Consider a variant of the voluntary contribution model for public goods that we considered in class. Let \( x \) be a private good and \( y \) be a public good. Assume agents \( A \) and \( B \) are endowed with one unit of the private good each and that the private good can be transformed to the public good on a one-to-one basis. Assume that the agents decide simultaneously how much to provide and let \( y_A \) be the contribution by \( A \) and \( y_B \) be the contribution by \( B \). Suppose that the preferences are

\[
U_A (x, y) = ax + (1 - a) y \\
U_B (x, y) = bx + (1 - b) y,
\]

where \( a > 0 \) and \( b > 0 \).

(a) Define a Nash equilibrium for this voluntary contribution model.

**Answer:** \((y^A, y^B)\) is a Nash equilibrium if \( y^A \) solves

\[
\max_{0 \leq y^B \leq 1} U_A (1 - y, y + y^B) = \max_{0 \leq y^B \leq 1} a (1 - y) + (1 - a) (y + y^B) \\
(\text{alternatively}) = \max_{0 \leq y^B \leq 1} a - ay + y - ay + (1 - a) y^B \\
(\text{alternatively}) = \max_{0 \leq y^B \leq 1} a + y [1 - 2a] + (1 - a) y^B
\]

and (symmetrically) \( y^B \) solves

\[
\max_{0 \leq y^A \leq 1} U_B (1 - y, y^A + y)
\]

(b) Define a Pareto optimal allocation for this environment.

**Answer:** \((y, x^A, x^B)\) solves

\[
\max_{(y, x^A, x^B)} U_A (x^A, y) \\
\text{s.t. } U_B (x^B, y) \geq b \\
x^A + x^B + y \leq 2
\]

(c) Is it possible to find values of \( a \) and \( b \) such that the Nash equilibrium of the voluntary contribution model is inefficient?

**Answer:** Yes, suppose that \( a, b > \frac{1}{2} \). Then, \( y^A = 0 \) solves

\[
\max_{0 \leq y^B \leq 1} a + y [1 - 2a] + (1 - a) y^B
\]

for any \( y^B \) and \( y^B = 0 \) solves

\[
\max_{0 \leq y^A \leq 1} b + y [1 - 2b] + (1 - b) y^A
\]

for any \( y^A \). Hence, the only equilibrium is \((y^A, y^B) = (0, 0)\) and the utility is

\[
U_A (1, 0) = a \\
U_B (1, 0) = b
\]
Suppose instead that we consider allocation \((y, x^A, x^B) = (2, 0, 0)\). Then the utility is

\[
U^A (0, 2) = 2(1 - a) \\
U^B (0, 2) = 2(1 - b)
\]

For \((2, 0, 0)\) to Pareto dominate the voluntary provision equilibrium we need that

\[
a < (1 - a) \\
b < (1 - b)
\]

Hence, we have that if \(a \in \left(\frac{1}{2}, \frac{2}{3}\right)\) and \(b \in \left(\frac{1}{2}, \frac{2}{3}\right)\), then the voluntary provision equilibrium is inefficient.

(d) Is it possible to find examples of \(a\) and \(b\) such that the Nash equilibrium of the voluntary contribution model is efficient? (If yes, give a concrete example. If no, explain/prove).

**Answer:** Yes. Suppose that \(a, b < \frac{1}{2}\). Then the voluntary provision equilibrium is \((1, 1)\) giving utility

\[
U^A (0, 2) = 2(1 - a) \\
U^B (0, 2) = 2(1 - b)
\]

Suppose this is inefficient. Then there exists \((x^A, x^B, y)\) such that

\[
U^A (x^A, y) = ax^A + (1 - a) y = x^A + (1 - a) (y - x^A) > 2(1 - a) \\
U^B (x^B, y) = bx^B + (1 - b) y = x^B + (1 - b) (y - x^B) > 2(1 - b)
\]

or

\[
x^A > (1 - a) [2 - y + x^A] > \frac{1}{2} [2 - y + x^A] \\
x^B > (1 - b) [2 - y + x^B] > \frac{1}{2} [2 - y + x^B]
\]

Sum

\[
x^A + x^B > 2 - y + \frac{1}{2} (x^A + x^B) \\
y + x^A + x^B > 2 + \frac{1}{2} (x^A + x^B) > 2,
\]

which is a contradiction against feasibility.