F^n = (F_1, \ldots, F_n)$ denote the vector of distributions for the agents in economy n. An economy of size n is thus fully described by the pair $(C(n), F^n)$. Also note that the first n agents in economy $(C(n+1), F^{n+1})$ are identical to the agents in the n-th economy, *i.e.* $F^{n+1} = (F^n, F_{n+1})$.

The rationale for adding agents one at a time is to make the limiting *per capita* monopoly profit well-defined, which is necessary in order to provide an asymptotic characterization. In addition, I need some regularity conditions on the distributions of valuations and the cost sequence:

Definition 1. $\{C(n), F^n\}_{n=1}^\infty$ is said to be a regular sequence of economies if:

(1) for each i, distribution F_i has a density f_i such that $x_i(\theta_i)$ is strictly increasing and $f_i(\theta_i) \geq k > 0$ on support $[\underline{\theta}_i, \overline{\theta}_i] \subset [\underline{\theta}, \overline{\theta}]$ for some uniform bounds $-\infty < \underline{\theta} < \overline{\theta} < \infty$,
(2) there exists $c^* > 0$ such that $\lim_{n\to\infty} C(n)/n = c^*$.

The restriction that $x_i(\theta_i)$ is strictly increasing is convenient because θ_i^* is then uniquely defined, but Propositions 2 and 3 continue to hold with minor modifications of the proofs if $x_i(\theta_i)$ is weakly increasing. However, I have not been successful in extending the analysis to the "irregular case", when $x_i(\theta_i)$ is non-monotonic.[7] The need for uniform bounds on the support of

7. The main obstacle is that, unlike a profit maximizer, a surplus maximizer may have a strict incentive to randomize in the irregular case.

f_i may be understood from the example where f_i is uniform on $[0, n]$, in which case $\theta_i^* = \frac{i}{2}$ for each i, implying that the *per capita* profit for a profit maximizing firm is $(n+1)/4$, which diverges.

The first result provides a condition for when it is “asymptotically impossible” to provide the public good:

Proposition 2. *Let* $\{C(n), F^n\}_{n=1}^{\infty}$ *be a regular sequence of economies and suppose that* $\lim_{n\to\infty} C(n)/n = c^* > \lim_{n\to\infty} \sum_i [1 - F_i(\theta_i^*)]\theta_i^*/n$, *where* θ_i^* *is defined in* (19). *Then,* $\lim_{n\to\infty} E\rho^n(\theta) = 0$ *for any sequence* $\{\rho^n, \eta^n\}_{n=1}^{\infty}$ *of feasible solutions to* (16).

Proposition 2 establishes that if a monopoly provider that is forced to provide the good for sure, but free to price any way it wants, would make a loss, then the provision probability in any incentive feasible mechanism tends to zero.

Proposition 2 is proved by observing that an upper bound on the provision probability is found by maximizing the *ex ante* probability of provision subject to the constraints in (16). No welfare considerations enter this problem, so η^n must maximize the expected transfer from each agent if the integral constraint (17) is binding. For any ρ^n, this is achieved by the rule

$$\eta_i^n(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \theta_i^* \\ 0 & \text{if } \theta_i < \theta_i^*. \end{cases} \tag{20}$$

Use inclusions are thus exactly as for the profit maximizing mechanism with sunk costs. This does *not* imply that type $\theta_i > \theta_i^*$ pays exactly θ_i^*. If the provision probability is increasing in type (as in the mechanism under consideration), the threat of non-provision may be used to extract more than θ_i^* from higher types when the good is provided. However, this extra revenue in addition to θ_i^* is negligible in a large economy. To understand this, note that (20) implies that $\{\eta_i^n(\theta)x_i(\theta_i)\}_{i=1}^n$ is a sequence of independent random variables. An integration by parts shows that

$$E[\eta_i^n(\theta)x_i(\theta_i)] = \int_{\theta_i^*}^{\overline{\theta}_i} x_i(\theta) f_i(\theta_i) d\theta_i = \theta_i^*[1 - F_i(\theta_i^*)], \tag{21}$$

and an application of Chebyshev's inequality shows that

$$\lim_{n\to\infty} \Pr\left[\left|\sum_{i=1}^n \frac{\eta_i^n(\theta)x_i(\theta_i)}{n} - \sum_{i=1}^n \frac{\theta_i^*[1 - F_i(\theta_i^*)]}{n}\right| \geq \varepsilon\right] = 0 \tag{22}$$

for every $\varepsilon > 0$. Conditional on the project being implemented for sure, (22) says that the maximal *per capita* revenue converges in probability to the “standard monopoly revenue”.

The rule that maximizes the probability of provision involves a threshold k_n such that the public good is provided if and only if $\sum_i \eta_i^n(\theta)x_i(\theta_i)/n \geq k_n$. The main argument in the proof is to show that, to satisfy (17), k_n must be set so that the provision probability goes to zero. Roughly, if the provision probability stays bounded away from zero, then the expectation of $\sum_i \eta_i^n(\theta)x_i(\theta_i)/n$ conditional on provision approaches the unconditional expectation of $\sum_i \eta_i^n(\theta)x_i(\theta_i)/n$. Conditional on provision, the *per capita* transfer revenue is thus near $\sum_i [1 - F_i(\theta_i^*)]\theta_i^*/n$, whereas the *per capita* cost is near c^*. A sequence of provision rules for which the provision probability stays bounded away from zero therefore eventually violates incentive feasibility.

Next, I consider the case when a profit maximizer would be willing to provide the good for sure. The asymptotic characterization of the constrained efficient mechanism then switches from almost no provision to provision with probability near one:

Proposition 3. *Let $\{C(n), F^n\}_{n=1}^{\infty}$ be a regular sequence of economies and suppose that $\lim_{n\to\infty} C(n)/n = c^* < \lim_{n\to\infty} \sum_i [1 - F_i(\theta_i^*)]\theta_i^*/n$. Then $\lim_{n\to\infty} E\rho^n(\theta) = 1$ for any sequence $\{\rho^n, \eta^n\}_{n=1}^{\infty}$ of optimal solutions to* (16).

Taken together, Propositions 2 and 3 show an intuitive but striking feature of the constrained efficient provision rule. If a monopolist can make a profit, then the good is provided asymptotically. If not, the provision probability converges to zero. This means that the underprovision of the public good associated with monopolistic provision (shown in (18)) is negligible in a large economy.[8]

To understand Proposition 3 it is necessary to consider problem (16) in some more detail. Lemma 2 implies that if (η^n, ξ^n) solves (16), then there exists some θ_i^n for every $i \in I$ such that

$$\eta_i^n(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \theta_i^n \\ 0 & \text{if } \theta_i < \theta_i^n. \end{cases} \tag{23}$$

As for the revenue maximizing rule (20), the optimal inclusion rule for agent i is independent of announcements by other agents, which makes it possible to conclude that,

$$\lim_{n\to\infty} \Pr\left[\left|\sum_{i=1}^{n} \frac{\eta_i^n(\theta)x_i(\theta_i)}{n} - \sum_{i=1}^{n} \frac{[1 - F_i(\theta_i^n)]\theta_i^n}{n}\right| \geq \varepsilon\right] = 0. \tag{24}$$

The interpretation of (24) is that *per capita* revenue converges in probability to $\sum_i [1 - F_i(\theta_i^n)]\theta_i^n/n$ if the project is undertaken for sure. We know that charging a fixed price θ_i^* for access from each agent and providing for sure is feasible. This mechanism generates a positive surplus, implying that, if the result fails, there must at least be some strictly positive probability of provision in the limit (of any subsequence). This in turn implies that $\lim_{n\to\infty} \sum_i [1 - F_i(\theta_i^n)]\theta_i^n/n = c^*$. It is therefore possible to construct inclusion rules near the hypothetical solution that allow provision with probability 1. Since the probability that the project is socially desirable approaches one, the deviation increases social surplus.

The main difficulty in the proof is that a *strict* budget surplus for inclusion rules close to the original is needed in the limit to guarantee feasibility for a large finite economy. This requires a convexification of the achievable expected revenues, which is achieved by randomizing over thresholds θ_i^n and θ_i^*.

3.2. A fixed fee mechanism is almost optimal

Propositions 2 and 3 provide a sharp characterization of the constrained efficient provision mechanism in a large economy. However, one may worry that the efficient mechanism is supported by unreasonably complicated transfer schemes. Therefore, I now consider a class of simple mechanisms that sets fixed user fees.

There are (at least) two possible notions of what could be considered a “fixed fee mechanism”. One possibility is that consumers pay fixed user fees regardless of whether the good is provided or not. The alternative is a mechanism where the fee is paid only if the good is actually provided, which is the case I consider in the formal proofs.

Definition 2. (ρ, η, ξ) is a fixed fee mechanism if, for each $i \in I$, there exists $\widetilde{\theta}_i \in \Theta_i$ such that $\eta_i(\theta) = 1$ if and only if $\theta_i \geq \widetilde{\theta}_i$, and if $\xi_i(\theta) = \eta_i(\theta)\rho(\theta)\widetilde{\theta}_i$ for each $i \in I$ and $\theta \in \Theta$.

8. This conclusion does not carry over to the more general model. See Section 4.2.

If the good is not provided, nobody pays anything. If the good is provided, then agent i gets access to the good if and only if she is willing to pay the fixed user fee $\widetilde{\theta}_i$. Notice that, since inclusion thresholds in the optimal mechanism in general are individualized, I allow for third degree price discrimination in the fixed fees. In general, Proposition 4 below would fail for uniform fees.

Intuitively, the logic driving Propositions 2 and 3 suggests that one may construct an incentive feasible fixed free mechanism by simply keeping (ρ^n, η^n) as in the optimal mechanism and charging the threshold valuation θ_i^n from agents who are given access. The problem is that the feasibility constraint binds: the surplus maximizing mechanism generates a zero profit and extracts slightly more than θ_i^n from each type $\theta_i > \theta_i^n$ when providing the good. Implementing the surplus maximizing decision rule using fixed fees therefore leads to a budget deficit.

The *per capita* deficit is negligible in a large economy, and can be eliminated by a small increase in the thresholds. One can therefore show that it is possible to approximate the *per capita* surplus of the constrained optimal mechanism by a fixed fee mechanism with inclusion rules near those of the optimal mechanism:

Proposition 4. *Let $\{C(n), F^n\}_{n=1}^{\infty}$ be a regular sequence of economies and suppose, in addition, that, (i) $\lim_{n\to\infty}\sum_i[1-F_i(\theta_i^*)]\theta_i^*/n \neq c^*$ and, (ii) $F_i \in \mathcal{F}$ for every i, where $\mathcal{F}$ is a finite set.*[9] *Then, for each $\varepsilon > 0$, there exists N such that, for every $n \geq N$, an incentive feasible fixed fee mechanism satisfying* ex post *budget balance and* ex post *voluntary participation exists, that generates a* per capita *surplus which is at most ε below a surplus maximizing mechanism.*

If all agents have valuations drawn from the same distribution, all inclusion thresholds coincide in the optimal mechanism. The simple mechanism in Proposition 4 generates a budget surplus, but, this is negligible in a large economy, so the uniform user fee converges to the average cost. Average cost pricing is thus asymptotically optimal in this case.

It is possible to simplify the mechanism even further and (as Schmitz, 1997 does for a profit maximizing provider) consider provision rules that are constant in θ. The only change in terms of the statement of Proposition 4 is that such a provision rule fails to balance the budget *ex post*.

As for optimal auctions, the constrained optimal mechanism may be supported by a multitude of transfer schemes. Nevertheless, the probability of a realization of θ such that the provision rule in the optimal mechanism differs from the fixed fee mechanism used in the proof of Proposition 4 is negligible for large n. The probability that a given agent i is treated differently in the exclusion decision is also vanishingly small. It therefore makes sense to say that the constrained efficient mechanism approaches a fixed fee mechanism.

