

# 1 Rational Preferences.

## 1.1 Preference Relations

While alternative theories are starting to challenge the standard approach, it is still the case that we in most economic theory assume that individual behavior is driven by *rational decisions*. This means is that it is assumed that a decision maker (often a consumer) (a) has some specified set of options to choose from (the budget set in consumer theory), (b) picks the best (or one out of a set of best options) of the set of available options.

To formalize this we need some notation. To begin with, we let  $X$  represent some set of options. The most typical application is to have  $X$  being a space of consumption bundles. In an economy with 2 commodities (say tea and coffee) a typical element of  $X$  is a pair  $x = (x_1, x_2)$ , where  $x_1$  represents the quantity of tea and  $x_2$  represents the quantity of coffee. With 3 commodities (tea, coffee, Dr. Pepper) a typical element is a triple  $x = (x_1, x_2, x_3)$ , where  $x_1$  and  $x_2$  still represent quantities of tea and coffee respectively and  $x_3$  is the quantity of Dr. Pepper. In other decision problems the choice set could be a discrete list of options, in which case I often write  $X = \{a, b, c, \dots\}$ , where  $a$  is the first option,  $b$  is the second option... (  $\{ \}$  is used to indicate that what is in the parenthesis is a set rather than a vector).

Now, imagine that the consumer considers two options from the set  $X$ , call them  $x'$  and  $x''$ . Suppose that the consumer asks the question, Is  $x'$  better than  $x''$  or is  $x''$  better than  $x'$ ? This introspection can lead to three possible responses:

1.  $x'$  is better than  $x''$  and  $x''$  is not better than  $x'$ . In this case we write  $x' \succ x''$  and say that  $x'$  is *strictly preferred to*  $x''$
2.  $x''$  is better than  $x'$  and  $x'$  is not better than  $x''$ . In this case we write  $x'' \succ x'$  and say that  $x''$  is *strictly preferred to*  $x'$
3.  $x'$  and  $x''$  are equally good. In this case we write  $x' \sim x''$  and say that the consumer is *indifferent* between  $x'$  and  $x''$ .

It is logically possible for the consumer to conclude that  $x'$  is better than  $x''$  and  $x''$  is better than  $x'$ , but they are not equally good, but this would be quite inconsistent with everyday usage of the words “better than”, so we rule this potential 4th possibility out right away. That is, we assume

**Assumption 1** Strict preferences are *asymmetric*: There is no pair  $x'$  and  $x''$  in  $X$  such that  $x' \succ x''$  and  $x'' \succ x'$ .

It is convenient to introduce the notation  $\succeq$  for weak preference. That is, we write  $x' \succeq x''$  if  $x'$  is either strictly preferred to or indifferent with  $x''$ , which we may read as “ $x'$  is weakly preferred to  $x''$ ” or alternatively “ $x'$  is at least as good as  $x''$ ”.

We say that  $\succ$  is the consumers *strict preference relation* and  $\succeq$  is the *weak preference relation*. For any pair of options  $x'$  and  $x''$  we have that either  $x' \succ x''$  or not. The same thing is true for the weak preference order, either  $x' \succeq x''$  or not.

Certain details that you can safely skip unless you are interested will be relegated to paragraphs such as this one: smaller type. Mathematically speaking a preference relation (strict or weak) in a binary relation. The defining property of a binary relation (reflected in the name) is that two objects either stands in a certain relation to each other or not. Given a binary relation  $B$  one writes  $xBy$  “ $x$  Bees  $y$ ” or “ $x$  stands in relation to  $y$ ” if  $x$  does fulfill the particular relation in question. If not one can simply write “not  $xBy$ ”. Some examples of binary relations that are not preference relations are:

- Let  $X$  be the real line (all real numbers) and  $B$  the relation  $\geq$ . Since for any two numbers  $x$  and  $y$  either  $x$  is greater than or equal to  $y$  or not, this is a binary relation.
- Let  $X$  be the 2-dimensional Euclidean space and let  $B$  be defined as  $xBy$  whenever  $x_1 \geq y_1$  and  $x_2 \geq y_2$  (that is, the relation holds if both the first and the second coordinate in the  $x$ -vector are bigger than in the  $y$ -vector).
- Let  $X$  be the set of integer numbers (1,2,3,...) and let  $B$  be the relation where  $xBy$  if the product of  $x$  and  $y$  is, say, 24 (in which case  $xBy$  for  $(x, y) = (1, 24), (2, 12) \dots$ ).

