1. Consider an economy with two goods, \( x \) and \( y \) and two agents, \( A \) and \( B \). Suppose that the agents are endowed with \( e^A = (e^A_x, e^A_y) \) and \( e^B = (e^B_x, e^B_y) \) units of the two goods. Also suppose that preferences over different consumption bundles are represented by utility functions \( u^A(x, y) \) and \( u^B(x, y) \) respectively.

(a) Define a competitive equilibrium.
(b) Define a Pareto optimal allocation.
(c) True/False: All competitive equilibria are Pareto Optimal. Explain using a wisely chosen graph. No credit unless explanation satisfactory.
(d) True/False: All Pareto optimal allocations are competitive equilibria. Explain using a wisely chosen graph. No credit unless explanation satisfactory.
(e) Suppose instead that preferences are given by \( u^A(x^A, x^B, y) \) and \( u^B(x^A, b^B, y) \). Would this affect the answers above? Why? Why not? No credit unless answer contains a reasonable explanation.

2. Consider a variant of the voluntary contribution model for public goods that we considered in class. Let \( x \) be a private good and \( y \) be a public good. Assume agents \( A \) and \( B \) are endowed with one unit of the private good each and that the private good can be transformed to the public good on a one-to-one basis. Assume that the agents decide simultaneously how much to provide and let \( y^A \) be the contribution by \( A \) and \( y^B \) be the contribution by \( B \). Suppose that the preferences are

\[
U^A(x, y) = ax + (1-a)y \\
U^B(x, y) = bx + (1-b)y,
\]

where \( a > 0 \) and \( b > 0 \).

(a) Define a Nash equilibrium for this voluntary contribution model.
(b) Define a Pareto optimal allocation for this environment.
(c) Is it possible to find values of \( a \) and \( b \) such that the Nash equilibrium of the voluntary contribution model is inefficient? (If yes, give a concrete example. If no, explain/prove).
(d) Is it possible to find examples of \( a \) and \( b \) such that the Nash equilibrium of the voluntary contribution model is efficient? (If yes, give a concrete example. If no, explain/prove).

3. Suppose that Bart and Lisa cares about their private consumption and a public good. The public good is “binary”, meaning that it is a project that can either be implemented or not. Write \( y \) for the public good. Given that it is a binary decision, we may then without loss of generality write \( y = 0 \) when the project is not implemented and \( y = 1 \) when it is implemented. Barts preferences are given by

\[
U_B(x, y) = \begin{cases} 
  x & \text{if } y = 0 \\
  x + b & \text{if } y = 1 
\end{cases}
\]
and Lisas preferences are

\[ U_L(x, y) = \begin{cases} 
  x & \text{if } y = 0 \\
  x + l & \text{if } y = 1 
\end{cases} \]

Assume that \( b > 0 \) and \( l > 0 \) and that Bart and Lisa have endowments \( e_B = e_L = 1 \) in units of the private good and that the public good costs 1 unit of the private good to implement.

(a) Under what conditions on \( b \) and \( l \) is the efficient decision to build the public good?

(b) Consider the following game. Bart and Lisa simultaneously decides how much to contribute. Hence, a strategy for Bart is to pick a contribution \( y_B \) between 0 and 1 (there is no point in contributing more). For Lisa, a strategy is some \( y_L \) between 0 and 1. Suppose that if the contributions are sufficient, that is if \( y_B + y_L \geq 1 \), then the project is implemented and that the difference between 1 and \( y_B + y_L \) is redistributed to the agents (you may for simplicity assume that Bart and Lisa gets 50% each). If the contributions are not sufficient, then they get returned to the players (i.e., Bart gets \( y_B \) back and Lisa gets \( y_L \) back).

Write down the relevant payoff function in terms of the strategic variables \( (y_B, y_L) \) for Bart and Lisa. Assuming that provision is efficient, is there always an equilibrium in the game where the public good is provided? Be careful to explain the argument carefully.

(c) Again assuming that provision is efficient and assume that \( b < 1 \) and \( l < 1 \), is there an equilibrium when the public good is not provided?