4. RESULTS FOR THE MORE GENERAL MODEL[10]

I now return to the more general model where the public good may be provided in any quantity between 0 and $\overline{y}$. Proposition 5 establishes that the quantity provided converges in probability to its expectation, so the provision level is asymptotically deterministic. Hence, there is no

9. The assumption that $\mathcal{F}$ is finite is made to assure that the limit of the average expected revenue is strictly in between the limit average expected revenue in the optimal and revenue maximizing mechanisms if thresholds are chosen in between θ_i^n and θ_i^*. This was not an issue for Proposition 3 since randomizations were used, whereas randomizations now are ruled out by definition of a fixed fee mechanism. I conjecture that the necessary convexification can be achieved generally by an alternative strategy of proof where agents pay a price that is *either* equal to the surplus maximizing inclusion threshold *or* the revenue maximizing threshold. However, this approach introduces some other difficulties since agents in the two "groups" cannot be picked arbitrarily.

10. I am grateful to a referee who suggested a more complete analysis of the material in this section than the initial draft and provided several useful conjectures.

significant loss to restrict attention to mechanisms that either provide some given positive level or not provide the good at all. Proposition 6 shows that the expected provision level converges to a constant under some additional regularity conditions. In conjunction, these two results imply that a mechanism that either provides zero or some appropriately chosen y^* is almost as good as the surplus maximizing mechanism in a large economy. This makes it possible to extend all results from the binary model also to this case. Proposition 7 gives conditions similar to Propositions 2 and 3 for when the provision level is asymptotically zero vs. strictly positive and Proposition 8 generalizes the asymptotic optimality of a fixed fee mechanism. Finally, Proposition 9 generalizes the asymptotic impossibility result for a non-excludable public good from Mailath and Postlewaite (1990).

An economy is now defined by the primitives $(\upsilon, [0, \overline{y}], C(n), F^n)$ and, since υ and $\overline{y}$ are held fixed, sequences of economies are constructed just like in Section 3.

Proposition 5. *Let* $\{\upsilon, [0, \overline{y}], C(n), F^n\}_{n=1}^{\infty}$ *be a regular sequence of economies and suppose that* υ *is strictly concave.*[11] *Then,* $\lim_{n\to\infty} \Pr(|y^n(\theta) - E(y^n(\theta))| \geq \varepsilon) = 0$ *for all* $\varepsilon > 0$ *and any sequence of constrained optimal mechanisms* $\{y^n, \eta^n\}$.

Proposition 5 says that the provision level is almost deterministic in a large economy. A crucial property of the optimization problem that is heavily exploited in the proof is that use exclusions continue to have the same characterization as in Section 3: a threshold rule that is independent of the realized types for all other agents. This independence, which ultimately comes from the linearity of the problem, makes it possible to use arguments relying on laws of large numbers (that is, (24) continues to hold).

Once the threshold inclusions are established, the rest of the argument is rather intuitive. Variation in $y(\theta)$ has two advantages: (i) the utility for any given y is increasing in θ_i, and (ii) making the level of the public good increasing in the announcements discourages high types from misreporting. However, the average type conditional on being above the inclusion threshold converges in probability, and incentives approach those of a fixed fee mechanism, so both these considerations favouring variability in y become negligible in a large economy. But, agents have an intrinsic distaste for variation in y, since υ is strictly concave. The latter effect remains of first order importance also when n is large.

Observe that there is no sense in which the constrained optimal solutions approach first best efficiency. Except in the trivial case when the first best provisions approach zero, $\sum_i (1 - F_i(\theta_i^n))\theta_i^n/n$ must converge to zero in order for the surplus to approach first best: but, then the *per capita* revenue, which is roughly $\upsilon(Ey^n(\theta)) \sum_i (1 - F_i(\theta_i^n))\theta_i^n/n$ in a large economy, also approaches zero. *Per capita* costs are bounded away from zero, so this violates feasibility.

4.1. The replica case

In order to describe the approximate solution for a large economy more constructively, I now add some extra structure to the model. The point is to assure that $Ey^n(\theta)$ has a well-defined limit. This, in turn, makes it possible to characterize conditions for provision vs. no provision in terms of conditions similar to those in the binary case.

This analysis introduces some new technical issues. In particular, it must now be established that problems with large n are "near each other" in the sense that if average surplus S is feasible in a particular large economy, then something near S is feasible for all sufficiently large economies. To deal with this I restrict attention to sequences of economies that are generated by replicating a given finite economy:

11. I continue to refer to sequences satisfying the conditions in Definition 1 as regular.

Definition 3. $\{v, [0, \overline{y}], C(n), F^n\}_{n=1}^{\infty}$ is said to be a sequence of replicas (of an economy with r agents) if there exists r such that $F_{i+r} = F_i$ for all i.[12]

Definition 4. A regular sequence of replica economies is a sequence of replicas $\{v, [0, \overline{y}], C(n), F^n\}_{n=1}^{\infty}$ such that (i) the regularity conditions in Definition 1 are satisfied, and, (ii) v is strictly concave.

The analytical advantage of making the restriction to replications of a finite economy is that there is a well-defined "limiting economy". For sequences of replicas one shows:

Proposition 6. *Let* $\{v, [0, \overline{y}], C(n), F^n\}_{n=1}^{\infty}$ *be a regular sequence of replica economies. Then, there exists some* $y^* \in [0, \overline{y}]$ *such that* $\lim_{n\to\infty} \Pr[|y^n(\theta) - y^*| \geq \varepsilon] = 0$ *for any* $\varepsilon > 0$ *and any sequence of optimal solutions to* (9).

To get an intuitive idea for why the replica structure helps, let r be the size of the economy being replicated and define

$$\widetilde{\eta}_i(\theta_i, \beta) \equiv \begin{cases} 1 & \text{if } (1-\beta)\theta_i + \beta x_i(\theta_i) \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

for $i = 1, \ldots, r$. Observe that if $\beta^n = \lambda^n/(1+\lambda^n)$ and λ^n is the multiplier associated with the optimal mechanism, then $\widetilde{\eta}_i(\cdot, \beta^n)$ is the inclusion rule for agent i in the optimal mechanism. Also define

$$\Phi(\beta) \equiv \int_{\theta_1} \cdots \int_{\theta_r} \frac{\sum_{i=1}^{r} \widetilde{\eta}_i(\theta_i, \beta)\theta_i}{r} \Pi_k f_k(\theta_k) d\theta_k \tag{26}$$

$$\Psi(\beta) \equiv \int_{\theta_1} \cdots \int_{\theta_r} \frac{\sum_{i=1}^{k} \widetilde{\eta}_i(\theta_i, \beta) x_i(\theta_i)}{r} \Pi_k f_k(\theta_k) d\theta_k \tag{27}$$

$$Q(\beta) \equiv \arg\max_{y\in[0,\overline{y}]} v(y)[(1-\beta)\Phi(\beta) + \beta\Psi(\beta)] - yc^*. \tag{28}$$

Let β be the limiting value of $\beta^n = \lambda^n/(1+\lambda^n)$, where λ_n is the Lagrange multiplier associated with the n-th solution in the sequence of optimal solutions. The basic idea behind Proposition 6 is then that $v(Q(\beta))\Phi(\beta) - Q(\beta)c^*$ is a good approximation of the *per capita* surplus and that the constraint in (9) is well approximated with $v(Q(\beta))\Psi(\beta) - Q(\beta)c^* \geq 0$.

Translating the problem to a binary problem by making a linear approximation of the constraint we may use Proposition 6 to get conditions for when there is non-negligible provision in a large economy.

Proposition 7. *Let* $\{v, [0, \overline{y}], C(n), F^n\}_{n=1}^{\infty}$ *be a regular sequence of replica economies and for each* i*, let* θ_i^* *be defined by* (19).

(a) *If* $v'(0)\sum_{i=1}^{r} \theta_i^*(1 - F_i(\theta_i^*))/r < c^*$*, then* $y^n(\theta)$ *converges in probability to* $y^* = 0$ *for any sequence of feasible solutions to* (9).[13]

(b) *If* $v'(0)\sum_{i=1}^{r} \theta_i^*(1 - F_i(\theta_i^*))/r > c^*$*, then* $y^n(\theta)$ *converges in probability to some* $y^* > 0$ *for any sequence of optimal solutions to* (9).

12. To minimize additional notation I continue to add agents one at a time rather than r, at a time.

13. For part (a), the replica structure is not needed. Replacing $\sum_{i=1}^{r} \theta_i^*(1-F_i(\theta_i^*))/r$ with $\lim_{n\to\infty}\sum_{i=1}^{n} \theta_i^*(1-F_i(\theta_i^*))/n$ in the statement of part (a), the result still holds and the proof is identical.

This proposition parallels Propositions 2 and 3 for the binary model. Again, the interpretation is that the conditions for when there is non-trivial provision in a large economy are the same for a benevolent planner as for a profit maximizing monopolist. Unlike the binary case, there will nevertheless be under-provision in the monopoly solution (see Section 4.2).

A useful way to interpret Proposition 7 is to think of the design problem as a binary one. For any given y, the results in Section 3 imply that y can be provided for sure (and is desirable to provide for sure) if

$$v(y)\lim_{n\to\infty}\frac{\sum_i[1-F_i(\theta_i^*)]\theta_i^*}{n} > yc^* \tag{29}$$

holds. Condition (29) is harder to satisfy the larger is y, so the dividing line between the case where $y^n(\theta)$ converges to zero in probability and where it remains strictly positive can be obtained by taking the limit of (29) as y approaches zero.

This relationship between the binary model and the set-up with a quantity decision also allows a generalization of the approximate efficiency of fixed fees to the set-up with a variable quantity. One shows that there is an incentive feasible fixed fee mechanism that takes on two values, 0 and y^*, generating a surplus that can be made arbitrary close to that of an optimal mechanism for large n. This is proven by using the exact same construction as in the proof of Proposition 4. Since there is (slight) variability in the provision level around y^*, there is an additional layer of approximations relative to the binary case, but the extension is straightforward and the proof omitted:

Proposition 8. *Let $\{v, [0,\overline{y}], C(n), F^n\}_{n=1}^{\infty}$ be a regular sequence of replica economies and let y^* be the asymptotic provision level corresponding to any sequence of optimal mechanisms. Then, for each $\varepsilon > 0$ there exists some $N \leq \infty$ and a sequence of fixed fee mechanisms with provision level y^* such that the difference in* per capita *surplus between the fixed fee mechanism and a constrained optimal mechanism is less than ε for every $n \geq N$. Moreover, truth-telling is a dominant strategy in the fixed fee mechanism.*

As in the binary case, Proposition 8 holds either with a provision rule that provides when the fees collected are sufficient to cover the costs or with a provision rule that provides the good for sure (or not at all, when the problem is asymptotically impossible). In the first case, the mechanism also satisfies *ex post* budget balance, while this fails when the good is provided for sure. In both cases, participation constraints hold *ex post*.

4.2. Comparison with the profit maximizing mechanism

Consider first a binary public good. A profit maximizing monopolist maximizes the value of the function in the constraint (17), subject only to the boundary constraints. It is intuitive that the inclusion threshold is set to θ_i^* defined in (19) for each i (see Cornelli (1996), Schmitz (1997)). The revenue conditional on provision converges in probability to $\lim_{n\to\infty}\sum_{i=1}^n \theta_i^*(1-F_i(\theta_i^*))/n$ and arguments along the same lines as for the surplus maximizing case establish that the *ex ante* probability of provision converges to zero or one depending on whether $\lim_{n\to\infty}\sum_{i=1}^n \theta_i^*(1-F_i(\theta_i^*))/n$ is greater than or smaller than c^*.

Constrained efficient and monopolistic provision thus coincide asymptotically, but the constrained efficient inclusion threshold θ_i^n is strictly smaller than θ_i^* for each i. This can be seen from observing that $\theta_i(1-F_i(\theta_i))$ is single-peaked with maximum at θ_i^* and that the optimal mechanism satisfies the feasibility constraint with equality. Social surplus is decreasing in the inclusion thresholds, so budget balance is achieved by lowering the thresholds relative to θ_i^*. Monopoly pricing therefore excludes too many consumers from a social point of view, but the provision decision is not distorted in the binary case.

