2 Consider a variant of the voluntary contribution model for public goods that we considered in class. Let x be a private good and y be a public good. Assume agents A and B are endowed with one unit of the private good each and that the private good can be transformed to the public good on a one–to-one basis. Assume that the agents decide simultaneously how much to provide and let y^A be the contribution by A and y^B be the contribution by B. Suppose that the preferences are

$$\begin{aligned} U^A(x,y) &= ax + (1-a)\,y \\ U^B(x,y) &= bx + (1-b)\,y, \end{aligned}$$

where $a > 0$ and $b > 0$.

(a) Define a Nash equilibrium for this voluntary contribution model.
Answer: $\left(y^A, y^B\right)$ is a Nash equilibrium if y^A solves

$$\begin{aligned} \max_{0\le y\le 1} U^A\left(1-y, y+y^B\right) &= \max_{0\le y\le 1} a(1-y) + (1-a)\left(y+y^B\right) \\ \text{(alternatively)} &= \max_{0\le y\le 1} a - ay + y - ay + (1-a)\,y^B \\ \text{(alternatively)} &= \max_{0\le y\le 1} a + y\,[1-2a] + (1-a)\,y^B \end{aligned}$$

and (symmetrically) y^B solves

$$\max_{0\le y\le 1} U^B\left(1-y, y^A + y\right)$$

(b) Define a Pareto optimal allocation for this environment.
Answer: $\left(y, x^A, x^B\right)$ solves

$$\begin{aligned} &\max_{(y,x^A,x^B)} U^A\left(x^A, y\right) \\ \text{s.t. } U^B\left(x^B, y\right) &\ge b \\ x^A + x^B + y &\le 2 \end{aligned}$$

(c) Is it possible to find values of a and b such that the Nash equilibrium of the voluntary contribution model is inefficient?
Answer: Yes, suppose that $a, b > \frac{1}{2}$. Then, $y^A = 0$ solves

$$\max_{0\le y\le 1} a + y\,[1-2a] + (1-a)\,y^B$$

for any y^B and $y^B = 0$ solves

$$\max_{0\le y\le 1} b + y\,[1-2b] + (1-b)\,y^A$$

for any y^A. Hence, the only equilibrium is $\left(y^A, y^B\right) = (0,0)$ and the utility is

$$\begin{aligned} U^A(1,0) &= a \\ U^B(1,0) &= b \end{aligned}$$

Suppose instead that we consider allocation $\left(y, x^A, x^B\right) = (2, 0, 0)$. Then the utility is

$$\begin{aligned} U^A(0,2) &= 2(1-a) \\ U^B(0,2) &= 2(1-b) \end{aligned}$$

For $(2,0,0)$ to Pareto dominate the voluntary provision equilibrium we need that

$$\begin{aligned} a &< 2(1-a) \\ b &< 2(1-b) \end{aligned}$$

Hence, we have that if $a \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and $b \in \left(\frac{1}{2}, \frac{2}{3}\right)$, then the voluntary provision equilibrium is inefficient.

(d) Is it possible to find examples of a and b such that the Nash equilibrium of the voluntary contribution model is efficient? (If yes, give a concrete example. If no, explain/prove).

Answer: Yes, Suppose that $a, b < \frac{1}{2}$. Then the volunatary provision equilibrium is $(1,1)$ giving utility

$$\begin{aligned} U^A(0,2) &= 2(1-a) \\ U^B(0,2) &= 2(1-b) \end{aligned}$$

Suppose this is inefficient. Then there exists $\left(x^A, x^B, y\right)$ such that

$$\begin{aligned} U^A\left(x^A, y\right) &= ax^A + (1-a)y = x^A + (1-\alpha)\left(y - x^A\right) > 2(1-a) \\ U^B\left(x^B, y\right) &= bx^B + (1-b)y = x^B + (1-b)\left(y - x^B\right) > 2(1-b) \end{aligned}$$

or

$$\begin{aligned} x^A &> (1-a)\left[2 - y + x^A\right] > \frac{1}{2}\left[2 - y + x^A\right] \\ x^B &> (1-b)\left[2 - y + x^B\right] > \frac{1}{2}\left[2 - y + x^B\right] \end{aligned}$$

Sum

$$\begin{aligned} x^A + x^B &> 2 - y + \frac{1}{2}\left(x^A + x^B\right) \\ y + x^A + x^B &> 2 + \frac{1}{2}\left(x^A + x^B\right) > 2, \end{aligned}$$

which is a contradiction against feasibility.