There are lots of properties a preference relation (or any binary relation) may or may not satisfy. Stated in terms of the weak preference ordering there are really only two central properties:

**Definition 1** We say that  $\succeq$  is complete if for all  $x'$  and  $x''$  in  $X$  either  $x' \succeq x''$  or  $x'' \succeq x'$  (both is OK, in which case the consumer is indifferent)

In words, all pairs can be compared.

**Definition 2** We say that  $\succeq$  is transitive if  $x' \succeq x''$  and  $x'' \succeq x'''$  implies that  $x' \succeq x'''$

Observe that a failure of this would mean that  $x'$  would be strictly worse than  $x'''$  in spite of being at least as good as  $x''$ , which in turn is at least as good as  $x'''$ . There is absolutely no way to use *logic* to rule this out, but a failure of transitivity seems “weird”. At a more practical level, I don’t know of any reasonable decision theory that can accommodate intransitive individual preferences.

Now, the word “rational” in everyday usage has a long list of different connotations (which may explain some of the usual non-economists complaints about economists being narrow in insisting on rational behavior). However, in the context of decision theory rationality is defined as having preferences that are complete and transitive. That is,

**Definition 3**  $\succeq$  is a rational preference ordering if it is complete and transitive.

Almost all economic theory assumes rational preferences. There are interesting exceptions, but this is advanced material that you will not see in this class.

**Example 4** Imagine a very simple choice problem where the choice is between 3 CDsrrrrrrrr

*Metallica*

*Ricki Martin*

*Hives*

Ask:

1. Which is better (which would you rather take in a pairwise choice), *Metallica* or *Ricki Martin*

2. Which is better, Metallica or Hives
3. Which is better, Hives or Ricki Martin

If

1. You can answer all questions: then preferences are complete
2. If answers are “consistent” in the sense that you don’t get pairwise rankings like

*Metallica better than Ricki Martin*

*Ricki Martin better than Hives*

*Hives better than Metallica,*

*or anything else that creates a cycle, then the preferences are transitive.*

## 1.2 Some Remarks about Objections to the Basic Axioms on Preferences

There are certain issues about the standard axioms on preferences that are truly problematic. In particular, while it may be reasonable to assume that the decision maker can decide upon a complete ranking of all possible alternatives in a choice problem with very few alternatives, it doesn’t really seem like a realistic descriptive model of more complex problems (think about all the possible consumption baskets you can put together in a supermarket). Human beings do not have infinite computational capacity or a perfect memory. However, to model these types of constraints in a satisfactory way is very hard.

However, several common gut-feeling objections to the axioms are nonsense. For example: one may argue that a beer may be preferred to a glass of red wine and a glass of red wine preferred to a scotch, but, if the next day is cool, gloomy and rainy, then it may be that a scotch is better than a beer. I agree that this is quite possible, *but this is not a violation of transitivity in preferences.* The reason is that a beer (glass of red wine, scotch) on a hot day is a different commodity than a beer (glass of red wine, scotch) on a cold and gloomy day.

That is, the notion of a good in economics doesn't only capture the physical characteristics, but also *when* and *under what circumstances* the good is consumed.