With a variable quantity, over-exclusions occur for the same reasons as in the binary case: it is always beneficial for a profit maximizer to raise the inclusion threshold from θ_i^n to θ_i^* for any provision rule $y^n(\theta)$. Moreover, a monopolist sets the quantity to maximize virtual surplus, rather than a weighted average of actual and virtual surplus. The expected virtual valuation is always below the expected valuation, so there is now under-provision as well as over-exclusions compared with the constrained efficient mechanism. The property that the only inefficiency is that a too high price discourages too many agents to participate is thus an artefact of the binary model.

4.3. Partial exclusions

The formulation of the problem (9) assumes that all agents that are given access to the public good consume at the maximum quantity. In some cases it may be more reasonable to also allow partial exclusions. That is, if y is the number of articles in an electronic library, a possibility is to sell a licence for a restricted number of downloads.

The simplest way to formulate such partial exclusions is to let $a_i(\theta) \in [0, 1]$ be the proportion of $y(\theta)$ that the consumer gets access to. Since exclusion possibilities are now continuous I ignore randomizations. The (relaxed) surplus maximization problem may then be written:

$$\begin{aligned} &\max_{\{y(\cdot),\{a_i(\cdot)\}_{i=1}^n\}} \int_{\theta_1} \cdots \int_{\theta_n} \left[\sum_i v(a_i(\theta)y(\theta))\theta_i - y(\theta)C(n)\right] \Pi_k f_k(\theta_k) d\theta_k \\ &\text{s.t. } 0 \le \int_{\theta_1} \cdots \int_{\theta_n} \left[\sum_i v(a_i(\theta)y(\theta))x_i(\theta_i) - y(\theta)C(n)\right] \Pi_k f_k(\theta_k) d\theta_k 0. \end{aligned} \tag{30}$$

Solutions to problem (30) have the same characterization as saddle points of the Lagrangian as (9). An optimal inclusion rule a_i may therefore be obtained by setting

$$a_i(\theta) \in \arg\max_{a\in[0,1]} v(a_i(\theta)y(\theta))[\theta_i + \lambda x_i(\theta_i)]. \tag{31}$$

The solution to (31) is to set $a_i(\theta) = 1$ if $\theta_i + \lambda x_i(\theta_i) \ge 0$ and $a_i(\theta) = 0$ otherwise. This optimality condition is identical to the optimality condition for $\eta_i(\theta)$ for every θ, so problem (30) has the same solutions as (9). Partial exclusions are thus never active in an optimal mechanism.

The same conclusion applies if the provider maximizes profits: the condition $\theta_i + \lambda x_i(\theta_i) \ge 0$ is replaced by the condition $x_i(\theta_i) \ge 0$, again implying that no partial exclusions are active in the solution. This may seem surprising at first. For a fixed level of provision, the model with partial exclusions may be reinterpreted as a standard problem of non-linear pricing, where the provision level $y(\theta)$ acts as a satiation point in preferences and costs of production are zero.

Usually, the quantity traded is strictly increasing in type in models of non-linear pricing, but, due to the absence of a marginal cost, the set-up is similar to Stokey (1979) and Riley and Zeckhauser (1983). Indeed, if v is a linear function, we may think of $a_i(\theta)$ as a probability of trade. Not using partial exclusions is then just like "foregoing the option of intertemporal price discrimination" in Stokey (1979) or "no haggling" in Riley and Zeckhauser (1983).

We conclude that partial exclusions may be ignored even if technologically feasible. This changes if there is a positive marginal cost, say a "distribution cost". The model then becomes a more conventional model of non-linear pricing. Moreover, these "all-or nothing" inclusions arise in a model where type is a scalar parameter. For some public goods problems with a "quantity" dimension it seems more appropriate to model type as a multidimensional object representing willingness to pay for different units.

4.4. Comparison with a non-excludable public good

It is of course an easy matter to remove the use exclusions from the model. In the binary case this reduces the model to the exact set-up of Mailath and Postlewaite (1990), but for the case with a variable quantity this fills a small gap in the literature. Without exclusion possibilities the set of feasible provision rules consists of all $y : \Theta \to R_+$ for which $E_{-i}v(y(\theta))$ is weakly increasing and

$$\int_{\theta_1} \cdots \int_{\theta_n} \left(\sum\nolimits_i v(y(\theta)) \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) - y(\theta)C(n) \right) \Pi_k f_k(\theta_k) d\theta_k \geq 0. \tag{32}$$

The following generalization of the asymptotic impossibility result in the Mailath and Postlewaite result is then very easy to prove:

Proposition 9. *Suppose* $\lim_{n\to\infty} \sum_{i=1}^{n} v'(0)\underline{\theta}_i/n - \lim_{n\to\infty} C(n)/n < 0$ *and that* $\{y^n\}$ *is a sequence of incentive feasible provision rules. Then* $Ey^n(\theta) \to 0$ *as* $n \to \infty$.

Making the typical assumption that $\underline{\theta}_i = 0$ is thus sufficient for the expected provision level to be near zero in a large economy. This demonstrates that the asymptotic impossibility for voluntary agreements in a large group to realize large potential gains is not an artefact of the collective decision being a binary choice.

If costs are kept constant as n goes out of bounds, Hellwig (2003) shows that the asymptotic properties depend critically on whether the level of the public good is bounded or not. *Ex post* efficiency can be achieved in the limit if the quantity is bounded. This is because the necessary *per capita* contribution is of order $1/n$ whereas the pivot probability in a mechanism that provides (the efficient level) if and only if at least m agents announce that they have a valuation above ε is of order $1/\sqrt{n}$. Hence it is possible to induce payments of order $1/\sqrt{n}$ from each agent (with valuation above some ε), so the aggregate transfers are of order $\sqrt{n}$, which is sufficient to cover cost for a large economy.

For the same reasons, the constrained efficient level of the public good approaches infinity if the efficient level is unbounded. At the same time, the ratio of the social surplus for the constrained efficient outcome and that of the *ex post* efficient rule converges to zero, thus providing an analogue Proposition 9. In a sense, the assumption that the efficient level of the public good is unbounded is similar to the assumption that $\lim_{n\to\infty} C(n)/n = c^* > 0$. Both assumptions ensure that transfers of order $\sqrt{n}$ are insufficient to achieve efficiency.

5. CONCLUDING REMARKS

The basic conclusions from this paper are that it is possible for a "private market" to provide non-trivial amounts of an excludable public goods, and that there are essentially no gains to exploit beyond third degree price discrimination. Unlike all contributions in the related literature except Hellwig (2003), the analysis allows for a variable quantity of the public good. In a sense, one may argue that the main insight from the analysis is that there is no significant qualitative difference between the binary case and the case where the provision level is variable. Besides robustness of conclusions drawn from the binary model, the fact that the asymptotic provisions are constant generates a tractable "limiting problem" associated with the full-blown mechanism design problem, that I believe may be a useful tool for more applied studies.

APPENDIX A. PROOFS

To conserve space I write $\int_\Theta$ rather than $\int_{\theta_1} \cdots \int_{\theta_n}$ in the proofs that follow and use $dF^n(\theta)$ as shorthand notation for $\Pi_{k=1}^n f_k(\theta)d\theta_k$.

A.1. *Proposition* 2

Proof. Let $\delta = \frac{c^* - \lim_{n\to\infty}\sum_i \theta_i^*[1-F_i(\theta_i^*)]}{2} > 0$. An upper bound on the *ex ante* probability that the good is provided is found by maximizing $\int_\Theta \rho^n(\theta)dF^n(\theta)$ subject to the constraints in (16). A modification of the argument in Proposition 3.1 in Hellwig shows that there exists λ^n such that (ρ^n, η^n) solves this problem if and only if

$$\eta_i^n(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \theta_i^* \\ 0 & \theta_i < \theta_i^* \end{cases} \qquad \rho^n(\theta) = \begin{cases} 1 & \text{if } 1 + \lambda^n\left(\sum_i \eta_i^n(\theta)x_i(\theta_i) - C(n)\right) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

To understand (A.1), note that whenever the constraint binds it is necessary that η^n maximizes revenue given ρ^n for (ρ^n, η^n) to maximize the probability of provision. An inclusion rule with threshold θ_i^* does exactly that: no matter what the provision rule is, transfer revenue is maximized using this critical value. Dividing by $\lambda^n n$ and defining $k_n \equiv C(n)/n - 1/(n\lambda^n)$ we can express the *ex ante* probability of provision in economy n as $E\rho^n(\theta) = \Pr\left[\sum_i \eta_i^n(\theta)x_i(\theta_i)/n \geq k_n\right]$.

Case 1: Suppose $k_n \geq \sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta$. We note that $E\eta_i^n(\theta)x_i(\theta_i) = \theta_i^*[1-F_i(\theta_i^*)]$ and that $\{\eta_i^n(\theta)x_i(\theta_i)\}_{i=1}^n$ is a sequence of n independent random variables, where $\eta_i^n(\theta)x_i(\theta_i) \in [0, \overline{\theta}]$ for every i and n. Hence there exists some $\sigma^2 < \infty$ such that the variance of $\eta_i^n(\theta)x_i(\theta_i)$ is less than σ^2 for all i so Chebyshev's inequality implies[14]

$$\Pr\left[\frac{\sum_i \eta_i^n(\theta)x_i(\theta_i)}{n} \geq k_n\right] \leq \Pr\left[\left|\sum_i \eta_i^n(\theta)x_i(\theta_i) - \sum_i \theta_i^*[1-F_i(\theta_i^*)]\right| \geq \delta n\right] \leq \frac{\sigma^2}{\delta^2 n}. \tag{A.2}$$

Case 2: If $k_n < \sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta$ we define $H(n) = \{\theta \mid \sum_i \eta_i^n(\theta)x_i(\theta_i)/n > \sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta\}$ and $L(n) = \{\theta \mid k_n \leq \sum_i \eta_i^n(\theta)x_i(\theta_i)/n \leq \sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta\}$. Since $\rho^n(\theta) = 1$ if and only if $\theta \in H(n) \cup L(n)$, the integral constraint evaluated at (ρ^n, η^n) is

$$\begin{aligned} 0 &\leq \int_{H(n)} \left(\frac{\sum_i \eta_i^n(\theta)x_i(\theta)}{n} - \frac{C(n)}{n}\right) dF^n(\theta) + \int_{L(n)} \left(\frac{\sum_i \eta_i^n(\theta)x_i(\theta)}{n} - \frac{C(n)}{n}\right) dF^n(\theta) \\ &\leq \overline{\theta}\Pr(H(n)) + \left(\sum_i \theta_i^*[1-F_i(\theta_i^*)]/n + \delta - C(n)/n\right)\Pr(L(n)), \end{aligned} \tag{A.3}$$

after observing that $\sum_i \eta_i^n(\theta)x_i(\theta_i)/n \leq \sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta$ for all $\theta \in L(n)$ and that $\eta_i^n(\theta)x_i(\theta_i) \leq \overline{\theta}$ for all i and θ_i. By the same application of Chebyshev's inequality as in (A.2), $\Pr(H(n)) \leq \frac{\sigma^2}{\delta^2 n}$. Moreover, $\lim_{n\to\infty}\sum_i \theta_i^*[1-F_i(\theta_i^*)]/n - C(n)/n = -2\delta$, so there exists N such that $\sum_i \theta_i^*[1-F_i(\theta_i^*)]/n+\delta-C(n)/n \leq -\frac{\delta}{2}$ and for $n \geq N$. Combining with (A.3) and rearranging shows that $\Pr(L(n)) \leq \frac{2\overline{\theta}}{\delta}\frac{\sigma^2}{\delta^2 n}$ for $n \geq N$. Hence,

$$E\rho^n(\theta) = \Pr(H(n)) + \Pr(L(n)) \leq \frac{\sigma^2}{\delta^2 n}\left(1 + \frac{2\overline{\theta}}{\delta}\right). \tag{A.4}$$

Cases 1 and 2 are exhaustive, so (A.2) and (A.4) imply that there is $N < \infty$ such that

$$E\rho^n(\theta) \leq \max\left[\frac{\sigma^2}{\delta^2 n}, \frac{\sigma^2}{\delta^2 n}\left(1+\frac{2\overline{\theta}}{\delta}\right)\right] \qquad \forall n \geq N. \tag{A.5}$$

Since the R.H.S. of (A.5) goes to zero as $n \to \infty$ it follows that $\lim_{n\to\infty} E\rho^n(\theta) = 0$. ‖

A.2. *Proposition* 3

Lemma A1. *Let (ρ^n, η^n) be a sequence of feasible mechanisms such that there exists θ_i^n such that $\eta_i^n(\theta_i) = 1$ if $\theta_i \geq \theta_i^n$ and $\eta_i^n(\theta_i) = 0$ for $\theta_i < \theta_i^n$ for each i and n, and ρ^n is the surplus maximizing provision rule conditional on η^n for every n. Then,* (a) *$E\rho^n(\theta) \to 1$ as $n \to \infty$ if $\lim_{n\to\infty}\sum_i \theta_i^n[1-F_i(\theta_i^n)]/n > c^*$,* (b) *$E\rho^n(\theta) \to 0$ as $n \to \infty$ if $\lim_{n\to\infty}\sum_i \theta_i^n[1-F_i(\theta_i^n)]/n < c^*$.*

Proof. (a) For any fixed η^n it is *ex post* optimal to provide if and only if $\sum_i \eta_i^n(\theta_i)\theta_i \geq C(n)$. Since $\theta_i \geq x_i(\theta_i)$ it follows that $\rho^n(\theta) = 1$ for all θ such that $\sum_i \eta_i(\theta_i)x_i(\theta_i) \geq C(n)$ and $\rho^n(\theta) \leq 1$ for all θ such that $\sum_i \eta_i(\theta_i)x_i(\theta_i) < C(n)$, so

$$\int_\Theta \left(\frac{\sum_i \eta_i^n(\theta)x_i(\theta_i) - C(n)}{n}\right)\rho^n(\theta)dF^n(\theta) = \sum_i \frac{\theta_i^n[1-F_i(\theta_i^n)]}{n} - \frac{C(n)}{n}. \tag{A.6}$$

14. The assumption that $f_i(\theta_i) \geq k > 0$ is not needed to assure existence of σ^2. Indeed, the assumption is not needed for Proposition 2. The condition is kept only to have the same set of regularity conditions on f_i for all results.

The limit of the R.H.S. is strictly positive by assumption, so the integral constraint (17) holds strictly for n large enough. We conclude that (ρ^n, η^n) is feasible. Since $\theta_i \geq x_i(\theta_i)$ it follows that $1 - E\rho^n(\theta) = \Pr[\sum_i \eta_i^n(\theta_i)\theta_i \leq C(n)] \leq \Pr[\sum_i \eta_i^n(\theta_i)x_i(\theta_i) \leq C(n)]$. By hypothesis, for each $\varepsilon > 0$ there exists N such that $C(n)/n \leq \sum_i \theta_i^n[1-F_i(\theta_i^n)]/n - \varepsilon$ for $n \geq N$. Since $f_i(\theta_i) \geq k > 0$ we have that $\eta_i^n(\theta_i)x_i(\theta_i) \in [\underline{\theta} - \frac{1}{k}, \overline{\theta}]$ for every i, so there exists a uniform upper bound σ^2 for the variance of $\eta_i^n(\theta_i)x_i(\theta_i)$. An application of Chebyshev's inequality completes the proof.
(b) The probability of provision for the surplus maximizing rule conditional on η^n is bounded above by the maximal probability of provision conditional on η^n. Although the inclusion rules are different from the ones in Proposition 2, ρ^n still has the same form as in (A.1). Replacing δ with $\delta' = 1/2(c^* - \lim_{n\to\infty} \sum_i \theta_i^n[1 - F_i(\theta_i^n)]/n)$ one can proceed as in the proof of Proposition 2. ‖

Proof of Proposition 3. Suppose for contradiction that $\{\rho^n, \eta^n\}$ is a sequence of solutions to (16) such that $E\rho^n(\theta)$ does not converge to 1. Every element of $\{\sum_{i=1}^n \theta_i^n[1 - F_i(\theta_i^n)]/n\}_{n=1}^\infty$ belongs to a compact set, since $\theta_i^n[1 - F_i(\theta_i^n)] < \overline{\theta}$, and optimality of the inclusion rules requires that $\theta_i^n[1 - F_i(\theta_i^n)] \geq 0$ (see (14)). The provision rule in the solution to (16) must optimize the objective function conditional on the inclusion rule in the optimal solution. Taking a subsequence if necessary, Lemma A1 implies that $\lim_{n\to\infty} \sum_{i=1}^n \theta_i^n[1 - F_i(\theta_i^n)]/n = c^*$ and that there is some $\delta > 0$ and $N < \infty$ such that $E\rho^n(\theta) < 1 - \delta$ if $E\rho^n(\theta)$ fails to converge to unity: $\lim_{n\to\infty} E\rho^n(\theta) = 1$ if $\lim_{n\to\infty} \sum_{i=1}^n \theta_i^n[1-F_i(\theta_i^n)]/n > c^*$ and $\lim_{n\to\infty} E\rho^n(\theta) = 0$ if $\lim_{n\to\infty} \sum_{i=1}^n \theta_i^n[1-F_i(\theta_i^n)]/n < c^*$. The latter cannot be optimal since the surplus converges to zero and the best mechanism with $\theta_i^n = \theta_i^*$ for every i and n generates a strictly positive surplus. Pick some $\varepsilon > 0$ and partition the set of θ for which provisions occur into $H(n) = \{\theta \mid \rho(\theta) = 1$ and $\sum_i \eta_i^n(\theta)\theta_i/n \geq \sum_i E\eta_i^n(\theta)\theta_i/n + \varepsilon\}$ and $L(n) = \{\theta \mid \rho(\theta) = 1$ and $\sum_i \eta_i^n(\theta)\theta_i/n < \sum_i E\eta_i^n(\theta)\theta_i/n + \varepsilon\}$. Then,

$$\begin{aligned} s_n(\rho^n, \eta^n) &= \int_{H(n)} \frac{\left[\sum_i \eta_i^n(\theta)\theta_i - C(n)\right]}{n} dF^n(\theta) + \int_{L(n)} \frac{\left[\sum_i \eta_i^n(\theta)\theta_i - C(n)\right]}{n} dF^n(\theta) \\ &\leq \Pr(H(n))\left[\overline{\theta} - \frac{C(n)}{n}\right] + \Pr(L(n))\left[\frac{\sum_i E\eta_i^n(\theta)\theta_i}{n} - \frac{C(n)}{n}\right] \end{aligned} \tag{A.7}$$

where $s_n(\rho^n, \eta^n)$ denotes the *per capita* surplus in the n-th economy. By Chebyshev's inequality, $\lim_{n\to\infty} \Pr(H(n)) = 0$ for any $\varepsilon > 0$ and $\Pr(L(n)) \leq 1 - \delta - \Pr(H(n))$, so for any $\varepsilon > 0$ there exists N such that $s_n(\rho^n, \eta^n) \leq (1-\delta)(\sum_i E\eta_i^n(\theta)\theta_i/n - C(n)/n) + \varepsilon$. Consider an alternative (sub-) sequence of mechanisms $\{\hat{\rho}^n, \hat{\eta}^n\}$ where

$$\hat{\rho}^n(\theta) = 1 \qquad \forall\theta \in \Theta \qquad \hat{\eta}_i^n(\theta_i) = \begin{cases} 1 & \text{if } \theta_i \geq \theta_i^* \\ 1-\delta & \text{if } \theta_i^n \leq \theta_i \leq \theta_i^* \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in I, n, \tag{A.8}$$

and θ_i^* is defined in (19). The expected *per capita* surplus in mechanism $(\hat{\rho}^n, \hat{\eta}^n)$ is

$$\begin{aligned} s_n(\hat{\rho}^n, \hat{\eta}^n) &= \frac{\sum_i E\eta_i^n(\theta)\theta_i}{n} - \delta\frac{\sum_i \int_{\theta_i^n}^{\theta_i^*} \theta_i f_i(\theta_i)d\theta_i}{n} - \frac{C(n)}{n} \\ &\geq s_n(\rho^n, \eta^n) - \varepsilon + \delta\left[\sum_i \int_{\theta_i^*}^{\overline{\theta}_i} \frac{\theta_i f_i(\theta_i)d\theta_i}{n} - \frac{C(n)}{n}\right]. \end{aligned} \tag{A.9}$$

The bracketed term in (A.9) is strictly positive (since $\int_{\theta_i^*}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i \geq \theta_i^*[1 - F_i(\theta_i^*)]$) and ε is arbitrarily small, so there exists N' such that $s_n(\rho^n, \eta^n) < s_n(\hat{\rho}^n, \hat{\eta}^n)$ for every $n \geq N'$. Moreover,

$$\int_\Theta \left(\sum_i \frac{\hat{\eta}_i^n(\theta)x_i(\theta_i) - C(n)}{n}\right)\hat{\rho}^n(\theta)dF^n(\theta) = \frac{\sum_i[(1-\delta)\theta_i^n[1 - F_i(\theta_i^n)] + \delta\theta_i^*[1 - F_i(\theta_i^*)]]}{n} - \frac{C(n)}{n}, \tag{A.10}$$

which, since $\lim_{n\to\infty} \sum_i \theta_i^*[1 - F_i(\theta_i^*)]/n > \lim_{n\to\infty} \sum_i \theta_i^n[1 - F_i(\theta_i^n)]/n = c^*$, is strictly positive for n large enough. Hence there is N'' such that $(\hat{\rho}^n, \hat{\eta}^n)$ is feasible for $n > N''$. Mechanism (A.8) is thus feasible and better than the hypothetical optimal mechanism for $n \geq \max\{N, N', N''\}$. ‖

A.3. *Proposition* 4

Proof. If $\lim_{n\to\infty} \sum_i \theta_i^*[1 - F_i(\theta_i^*)]/n < c^*$, Proposition 4 is trivial since the *per capita* surplus converges to zero in the efficient mechanism. Hence, I now assume that $\lim_{n\to\infty} \sum_i \theta_i^*[1 - F_i(\theta_i^*)]/n > c^*$. Let $\gamma \in (0, 1)$ and define $\widetilde{\theta}_i^n(\gamma) \equiv \gamma\theta_i^n + (1-\gamma)\theta_i^*$, where θ_i^n is the threshold from the surplus maximizing mechanism for every i and n and θ_i^* is defined in (19) for every i. Consider a sequence $\{\hat{\rho}^n, \hat{\eta}^n, \hat{\xi}^n\}$ of fixed fee mechanisms where for each n

$$\hat{\eta}_i^n(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \widetilde{\theta}_i^n(\gamma) \\ 0 & \text{otherwise} \end{cases} \qquad \hat{\rho}^n(\theta) = \begin{cases} 1 & \text{if } \sum_i \hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma) \geq C(n) \\ 0 & \text{otherwise,} \end{cases} \tag{A.11}$$

and $\hat{\xi}_i^n(\theta) = \widetilde{\theta}_i^n(\gamma)\hat{\eta}_i^n(\theta)\hat{\rho}^n(\theta)$. The feasibility constraint (4) holds (also *ex post*) by construction and i's pay-off from announcements $\hat{\theta}$ is

$$\hat{\rho}^n(\hat{\theta})\hat{\eta}_i^n(\hat{\theta})(\theta_i - \hat{\xi}_i^n(\hat{\theta})) = \begin{cases} \hat{\rho}^n(\hat{\theta})(\theta_i - \hat{\theta}_i) & \text{if } \hat{\theta}_i \geq \widetilde{\theta}_i^n(\gamma) \\ 0 & \text{otherwise.} \end{cases} \tag{A.12}$$

The participation constraint (3) is thus satisfied and truth-telling is a dominant strategy and therefore also incentive compatible in the Bayesian sense. Mechanism $(\hat{\rho}^n, \hat{\eta}^n, \hat{\xi}^n)$ is thus incentive feasible for every n, with probability of provision given by $\Pr[\sum_i \hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma)/n \geq C(n)/n]$. For each n let λ^n be the Lagrange multiplier associated with the surplus maximizing mechanism.