### 1.3 From Preference to Rational Choice

The preference relation  $\succ$  (or  $\succeq$ ) tells us how the decision maker ranks the alternatives in  $X$ . Now suppose that  $A$  is a set of feasible alternatives for the decision maker (which may coincide with  $X$ , but in the typical application it is just a part (aka subset) of  $X$  : a consumer may have well-specified preferences with a Ferrari in the most preferred bundle without being able to afford a Ferrari). The behavioral assumption in economics and all other rational choice theory is that *THE CONSUMER PICKS SOME ALTERNATIVE  $x^*$  IN  $A$  WHICH IS SUCH THAT THERE IS NO OTHER ALTERNATIVE IN  $A$  THAT IS STRICTLY BETTER THAN  $x^*$* . Formally:

**Definition 5**  $x^*$  is a rational choice given preference relation  $\succ$  and choice set  $A$  if there exists no  $x'$  in  $A$  such that  $x' \succ x^*$ .

Observe here that we started with a postulated preference relation and derived the predicted choice. From a scientific point of view it is somewhat backwards: it would make more sense to start with the behavior (what people do is observable) and ask when the behavior is consistent with the preference based model of rational choice. This can be done (in principle and in practice in a somewhat unsatisfactory way). We will discuss this issue (revealed preference) later in the class in the context of traditional consumer theory.

### 1.4 Utility Functions

Preference relations are useful mainly to explain what it means to be a rational decision maker in the economics lingo. However, economists rarely use them except to prove abstract theorems. The reason is that the information contained in a rational preference ordering can be summarized by a numerical representation that goes under the name the *utility function*. Analytically, the utility function is far more convenient to use than a preference relation



represent exactly the same preferences since they all imply that  $a$  is strictly better than  $b$  which is strictly better than  $c$ . Conceptually, there is nothing wrong with the idea that intensity of preferences could be different (for example, that preferences for the outcome in a political race would differ not only according to who is the favored candidate, but also in how much difference it makes to the voter if their favorite candidate wins). However, treating the scale of the utility function as meaningful (in economics lingo-using the cardinal information of the utility function) leads to methodological difficulties and is usually not needed, so the orthodox view that only the ordinal ranking is meaningful makes lots of sense. We will return to this later and discuss in some more detail what it means for preferences to be ordinal.

The general case is not much harder to understand than the example. Utility functions are functions that “measure” objects of choice on a numerical scale,  $u(x)$  is a *number* (not a vector or anything even fancier) given any alternative  $x$  in  $X$ .

**Definition 7** *The utility function  $u(\cdot)$  is said to represent preferences  $\succeq$  if  $u(x') \geq u(x'')$  if and only if  $x' \succeq x''$ .*

So far we have only defined stuff. However, the sceptic may ask:

1. For a given preference relation, can we be sure that there is a numerical representation?
2. Conversely, for a given utility function, do we know that it represents a rational preference ordering?

To answer the first question in general one needs to use rather fancy mathematics. Basically, the answer is yes if one rules out certain “weird cases”. However, if  $X$  is a finite set one can avoid the hard math and while it is somewhat tedious to formally prove it in this case it should be quite apparent that the answer is indeed yes. To get the idea, let  $X = \{x^1, x^2, \dots, x^n\}$  be the list of alternatives and suppose for simplicity that all preferences

are strict. Then it is kind of clear that the utility function  $u$ , where

$$\begin{aligned} u(x^i) &= n \text{ for the best option} \\ u(x^j) &= n - 1 \text{ for the next best} \\ &\dots \\ u(x^k) &= 1 \text{ for the worst,} \end{aligned}$$

represents the preferences. If there are indifferences, the same number has to be assigned to options that are equally good and to actually describe the algorithm takes some tedious work, but it is sort of clear that it works.

For the second question, we can actually prove that the answer is yes. Here we start with a utility function  $u$  defined over a set of options  $X$  and construct a preference ordering  $\succsim$  by letting

$$x' \succsim x'' \text{ whenever } u(x') \geq u(x'')$$

**Question:** Starting with some given function  $u$ , will the preference ordering constructed as above be complete and transitive?