Case 1: Suppose that (taking a subsequence if necessary) $\lim_{n\to\infty} \lambda^n/(1+\lambda^n) = 1$. Then we see from (14) that $\lim_{n\to\infty} \theta_i^n = \theta_i^*$ for every i, implying that $\lim_{n\to\infty} \widetilde{\theta}_i^n(\gamma) = \theta_i^*$ for any $\gamma \in [0,1]$. Hence, $\lim_{n\to\infty} \sum_i \widetilde{\theta}_i^n(\gamma)[1 - F_i(\widetilde{\theta}_i^n(\gamma))]/n = \lim_{n\to\infty} \theta_i^*[1 - F_i(\theta_i^*)]/n > c^*$ and there exists $\varepsilon > 0$ and finite N such that $\sum_i \widetilde{\theta}_i^n(\gamma)[1 - F_i(\widetilde{\theta}_i^n(\gamma))]/n - C(n)/n \geq \varepsilon$ for all $n \geq N$. The probability of provision for mechanism (A.11) is $\Pr[\sum_i \hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma)/n \geq C(n)/n]$ and $E\hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma) = \widetilde{\theta}_i^n(\gamma)[1 - F_i(\widetilde{\theta}_i^n(\gamma))]$, so another application of Chebyshev's inequality implies that

$$\begin{aligned} 1 - E\hat{\rho}^n(\theta) &= \Pr\left[\sum\nolimits_i \hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma) \leq C(n)\right] \\ &\leq \Pr\left[\left|\sum\nolimits_i \hat{\eta}_i^n(\theta)\widetilde{\theta}_i^n(\gamma) - \sum\nolimits_i \widetilde{\theta}_i^n(\gamma)(1 - F_i(\widetilde{\theta}_i^n(\gamma)))\right| \leq \varepsilon n\right] \leq \frac{\sigma^2}{\varepsilon^2 n}. \end{aligned} \tag{A.13}$$

Thus, $\lim_{n\to\infty} E\hat{\rho}^n(\theta) = 1$. Since $\lim_{n\to\infty} \widetilde{\theta}_i^n(\gamma) = \lim_{n\to\infty} \theta_i^n = \theta_i^*$ it is easy to check that the *per capita* surplus in the fixed fee mechanism approaches that of the optimal mechanism for any γ.

Case 2: Suppose instead (taking a subsequence if necessary) that $\lim_{n\to\infty} \lambda^n/(1+\lambda^n) = \beta < 1$. With some abuse of notation, let $\theta_i(\lambda)$ be the inclusion threshold from (14) associated with multiplier λ. We notice that $\lim_{n\to\infty} \theta_i^n = \theta_i(\lambda) < \theta_i^*$. Therefore, $\lim_{n\to\infty} \sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n = \lim_{n\to\infty} \sum_i \int_{\theta_i^n(\lambda)}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n$ and $\lim_{n\to\infty} \sum_i \int_{\widetilde{\theta}_i^n(\gamma)}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n = \lim_{n\to\infty} \sum_i \int_{\gamma\theta_i^n(\lambda)+(1-\gamma)\theta_i^*}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n$ for any γ. Moreover, the second limit converges to the first as $\gamma \to 1$. Together, this implies that for any $\varepsilon > 0$ there exists $\gamma < 1$ and $N < \infty$ such that

$$\sum\nolimits_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n - \sum\nolimits_i \int_{\widetilde{\theta}_i^n(\gamma)}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n < \varepsilon/2 \tag{A.14}$$

for all $n \geq N$. There also exists $\delta > 0$ and $N' < \infty$ such that $(\widetilde{\theta}_i^n(\gamma) - \theta_i^n) = (1-\gamma)(\theta_i^* - \theta_i^n) > \delta$ for $n \geq N'$. Since $\theta_i(1 - F_i(\theta_i))$ is strictly increasing on $[\underline{\theta}_i, \theta_i^*]$ this in turn implies that there is $\varepsilon_i > 0$ such that $\widetilde{\theta}_i^n(\gamma)[1 - F_i(\widetilde{\theta}_i^n(\gamma))] \geq \theta_i^n[1 - F_i(\theta_i^n)] + 2\varepsilon_i$. Since $\theta_i^n = \theta_j^n$ and $\theta_i^* = \theta_j^*$ for any (i, j) such that $F_i = F_j$ and $F_i \in \mathcal{F}$ for any i, where $\mathcal{F}$ is finite $\{\varepsilon_1, \ldots, \varepsilon_n\}$ takes on at most as many values as the cardinality of $\mathcal{F}$. Therefore, there exists $\widetilde{\varepsilon} > 0$ such that $\widetilde{\varepsilon} \leq \varepsilon_i$ for all i. Lemma A1 implies that $\lim_{n\to\infty} \sum_i \theta_i^n[1 - F_i(\theta_i^n)]/n \geq c^*$ since otherwise there is no provision in the limit. Hence there exists N'' such that $\sum_i \widetilde{\theta}_i^n(\gamma)[1 - F_i(\widetilde{\theta}_i^n(\gamma))]/n \geq C(n)/n + \widetilde{\varepsilon}$ for every $n \geq N''$. Applying (A.13) again we conclude that $\lim_{n\to\infty} E\hat{\rho}^n(\theta) = 1$ also along such subsequence. Let $s_n(\rho^n, \eta^n)$ and $s_n(\hat{\rho}^n, \hat{\eta}^n)$ be the *per capita* surplus generated by the optimal and fixed fee mechanisms respectively. Since the probability of provision converges to one in both mechanisms it is easy to verify that there exists N''' such that

$$s_n(\rho^n, \eta^n) - s_n(\hat{\rho}^n, \hat{\eta}^n) \leq \sum\nolimits_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n - \sum\nolimits_i \int_{\widetilde{\theta}_i^n(\gamma)}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i/n + \varepsilon/2 < \varepsilon, \tag{A.15}$$

where the second inequality comes from (A.14). Since ε was arbitrary the result follows. ‖

A.4. *Proposition* 5

For the proofs of Propositions 5–7 it is convenient to define the functions $s_n(y^n, \eta^n) \equiv S_n(y^n, \eta^n)/n$ and $g_n(y^n, \eta^n) \equiv G_n(y^n, \eta^n)/n$, where S_n and G_n are defined in (10) and (11). Since these are *per capita* expressions of the objective and the function in the constraint to (9) we may treat $s_n(y^n, \eta^n)$ as the maximand and $g_n(y^n, \eta^n)$ as the constraint.

Lemma A2. *Let $\{y^n, \eta_i^n\}_{n=1}^{\infty}$ be a sequence of incentive feasible mechanisms, where for each n and $i \leq n$, η_i^n is a threshold rule with inclusion threshold θ_i^n (independent of θ_{-i}). Then, for each $\varepsilon > 0$ there exists some finite N such that*

$$v(Ey^n(\theta))\frac{\sum_i \theta_i^n(1-F_i(\theta_i^n))}{n} \geq Ey^n(\theta)\frac{C(n)}{n} - \varepsilon \qquad \forall n \geq N. \tag{A.16}$$

Proof. Fix $\varepsilon > 0$ and let $H(n) = \{\theta \mid \sum_i \eta_i^n(\theta)x_i(\theta_i)/n - \sum_i \theta_i^n[1-F_i(\theta_i^n)]/n > \delta\}$, where $\delta = \varepsilon/2v(\overline{y})(\overline{\theta}+1) > 0$. By Chebyshev's inequality, there exists N such that $\Pr(H(n)) < \delta$ for $n \geq N$. Since $v(y^n(\theta)) \leq v(\overline{y})$ and $\eta_i(\theta)x_i(\theta_i) \leq \overline{\theta}$ for every $\theta \in \Theta$, $i \in I$ and $\delta < \varepsilon/2v(\overline{y})\overline{\theta}$ it follows that

$$\int_{H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) \leq v(\overline{y})\overline{\theta}\Pr(H(n)) < \frac{\varepsilon}{2} \tag{A.17}$$

for $n \geq N$. Decomposing $g_n(y^n, \eta^n)$, (A.17) implies that, for $n \geq N$

$$\begin{aligned} g_n(y^n, \eta^n) &= \int_{H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) + \int_{\Theta\setminus H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) - \frac{C(n)}{n}Ey^n(\theta) \\ &< \frac{\varepsilon}{2} + \int_{\Theta\setminus H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) - \frac{C(n)}{n}Ey^n(\theta). \end{aligned} \tag{A.18}$$

Observing that $\sum_i \eta_i(\theta)x_i(\theta_i)/n \leq \sum_i \theta_i^n[1-F_i(\theta_i^n)]/n + \delta$ for $\theta \in \Theta \setminus H(n)$ we have that

$$\begin{aligned} \int_{\Theta\setminus H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) &\leq \left(\frac{\sum_i \theta_i^n[1-F_i(\theta_i^n)]}{n} + \delta\right)\int_{\Theta\setminus H(n)} v(y^n(\theta))dF^n(\theta) \\ &\leq \left(\frac{\sum_i \theta_i^n[1-F_i(\theta_i^n)]}{n} + \delta\right)\int_{\Theta} v(y^n(\theta))dF^n(\theta) \\ &\leq v(Ey^n(\theta))\left(\frac{\sum_i \theta_i^n[1-F_i(\theta_i^n)]}{n} + \delta\right) \end{aligned} \tag{A.19}$$

where the last inequality comes from concavity of v. Since $\delta > \varepsilon/2v(\overline{y})$ and v is increasing, so $\delta v(Ey^n(\theta)) \leq \delta v(\overline{y}) \leq \frac{\varepsilon}{2}$. Combined with (A.19) this implies that

$$\int_{\Theta\setminus H(n)} v(y^n(\theta))\frac{\sum_i \eta_i(\theta)x_i(\theta_i)}{n}dF^n(\theta) \leq v(Ey^n(\theta))\frac{\theta_i^n[1-F_i(\theta_i^n)]}{n} + \frac{\varepsilon}{2}. \tag{A.20}$$

The conclusion follows by substituting (A.20) back into (A.18) and noting that $g_n(y^n, \eta^n) \leq 0$ for feasibility. ‖

Lemma A3. *Fix any $\varepsilon > 0$ and let $\{\eta^n\}_{n=1}^{\infty}$ be a sequence of threshold inclusion rules. Then there exists some finite N such that $s_n(y^n, \eta^n) > s_n(y'^n, \eta^n)$ for every $n \geq N$ and any provision rules y^n, y'^n satisfying*

$$\frac{\sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i}{n}(Ev(y^n(\theta)) - Ev(y'^n(\theta))) - \frac{C(n)}{n}(Ey^n(\theta)) - Ey'^n(\theta))) > \varepsilon. \tag{A.21}$$

Proof. Fix $\varepsilon > 0$ and define $H(n) = \{\theta \mid \sum_i \eta_i^n(\theta)\theta_i/n \geq \sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i/n - \delta\}$, where $\delta = \varepsilon/2v(\overline{y})(\overline{\theta}+1)$. Notice that

$$\int_{H(n)} v(y^n(\theta))dF^n(\theta) = Ev(y^n(\theta)) - \int_{\Theta\setminus H(n)} v(y^n(\theta))dF^n(\theta) \geq Ev(y^n(\theta)) - v(\overline{y})(1-\Pr(H(n))), \tag{A.22}$$

and that the *per capita* surplus for mechanism (y^n, η^n) can be decomposed as

$$\begin{aligned} s_n(y^n, \eta^n) &= \int_{H(n)} \frac{\sum_i \eta_i^n(\theta)\theta_i v(y^n(\theta))}{n}dF^n(\theta) + \int_{\Theta\setminus H(n)} \frac{\sum_i \eta_i^n(\theta)\theta_i v(y^n(\theta))}{n}dF^n(\theta) - Ey^n(\theta)\frac{C(n)}{n} \\ &\geq \left(\sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \frac{\theta_i f_i(\theta_i)d\theta_i}{n} - \delta\right)\int_{H(n)} v(y^n(\theta))dF^n(\theta) - Ey^n(\theta)\frac{C(n)}{n}. \end{aligned} \tag{A.23}$$

Together, (A.22) and (A.23) imply

$$s_n(y^n, \eta^n) \geq \left(\sum_i \frac{\int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i}{n} - \delta\right)[Ev(y^n(\theta)) - v(\overline{y})(1-\Pr(H(n)))] - Ey^n(\theta)\frac{C(n)}{n}. \tag{A.24}$$

Let $L(n) = \{\theta \mid \sum_i \eta_i^n(\theta)\theta_i/n \le \sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i/n + \delta\}$. A symmetric argument shows

$$\begin{aligned} s_n(y'^n, \eta^n) &\le \frac{1}{n}\left[\sum\nolimits_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i + \delta\right]\int_{L(n)} v(y'^n(\theta))dF^n(\theta) + \frac{\sum_i \overline{\theta}_i v(\overline{y})}{n}[1 - \Pr(L(n))] - Ey'^n(\theta)\frac{C(n)}{n} \\ &\le \frac{1}{n}\left[\sum\nolimits_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i + \delta\right]Ev(y'^n(\theta)) + \frac{\sum_i \overline{\theta}_i v(\overline{y})}{n}[1 - \Pr(L(n))] - Ey^n(\theta)\frac{C(n)}{n}. \end{aligned} \quad \text{(A.25)}$$

Now, $\delta > 0$ and $E\{\sum_i \eta_i^n(\theta)\theta_i\} = \sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i/n$, so, by Chebyshev's inequality, there exists N such that $\Pr(H(n)) \ge 1 - \delta$ for all $n \ge N$ and $\Pr(L(n)) \ge 1 - \delta$ for all $n \ge N$. Together with (A.21), (A.24), (A.25) and our choice of $\delta = \varepsilon/2v(\overline{y})(\overline{\theta} + 1)$ this implies that

$$\begin{aligned} s_n(y^n, \eta^n) - s_n(y'^n, \eta^n) \ge{} & \underbrace{\frac{\sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i}{n}[Ev(y^n(\theta)) - Ev(y'^n(\theta))] - \frac{C(n)}{n}[Ey^n(\theta)) - Ey'^n(\theta))]}_{>\varepsilon \text{ by (A.21)}} \\ & - \underbrace{\delta[Ev(y^n(\theta)) - v(\overline{y})(1 - \Pr(H(n)))]}_{<\delta[v(\overline{y})\Pr(H(n))]<\delta v(\overline{y})} - v(\overline{y})\underbrace{(1 - \Pr(H(n)))}_{<\delta}\underbrace{\sum\nolimits_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i)d\theta_i/n}_{<\overline{\theta}} \\ & - \delta\underbrace{Ev(y'^n(\theta))}_{<v(\overline{y})} - \underbrace{\frac{\sum_i \overline{\theta}_i v(\overline{y})}{n}}_{<\overline{\theta}v(\overline{y})}\underbrace{[1 - \Pr(L(n))]}_{<\delta} > \varepsilon - \delta v(\overline{y}) - \delta\overline{\theta}v(\overline{y}) - \delta v(\overline{y}) - \delta\overline{\theta}v(\overline{y}) \\ ={} & 0. \end{aligned} \quad \text{(A.26)}$$

Since ε was arbitrary, the result follows. ||

Lemma A4. *Suppose v is strictly concave. Then, for each $\varepsilon_1, \varepsilon_2 > 0$ there exists some $\delta > 0$ such that $v(Ey^n(\theta)) \ge Ev(y^n(\theta)) + \delta$ for every $y^n(\cdot)$ such that* $\Pr(|y^n(\theta) - E(y^n(\theta))| \ge \varepsilon_1) \ge \varepsilon_2$.

Proof. Omitted. ||

Lemma A5. *Consider a sequence of incentive feasible mechanisms $\{y^n, \eta^n\}$. Suppose there are $\varepsilon_1, \varepsilon_2 > 0$ and N such that* $\Pr(|y^n(\theta) - Ey^n(\theta)| \ge \varepsilon_1) \ge \varepsilon_2$ *for every $n \ge N$. Consider the alternative sequence $\{\overline{y}^n, \eta^n\}$ where $\overline{y}^n(\theta) = Ey^n(\theta)$ for all $\theta \in \Theta$ and every n and the inclusion rules are unchanged. Then, there exists N' such that $\{\overline{y}^n, \eta^n\}$ is incentive feasible for every $n \ge N'$.*

Proof. By Lemma A4 there exists $\delta > 0$ such that $v(Ey^n(\theta)) \ge Ev(y^n(\theta)) + \delta$. Moreover under the hypothesis of the Lemma there exists δ' such that $Ey^n(\theta) > \delta'$ for all $n \ge N$. Applying Lemma A2 this implies that there exists $K > 0$ such that $\sum_i \theta_i^n[1 - F_i(\theta_i^n)]/n \ge K$ for all $n \ge N$ since otherwise $Ey^n(\theta) \to 0$ (since (A.16) would be violated otherwise). Define $\widetilde{\delta} = \delta K/\overline{\theta}v(\overline{y}) > 0$ and let $\widetilde{H}(n) = \{\theta \mid \sum_i \eta_i^n(\theta)x_i(\theta_i)/n \ge \sum_i \theta_i^n[1 - F_i(\theta_i^n)]/n + \widetilde{\delta}\}$. A decomposition of $g_n(y^n, \eta^n)$ along the same lines as the decomposition of $s_n(y^n, \eta^n)$ in Lemma A3 yields

$$g_n(y^n, \eta^n) \le Ev(y^n(\theta))\left(\frac{\sum_i \theta_i^n[1 - F_i(\theta_i^n)]}{n} + \widetilde{\delta}\right) + \overline{\theta}v(\overline{y})\Pr(\widetilde{H}(n)) - Ey^n(\theta)\frac{C(n)}{n}. \quad \text{(A.27)}$$

For the mechanism $\{\overline{y}^n, \eta^n\}$, a direct calculation shows that

$$g_n(\overline{y}^n, \eta^n) = v(Ey^n(\theta))\frac{\sum_i \theta_i^n[1 - F_i(\theta_i^n)]}{n} - Ey^n(\theta)\frac{C(n)}{n}. \quad \text{(A.28)}$$

$E\sum_i \eta_i^n(\theta)x_i(\theta_i) = \sum_i \theta_i^n[1 - F_i(\theta_i^n)]$, so an application of Chebyshev's inequality shows that there exists $N \le N' < \infty$ such that $\Pr(\widetilde{H}(n)) < \widetilde{\delta}$. Using this and that $v(Ey^n(\theta)) \ge Ev(y^n(\theta)) + \delta$ together with (A.28) shows that for $n \ge N'$

$$\begin{aligned} g_n(\overline{y}^n, \eta^n) &\ge g_n(y^n, \eta^n) + \delta\underbrace{\frac{\sum_i \theta_i^n[1 - F_i(\theta_i^n)]}{n}}_{>K} - \widetilde{\delta}\underbrace{Ev(y^n(\theta))}_{<v(\overline{y})} - \overline{\theta}v(\overline{y})\underbrace{\Pr(\widetilde{H}(n))}_{<\widetilde{\delta}} \\ &> g_n(y^n, \eta^n) + \delta K - \widetilde{\delta}v(\overline{y})(1 + \overline{\theta}) = g_n(y^n, \eta^n) \ge 0, \end{aligned} \quad \text{(A.29)}$$

where the final inequality follows because (y^n, η^n) is incentive feasible. Hence $(\overline{y}^n, \eta^n)$ is also incentive feasible for $n \geq N'$. ||

Proof of Proposition 5. If the proposition would fail, then (taking a subsequence if necessary) there exists some $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\Pr(|y^n(\theta) - Ey^n(\theta)| \geq \varepsilon_1) \geq \varepsilon_2$ for all n. Consider a sequence of alternative mechanisms $\{\overline{y}^n, \eta^n\}$, where for each n the only difference with the initial mechanism is that $\overline{y}^n(\theta) = Ey^n(\theta)$. By Lemma A4 we have that there exists δ such that $v(Ey^n(\theta)) \geq Ev(y^n(\theta)) + \delta$ holds for every n in the sequence. Moreover, for the same reasons as in Lemma A5 there exists $K > 0$ such that $\sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i / n \geq K$. Hence,

$$\frac{\sum_i \int_{\theta_i^n}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i}{n} (Ev(\overline{y}^n(\theta)) - Ev(y^n(\theta))) - \frac{C(n)}{n} (E\overline{y}^n(\theta)) - Ey^n(\theta))) > \delta K > 0, \tag{A.30}$$

which by application of Lemma A3 implies that there exists N such that $(\overline{y}^n, \eta^n)$ generates a higher surplus than (y^n, η^n). By Lemma A5 there exists N' such that $(\overline{y}^n, \eta^n)$ is incentive feasible for $n \geq N'$. This contradicts that $\{y^n, \eta^n\}$ is a sequence of optimal mechanisms. ||

A.5. *Proposition* 6

Lemma A6. *$\Phi(\cdot)$, $\Psi(\cdot)$ and $Q(\cdot)$ defined in* (26), (27) *and* (28) *satisfy the following properties:*

(a) *$\Phi(\cdot)$, $\Psi(\cdot)$ and $Q(\cdot)$ are continuous functions of β*
(b) *$\Phi(\beta) > \Psi(\beta)$ for every $\beta \in [0, 1]$*
(c) *$\Phi(\cdot)$, $\Psi(\cdot)$ and $Q(\cdot)$ are weakly decreasing in β*
(d) *$\Phi(\beta)v(Q(\beta)) - Q(\beta)c^*$ is weakly decreasing in β. Moreover, if $\beta' < \beta''$ is such that $Q(\beta') > Q(\beta'')$, then $\Phi(\beta')v(Q(\beta')) - Q(\beta')c^* > \Phi(\beta'')v(Q(\beta'')) - Q(\beta'')c^*$.*

Proof. (a) $(1-\beta)\theta_i + \beta x_i(\theta_i)$ is strictly increasing in θ_i, implying that there is a unique threshold $\widetilde{\theta}_i(\beta) \in [\underline{\theta}_i, \overline{\theta}_i]$ such that $(1-\beta)\theta_i + \beta x_i(\theta_i) \geq 0$ if and only if $\theta_i \geq \widetilde{\theta}_i(\beta)$. Moreover, $(1-\beta)\theta_i + \beta x_i(\theta_i) < 0$ for every $\theta_i < \widetilde{\theta}_i(\beta)$ and $(1-\beta)\theta_i + \beta x_i(\theta_i) > 0$ for every $\theta_i > \widetilde{\theta}_i(\beta)$, so if $\{\beta^n\}$ is a sequence with $\lim_{n\to\infty} \beta^n \to \beta$, then $\lim_{n\to\infty} \widetilde{\eta}_i(\theta_i, \beta^n) = \widetilde{\eta}_i(\theta_i, \beta)$ for all $\theta_i \neq \widetilde{\theta}_i(\beta)$. Continuity of $\Phi(\cdot)$ and $\Psi(\cdot)$ follows by Lebesgues' monotone convergence theorem and the theorem of the maximum guarantees that $Q(\cdot)$ is an upper-hemi-continuous correspondence. Strict concavity of v implies that $Q(\beta)$ is unique for every $\beta \in [0, 1]$, implying that $Q(\cdot)$ is a continuous function.
(b) Each inclusion rule $\widetilde{\eta}_i(\cdot, \beta)$ can be characterized by an inclusion threshold $\widetilde{\theta}_i(\beta)$ (possibly equal to $\underline{\theta}_i$) for each β. Since $x_i(\theta_i)$ is continuous and $x_i(\overline{\theta}_i) = \overline{\theta}_i - (1 - F_i(\overline{\theta}_i))/f_i(\theta_i) = \overline{\theta}_i$ there exists $\theta_i' < \overline{\theta}_i$ such that $x_i(\theta_i) > 0$ for all $\theta_i \geq \theta_i'$ and $\widetilde{\theta}_i(\beta) \leq \widetilde{\theta}_i(1) \leq \theta_i'$ for every $\beta \in [0, 1]$. Hence,

$$\begin{aligned}\Phi(\beta) - \Psi(\beta) &= \int_{\theta_1} \cdots \int_{\theta_r} \frac{\sum_{i=1}^r \widetilde{\eta}_i(\theta_i, \beta)(\theta_i - x_i(\theta_i))}{r} dF^n(\theta) \\ &= \frac{1}{r} \sum_{i=1}^r \int_{\widetilde{\theta}_i(\beta)}^{\overline{\theta}_i} (\theta_i - x_i(\theta_i)) f_i(\theta_i) d\theta_i = \frac{1}{r} \sum_{i=1}^r \int_{\widetilde{\theta}_i(\beta)}^{\overline{\theta}_i} (1 - F_i(\theta_i)) d\theta_i > 0. \end{aligned} \tag{A.31}$$