**Answer:** Yes!

**Argument:** verify completeness and transitivity. Pick any  $x'$  and  $x''$

1. Pick any  $x'$  and  $x''$  in  $X$  and note that:

$u(x')$  is some number

$u(x'')$  is some number

$\Rightarrow$  either  $u(x') \geq u(x'')$  or  $u(x'') \geq u(x')$  (both OK, in which case  $u(x') = u(x'')$ )

$\Rightarrow$  either  $x' \succsim x''$  or  $x'' \succsim x'$  ( $x' \succsim x''$  and  $x'' \succsim x'$  if  $u(x') = u(x'')$  in which case we write  $x' \sim x''$  "indifference")

Thus, preferences are complete

2.  $x' \succsim x'' \Leftrightarrow u(x') \geq u(x'')$

$x'' \succsim x''' \Leftrightarrow u(x'') \geq u(x''')$

But,  $u(x') \geq u(x'')$  and  $u(x'') \geq u(x''') \Rightarrow u(x') \geq u(x''')$   
 $\Rightarrow x' \succeq x'''$ .

Thus, preferences are transitive.

Thus, any utility function which assigns a number (a happiness index) to every conceivable choice represents a rational preference ordering. Hence, when we assume that a consumer has a utility function we simply say that the consumer is rational. Don't worry about the proofs, but try to get an intuitive grasp of this! Asking a person what his utility function is and the person probably think you're an idiot. The arguments above (which can be generalized to choice sets that fits standard consumer theory) shows that this is just a convenient hypothetical construct.

## 1.5 Ordinal Preferences

Utility functions give "happiness index" for each choice. This number **has absolutely no meaning**. Only the ordering matters and you will see this more concretely when we start to actually solve problems of constrained optimization. Makes no sense to say "a Porsche is 15 utils better than a Mazda". Concisely:

1. Preferences is the fundamental object.
2. We use utility functions to describe preferences
3. The particular index has no meaning. Only "ordinal information" or "ranking" matters.

**Example 8** Consider a consumer who cares only for "Can't Believe It's Not Butter". The more the better. Let  $q$  be the quantity "Can't Believe It's Not Butter". Consider the following utility functions

- 1)  $u(q) = q$  for all  $q \geq 0$
- 2)  $\tilde{u}(q) = \sqrt{q}$  for all  $q \geq 0$

Since (we are only considering positive numbers)  $q > q' \Leftrightarrow \sqrt{q} > \sqrt{q'}$  we have that both utility functions represent the same preferences.

In fact, **any monotonically increasing function represents the “more is always better” preferences we assumed.** For the idea, just look at Figure ??, where the point is that for any fixed  $q'$  the set of options that is better than  $q'$  is the same for any two pair of monotonically increasing functions.

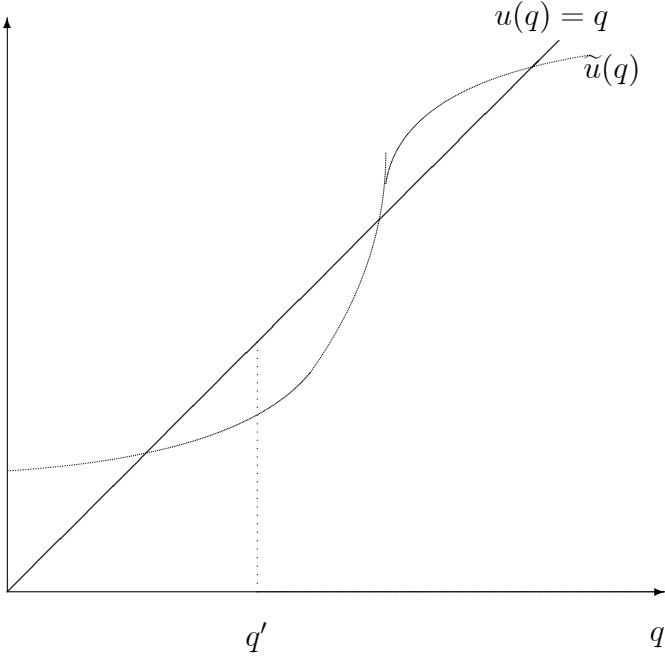


Figure 1: Two Functions Representing the Same Preferences