(c) Omitted (see Norman, 2003).
(d) To show that $\Phi(\beta)v(Q(\beta)) - Q(\beta)c^*$ is weakly decreasing, suppose for contradiction that there exists $\beta' < \beta''$ such that $\Phi(\beta')v(Q(\beta')) - Q(\beta')c^* < \Phi(\beta'')v(Q(\beta'')) - Q(\beta'')c^*$. Then,

$$\begin{aligned} v(Q(\beta'))[(1-\beta')\Phi(\beta') + \beta'\Psi(\beta')] - Q(\beta')c^* &< \Phi(\beta'')v(Q(\beta'')) - Q(\beta'')c^* + v(Q(\beta'))\beta' \underbrace{(\Psi(\beta') - \Phi(\beta'))}_{<0} \\ &\leq \Phi(\beta'')v(Q(\beta'')) - Q(\beta'')c^* + v(Q(\beta''))\beta'(\Psi(\beta') - \Phi(\beta')) \\ &= v(Q(\beta''))[(1-\beta')\Phi(\beta') + \beta'\Psi(\beta')] - Q(\beta'')c^*, \end{aligned} \tag{A.32}$$

contradicting the assumption that $Q(\beta')$ solves (28) for $\beta = \beta'$. Finally, if $Q(\beta') > Q(\beta'')$ then $v(Q(\beta'))\beta'(\Psi(\beta') - \Phi(\beta')) < v(Q(\beta''))\beta'(\Psi(\beta') - \Phi(\beta'))$, implying that (A.32) generates a contradiction also if from a weak inequality, thus validating the final claim. ||

Lemma A7. *Consider a sequence of replications of a given finite economy $\{v, [0, \overline{y}], C(r), F^r\}$. Let $\{y^n, \eta^n\}_{n=1}^\infty$ be a sequence of optimal mechanisms and λ^n be the Lagrange multiplier associated with the mechanism (y^n, η^n) and*

$\beta^n = \frac{\lambda^n}{1+\lambda^n}$ *for each* n. *Then, for any subsequence* $\{n_k\}$ *such that* $\lim_{n_k\to\infty} \beta^{n_k} = \beta$ *we have that*

$$\lim_{n_k\to\infty} \int_\Theta \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta)d\theta = \Phi(\beta) \tag{A.33}$$

$$\lim_{n_k\to\infty} \int_\Theta \frac{\sum_i \eta_i^{n_k}(\theta)x_i(\theta_i)}{n_k} dF^n(\theta)d\theta = \Psi(\beta). \tag{A.34}$$

Proof. Omitted (see Norman, 2003). ‖

Lemma A8. *Suppose the hypotheses of Lemma* A7 *are fulfilled. Then* $\lim_{n_k\to\infty} Ey^{n_k}(\theta) = Q(\beta)$ *for any subsequence such that* $\lim_{n_k\to\infty} \beta^{n_k} = \beta$.

Proof. Define $\chi_i(\beta,\theta_i) = (1-\beta)\theta_i + \beta x_i(\theta_i)$ and let

$$\begin{aligned} Y^{n_k} &\equiv \arg\max_{y\in[0,\overline{y}]} \frac{v(y)E\left(\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i)\right) - yC(n_k)}{n_k} \\ y^{n_k}(\theta) &= \arg\max_{y\in[0,\overline{y}]} \frac{\sum_{i=1}^{n_k} v(y)\eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i) - yC(n_k)}{n_k}, \end{aligned} \tag{A.35}$$

where Y^{n_k} is well-defined for every n_k by strict concavity of v and $y^{n_k}(\theta)$ is an alternative way to express the provision rule in (13) for economy n_k. Pick an arbitrary $\varepsilon > 0$ and let $m = 2(\overline{y}+1) < \infty$. By continuity of $y^{n_k}(\theta)$ in the parameters of the problem (the theorem of the maximum) there exists $\delta > 0$ such that $|y^{n_k}(\theta) - Y^{n_k}| < \varepsilon/m$ for all θ such that $|C(n)/n - c^*| \le \delta$ and

$$\left| \frac{\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i) - E\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i)}{n_k} \right| \le \delta. \tag{A.36}$$

Moreover, $\{\chi_i(\beta^{n_k},\theta_i)\}_{i=1}^{n_k}$ is a sequence of independent random variables with bounded variance, so an application of Chebyshev's inequality implies that there exists N such that

$$\Pr\left[\left|\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i) - E\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i)\right| \ge \delta n_k\right] < \frac{\varepsilon}{m} \tag{A.37}$$

for every $n_k \ge N$. It follows that $y^{n_k}(\theta) \ge Y^{n_k} - \frac{\varepsilon}{m}$ with probability of at least $(1-\frac{\varepsilon}{m})$, so $Ey^{n_k} \ge (1-\frac{\varepsilon}{m})(Y^{n_k} - \frac{\varepsilon}{m})$ for $n_k \ge N$. Symmetrically, $y^{n_k}(\theta) \le Y^{n_k} + \frac{\varepsilon}{m}$ with probability of at least $(1-\frac{\varepsilon}{m})$ and $y^{n_k}(\theta) \le \overline{y}$ for all θ, so $Ey^{n_k} \le (1-\frac{\varepsilon}{m})(Y^{n_k} + \frac{\varepsilon}{m}) + \frac{\varepsilon}{m}\overline{y}$ for all $n_k \ge N$. Since $m = 2(\overline{y}+1)$ we have that

$$\begin{aligned} Ey^{n_k} &\ge \left(1-\frac{\varepsilon}{m}\right)\left(Y^{n_k} - \frac{\varepsilon}{m}\right) \Leftrightarrow Ey^{n_k} - Y^{n_k} \ge -\frac{\varepsilon}{m}\left(Y^{n_k} - 1 - \frac{\varepsilon}{m}\right) \ge -\frac{\varepsilon}{m}Y^{n_k} \ge -\frac{\varepsilon}{2(\overline{y}+1)}Y^{n_k} \ge -\frac{\varepsilon}{2} \\ Ey^{n_k} &\le \left(1-\frac{\varepsilon}{m}\right)\left(Y^{n_k} + \frac{\varepsilon}{m}\right) + \frac{\varepsilon}{m}\overline{y} \Leftrightarrow Ey^{n_k} - Y^{n_k} \le \frac{\varepsilon}{m}\left(\overline{y} - Y^{n_k} - \frac{\varepsilon}{m} + 1\right) \le \frac{\varepsilon}{m}(\overline{y}+1) \\ &\le \frac{\varepsilon}{2(\overline{y}+1)}(\overline{y}+1) \le \frac{\varepsilon}{2}, \end{aligned} \tag{A.38}$$

so $|Ey^{n_k} - Y^{n_k}| \le \frac{\varepsilon}{2}$ for all $n_k \ge N$. To complete the argument we observe that given a subsequence $\{\beta^{n_k}\}$ such that $\beta^{n_k} \to \beta$ as $n_k \to \infty$ we may apply Lemma A7 to conclude that

$$\lim_{n_k\to\infty} \frac{E\sum_{i=1}^{n_k} \eta_i^{n_k}(\theta)\chi_i(\beta^{n_k},\theta_i)}{n_k} = (1-\beta)\Phi(\beta) + \beta\Psi(\beta) \tag{A.39}$$

and $\lim_{n_k\to\infty} C(n_k)/n_k = c^*$ by assumption. The theorem of the maximum assures that there exists some finite N' such that $|Y^{n_k} - Q(\beta)| \le \varepsilon/2$ for $n_k \ge N'$. Picking $N'' = \max\{N, N'\}$ the triangle inequality implies that $|Ey^{n_k} - Q(\beta)| \le \varepsilon$. Since $\varepsilon > 0$ was arbitrary the result follows. ‖

Lemma A9. *Suppose the hypotheses of Lemma* A7 *are fulfilled. Then* $\lim_{n_k\to\infty} s_{n_k}(y^{n_k}, \eta^{n_k}) = \Phi(\beta)v(Q(\beta)) - Q(\beta)c^*$ *for any subsequence* $\{n_k\}$ *such that* $\lim_{n_k\to\infty} \beta^{n_k} = \beta$.

Proof. Pick an arbitrary $\varepsilon > 0$ and let

$$\widetilde{s}_{n_k} = v(Ey^{n_k}(\theta)) \int_\Theta \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta) - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k}. \tag{A.40}$$

Combining (A.33) in Lemma A7 with Lemma A8 we conclude there exists some finite N such that $|\widetilde{s}_{n_k} - \Phi(\beta)v(Q(\beta)) - Q(\beta)c^*| < \varepsilon/2$ for all $n_k \geq N$. Let $m = 2\overline{\theta}(1 + v(\overline{y})) < \infty$. Continuity of v implies that there exists $\delta > 0$ such that $|v(Ey^{n_k}(\theta)) - v(y)| < \frac{\varepsilon}{m}$ for all y such that $|y - Ey^{n_k}(\theta)| \leq \delta$. Moreover, Proposition 5 guarantees that for every $\delta > 0$ there exists N' such that $\Pr(|y^{n_k}(\theta) - Ey^{n_k}| \geq \delta) \leq \frac{\varepsilon}{m}$ for all $n_k \geq N'$. For each n_k let $H(n_k) = \{\theta \mid y^{n_k}(\theta) \geq Ey^{n_k}(\theta) - \delta\}$ and observe that $\Pr(H(n_k)) \geq 1 - \frac{\varepsilon}{m}$ and $\frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} \leq \overline{\theta}$ for every $\theta \in \Theta$, so

$$\begin{aligned}\int_{H(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta) &= \int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta) - \int_{\Theta \setminus H(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta) \\ &\geq \int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^n(\theta) - \frac{\varepsilon\overline{\theta}}{m}. \end{aligned} \tag{A.41}$$

Since $y^{n_k}(\theta) \geq Ey^{n_k}(\theta) - \delta$ for every $\theta \in H(n_k)$ it follows that for all $n_k \geq N'' = \max\{N, N'\}$,

$$\begin{aligned} s_{n_k}(y^{n_k}, \eta^{n_k}) &= \int_{\Theta} \left(\frac{\sum_i \eta_i^{n_k}(\theta)\theta_i v(y^{n_k}(\theta)) - y^{n_k}(\theta)C(n_k)}{n_k} \right) dF^{n_k}(\theta) \\ &\geq v(Ey^{n_k}(\theta) - \delta) \int_{H(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k} \\ &\geq \left[v(Ey^{n_k}(\theta)) - \frac{\varepsilon}{m} \right] \int_{H(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k} \\ /(\text{A.41})/ &\geq \left[v(Ey^{n_k}(\theta)) - \frac{\varepsilon}{m} \right] \left[\int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) - \frac{\varepsilon\overline{\theta}}{m} \right] - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k} \\ &= \widetilde{s}_{n_k} - \frac{\varepsilon}{m} \left(\int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) + \overline{\theta}v(Ey^{n_k}(\theta)) - \frac{\varepsilon\overline{\theta}}{m} \right) \geq \widetilde{s}_{n_k} - \frac{\varepsilon\overline{\theta}[1 + v(\overline{y})]}{m} = \widetilde{s}_{n_k} - \frac{\varepsilon}{2}, \end{aligned} \tag{A.42}$$

where the last inequality comes from $m = 2\overline{\theta}(1 + v(\overline{y}))$. Next, let $L(n_k) = \{\theta \mid y^{n_k}(\theta) \leq Ey^{n_k} + \delta\}$. Observing that $\Pr(L(n_k)) \geq 1 - \frac{\varepsilon}{m}$ for $n_k \geq N''$, $y^{n_k}(\theta) \leq \overline{y}$ for all θ, and that $\sum_i \eta_i^{n_k}(\theta)\theta_i / n_k \leq \overline{\theta}$ for all θ we have that

$$\begin{aligned} s_{n_k}(y^{n_k}, \eta^{n_k}) &= \int_{\Theta} \left(\frac{\sum_i \eta_i^{n_k}(\theta)\theta_i v(y^{n_k}(\theta)) - y^{n_k}(\theta)C(n_k)}{n_k} \right) dF^{n_k}(\theta) \\ &\leq v(Ey^{n_k}(\theta) + \delta) \int_{L(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) \\ &\quad + v(\overline{y}) \int_{\Theta \setminus L(n_k)} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k} \\ &\leq \left[v(Ey^{n_k}(\theta)) + \frac{\varepsilon}{m} \right] \int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) + \frac{\overline{\theta}v(\overline{y})\varepsilon}{m} - Ey^{n_k}(\theta)\frac{C(n_k)}{n_k} \\ &= \widetilde{s}_{n_k} + \frac{\varepsilon}{m} \left(\int_{\Theta} \frac{\sum_i \eta_i^{n_k}(\theta)\theta_i}{n_k} dF^{n_k}(\theta) + \overline{\theta}v(\overline{y}) \right) \leq \widetilde{S}^{n_k} + \frac{\varepsilon\overline{\theta}(1 + v(\overline{y}))}{m} \\ &\leq \widetilde{s}_{n_k} + \frac{\varepsilon}{2}. \end{aligned} \tag{A.43}$$

Hence, $|s_{n_k}(y^{n_k}, \eta^{n_k}) - \widetilde{s}_{n_k}| \leq \frac{\varepsilon}{2}$ for all $n_k \geq N''$ and since $|\widetilde{s}_{n_k} - [\Phi(\beta)v(Q(\beta)) - Q(\beta)c^*]| < \frac{\varepsilon}{2}$ for all $n_k \geq N''$ it follows from the triangle inequality that $|s_{n_k}(y^{n_k}, \eta^{n_k}) - [\Phi(\beta)v(Q(\beta)) - Q(\beta)c^*]| < \varepsilon$. Since ε was arbitrary the result follows. ‖

Proof of Proposition 6. If there exists no y^* such that $\lim_{n\to\infty} Ey^n(\theta) = y^*$ Lemma A8 implies that $\{\beta^n\}_{n=1}^{\infty}$ cannot be a converging sequence. Since each $\beta^n \in [0, 1]$, a compact set, there must then exist at least two accumulation points β^L, β^H and corresponding subsequences $\{\beta^{n_{k(L)}}\}$, $\{\beta^{n_{k(H)}}\}$ with $\lim_{n_{k(J)}\to\infty} \beta^{n_{k(J)}} = \beta^J$ for $J = L, H$ where $Q(\beta^L) \neq Q(\beta^H)$. Label β^L, β^H such that $\beta^L < \beta^H$ and observe that parts 3 and 4 of Lemma A6 imply that $Q(\beta^L) > Q(\beta^H)$ and

$$\Phi(\beta^L)v(Q(\beta^L)) - Q(\beta^L)c^* > \Phi(\beta^H)v(Q(\beta^H)) - Q(\beta^H)c^*. \tag{A.44}$$

Pick some $\lambda \in (0, 1)$ and let $Q^\lambda = \lambda Q(\beta^H) + (1-\lambda)Q(\beta^L)$. Consider a sequence of mechanisms $\{\overline{y}^n, \overline{\eta}^n\}_{n=1}^{\infty}$, where $\overline{y}^n(\theta) = Q^\lambda$ and $\overline{\eta}_i^n(\theta) = \eta_i^n(\theta_i, \beta^L)$ for all $\theta \in \Theta$ and each n. This mechanism generates a *per capita* surplus

$$s_n(\overline{y}^n, \overline{\eta}^n) = v(Q^\lambda)\int_\Theta \frac{\sum_i \eta_i^n(\theta, \beta^L)\theta_i}{n} dF^n(\theta) - Q^\lambda \frac{C(n)}{n}, \tag{A.45}$$

implying (by Lemma A7) that $\lim_{n\to\infty} s_n(\overline{y}^n, \overline{\eta}^n) = v(Q^\lambda)\Phi(\beta^L) - Q^\lambda c^*$, while

$$\lim_{n_{k(H)}\to\infty} s_{n_k(H)}(y^{n_{k(H)}}, \eta^{n_{k(H)}}) = \Phi(\beta^H)v(Q(\beta^H)) - Q(\beta^H)c^*. \tag{A.46}$$

Strict concavity implies that

$$\begin{aligned} v(Q^\lambda)\Phi(\beta^L) - Q^\lambda c^* &> \lambda[v(Q(\beta^H)) + (1-\lambda)v(Q(\beta^L))]\Phi(\beta^L) - Q^\lambda c^* \\ &= \lambda[\Phi(\beta^L)v(Q(\beta^H)) - Q(\beta^H)c^*] + (1-\lambda)[\Phi(\beta^L)v(Q(\beta^L)) - Q(\beta^L)c^*] \\ /\Phi(\beta^L) \geq \Phi(\beta^H)/ &\geq \lambda[\Phi(\beta^H)v(Q(\beta^H)) - Q(\beta^H)c^*] + (1-\lambda)[\Phi(\beta^L)v(Q(\beta^L)) - Q(\beta^L)c^*] \\ &\geq \Phi(\beta^H)v(Q(\beta^H)) - Q(\beta^H)c^*, \end{aligned} \tag{A.47}$$

where the final inequality comes from (A.44). Hence there exists N such that $s_{n_k(H)}(\overline{y}^{n_{k(H)}}, \overline{\eta}^{n_{k(H)}}) > s_{n_k(H)}(y^{n_{k(H)}}, \eta^{n_{k(H)}})$ for all $n_{k(H)} \geq N$. Finally, we will establish feasibility of $(\overline{y}^n, \overline{\eta}^n)$ for large enough n. Since $\{y^{n_{k(L)}}, \eta^{n_{k(L)}}\}$ and $\{y^{n_{k(H)}}, \eta^{n_{k(H)}}\}$ are sequences of optimal mechanisms, they must also be sequences of feasible mechanisms, which by Lemma A2 requires that for every $\varepsilon > 0$ there exists a finite N such that (A.16) holds for $j = L, H$ and every $n_{k(J)} \geq N$. From (A.34) in Lemma A7 we have that $\lim_{n_{k(J)}\to\infty} \frac{\theta_i^{n_{k(J)}}(1-F_i(\theta_i^{n_{k(J)}}))}{n_{k(J)}} = \Psi(\beta^J)$ and $\lim_{n_{k(J)}\to\infty} Ey^{n_{k(J)}} = Q(\beta^J)$ by virtue of Lemma A8. Combining this with (A.16) it follows that a necessary condition for $\{y^{n_{k(L)}}, \eta^{n_{k(L)}}\}$ and $\{y^{n_{k(H)}}, \eta^{n_{k(H)}}\}$ to satisfy feasibility everywhere in each sequence is that

$$v(Q(\beta^J))\Psi(\beta^J) \geq Q(\beta^J)c^* \tag{A.48}$$

holds for $J = L, H$. The mechanism $(\overline{y}^n, \overline{\eta}^n)$ is feasible if

$$v(Q^\lambda)\int_\Theta \frac{\sum_i \eta_i^n(\theta, \beta^L)x_i(\theta_i)}{n} dF^n(\theta) - Q^\lambda \frac{C(n)}{n} \geq 0, \tag{A.49}$$

where (A.34) in Lemma A7 implies that $\lim_{n\to\infty}\int_\Theta[\sum_i \eta_i^n(\theta, \beta^L)x_i(\theta_i)/n]dF^n(\theta) = \Psi(\beta^L)$ and $\Psi(\beta^L) \geq \Psi(\beta^H)$, so from (A.48) it follows that

$$\begin{aligned} &v(Q(\beta^H))\Psi(\beta^L) \geq v(Q(\beta^H))\Psi(\beta^H) \geq Q(\beta^H)c^* \text{ and } v(Q(\beta^L))\Psi(\beta^L) \geq Q(\beta^L)c^* \\ &\Rightarrow v(Q^\lambda)\Psi(\beta^L) > \lambda v(Q(\beta^H))\Psi(\beta^L) + (1-\lambda)v(Q(\beta^L))\Psi(\beta^L) \\ &\qquad \geq \lambda Q(\beta^H)c^* + (1-\lambda)Q(\beta^L)c^* = Q^\lambda c^*. \end{aligned} \tag{A.50}$$

Using (A.50) we see that the limit of the L.H.S. in (A.49) is strictly positive, so there exists N' such that $(\overline{y}^n, \overline{\eta}^n)$ is feasible for $n \geq N'$. Hence $(\overline{y}^{n_{k(H)}}, \overline{\eta}^{n_{k(H)}})$ is both feasible and better than the hypothetical optimal solution for $n_{k(H)} \geq N'' = \max\{N, N'\}$. The result follows. ‖

A.6. *Proposition* 7

Proof. (a) Concavity implies that $v(y^n(\theta)) \leq v(0) + v'(0)y^n(\theta) = v'(0)y^n(\theta)$. Hence,

$$g_n(y^n, \eta^n) \leq \int_\Theta \left(v'(0)y^n(\theta)\frac{\sum_i \eta_i^n(\theta)x_i(\theta_i)}{n} - \frac{y^n(\theta)C(n)}{n}\right)dF^n(\theta). \tag{A.51}$$

Define $p^n(\theta) = y^n(\theta)/\overline{y}$, $\widetilde{C}(n) = \frac{1}{v'(0)}C(n)$, and $c^{**} = \frac{1}{v'(0)}c^*$. Since the integral constraint to (9) is equivalent to $g_n(y^n, \eta^n) \geq 0$, a necessary condition for the integral constraint to hold is that $\int_\Theta(\sum_i \eta_i^n(\theta)x_i(\theta_i) - \widetilde{C}(n))p^n(\theta)dF^n(\theta) \geq 0$. Moreover, $\lim_{n\to\infty}\sum_i \theta_i^*(1 - F_i(\theta_i^*))/n = \sum_{i=1}^r \theta_i^*(1 - F_i(\theta_i^*)) < c^{**}$, where the inequality is from the hypothesis of the result. Since $\widetilde{C}(n)$ approaches c^{**} as $n \to \infty$ and $p^n(\theta) \in [0, 1]$ for all θ we can apply Proposition 2 to conclude that $Ep^n(\theta) \to 0$ as $n \to \infty$. The conclusion follows since $Ey^n(\theta) \leq \overline{y}Ep^n(\theta)$ and $\overline{y}$ is finite.

(b) By Proposition 6 $y^n(\theta)$ converges in probability to some $y^* \in [0, \overline{y}]$. If the claim fails it must therefore be that the provision level converges to zero in probability, in which case the *per capita* surplus goes to zero as $n \to \infty$. But, if $\sum_{i=1}^r v'(0)\theta_i^*(1 - F_i(\theta_i^*))/n > c^*$ there exists $\varepsilon > 0$ such that $\sum_{i=1}^r v'(\varepsilon)\theta_i^*(1 - F_i(\theta_i^*))/n > c^*$. Moreover,

$v(\varepsilon) \geq v'(\varepsilon)\varepsilon$ by concavity. We may thus consider the sequence of mechanisms where $\hat{y}^n(\theta) = \varepsilon$ for all θ and every n and where $\hat{\eta}_i^n(\theta) = 1$ if and only if $\theta_i \geq \theta_i^*$ for each i and n. Evaluating the function g_n for mechanism (y^n, η^n) we have that

$$g_n(\hat{y}^n, \hat{\eta}^n) = v(\varepsilon)\frac{\sum_i \theta_i^*(1 - F_i(\theta_i^*))}{n} - \frac{\varepsilon C(n)}{n} > \varepsilon\left[v'(\varepsilon)\frac{\sum_i \theta_i^*(1 - F_i(\theta_i^*))}{n} - \frac{C(n)}{n}\right]. \tag{A.52}$$

Taking the limit as $n \to \infty$ we find that $g_n(\hat{y}^n, \hat{\eta}^n) > 0$ for n sufficiently large. Since the associated *per capita* surplus is strictly positive the result follows. ‖

A.7. *Proposition* 9

Proof. Set $\eta_i^n(\theta) = 1$ for all i and all θ and proceed as in Part 1 of the proof of Proposition 7. ‖

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