3 The Utility Maximization Problem

We have now discussed how to describe preferences in terms of utility functions and how to formulate simple budget sets. The rational choice assumption, that consumers pick the best affordable bundle, can then be described as an *optimization problem.* The problem is to find a bundle (x_1^*, x_2^*), which is in the budget set, meaning that

$$p_1 x_1^* + p_2 x_2^* \leq m \text{ and } x_1^* \geq 0, x_2^* \geq 0,$$

which is such that $u(x_1^*, x_2^*) \geq u(x_1, x_2)$ for all (x_1, x_2) in the budget set.

It is convenient to introduce some notation for this type of problems. Using rather standard conventions I will write this as

$$\begin{aligned} &\max_{x_1, x_2} u\left(x_1, x_2\right) \\ \text{subj. to } \quad & p_1 x_1 + p_2 x_2 \leq m \\ & x_1 \geq 0 \qquad , \\ & x_2 \geq 0 \end{aligned}$$

which is an example of a *problem of constrained optimization.*

A common tendency of students is to skip the step where the problem is written down. This is a bad idea. The reason is that we will often study variants of optimization problems that differ in what seems to be small "details". Indeed, often times the difficult step when thinking about a problem is to formulate the right optimization problem. For this reason I want you to:

1. Write out the "max" in front of the utility function (the *maximand*, or, *objective function*). This clarifies that the consumer is supposed to solve an optimization problem.

2. Below the max, it is a good idea to indicate what the *choice variables* are for the consumer (x_1 and x_2 in this example). This is to clarify the difference between the variables that are under control of the decision maker and variables that the decision maker has no control over, which are referred to as *parameters.* In the application above p_1, p_2 and m are parameters.

3. Finally, it is important that it is clear what the *constraints* to the problem are. I good habit is to write "subject to" or, more concisely, s.t. and then list whatever constraints there are, as in the problem above.

3.1 Solving the Utility Maximization Problem

Optional Reading: You may want to look at the appendix to chapter 5 in Varian (pages 90-94 in the most recent Edition).

We seek to solve the *consumer problem*

$$\begin{aligned} &\max_{x_1,x_2} u(x_1,x_2) \\ \text{subj. to} \quad &p_1x_1 + p_2x_2 \leq m \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned} \quad .$$

Given that we are willing to assume that preferences are *monotonic* (which we are) we first make the simple observation that we may replace the inequality

$$p_1x_1 + p_2x_2 \leq m$$

with

$$p_1x_1 + p_2x_2 = m.$$

To understand this one just needs to draw a graph. Suppose we would be able to find sine optimal solution $x^* = (x_1^*, x_2^*)$ to the consumer problem such that $p_1x_1^* + p_2x_2^* < m$ and that preferences are monotonic. Then we observe that we can increase good 1 (or good 2 or both) a little bit without violating the budget constraint. But, by the monotonicity, this increases the utility of the consumer, which means that x^* wasn't optimal. Hence we conclude that no optimal solution can be interior in the budget set.

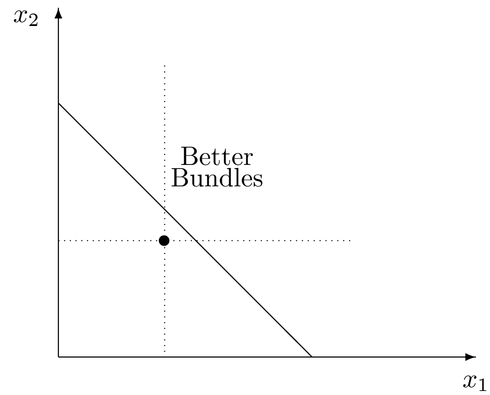

Figure 1: Interior Bundles Can't be Optimal with Monotonic Preferences

Thus, we can rule out all *interior* points in the budget set and solve

$$\begin{aligned} &\max_{x_1,x_2} u(x_1,x_2) \\ \text{subj. to} \quad & p_1x_1+p_2x_2=m \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned} \quad .$$

But since the constraint must hold with equality we know that in any optimal solution it must be the case that

$$x_2 = \frac{m-p_1x_1}{p_2}.$$

We can thus simply plug in the constraint into the objective function and solve the simpler problem

$$\max_{0\leq x_1\frac{m}{p_1}} u\left(x_1, \frac{m-p_1x_1}{p_2}\right),$$

WHICH IS A MAXIMIZATION PROBLEM WITH A SINGLE VARIABLE (exactly on the form $\max_x f(x)$ s.t. $a \leq x \leq b$ which we had in our discussion on maxima). Ignoring for now the possibility of corner solutions we can just differentiate this and set the derivative to zero to get the first order condition.

3.1.1 Optimality Conditions for Consumer Choice Problem

Using the chain rule, differentiate the utility function with respect to the choice variable x_1 to get

$$\frac{\partial u\left(x_1^*, \frac{m-p_1x_1^*}{p_2}\right)}{\partial x_1} + \frac{\partial u\left(x_1^*, \frac{m-p_1x_1^*}{p_2}\right)}{\partial x_2}\left(-\frac{p_1}{p_2}\right) = 0$$

Since the budget constraint is satisfied with equality we can now substitute back $x_2^* = \frac{m-p_1x_1^*}{p_2}$, which leads to the condition

$$\frac{\frac{\partial u\left(x_1^*,x_2^*\right)}{\partial x_1}}{\frac{\partial u\left(x_1^*,x_2^*\right)}{\partial x_2}} = \frac{p_1}{p_2}.$$

This condition, which we interpret as a tangency condition below, is a necessary condition for an interior optimum.

3.2 Interpretation of The Optimality Condition

The optimality condition has a useful geometric interpretation as a tangency condition between the indifference curve and the budget line and if often referred to as saying that

slope of indifference curve=MRS=slope of budget line

This geometric interpretation is useful since it will allow us to go back and forth between pictures and math.

The first thing to realize is that

$$-\frac{\frac{\partial u\left(x_1',x_2'\right)}{\partial x_1}}{\frac{\partial u\left(x_1',x_2'\right)}{\partial x_2}}$$

is the slope of the indifference curve for any point (x_1', x_2')(Remark about arguments & notational sloppiness). To see this, pick a point (x_1', x_2') and look for all (x_1, x_2) such that the consumer is indifferent between these points and (x_1', x_2') . That is

$$u\left(x_1, x_2\right) = \underbrace{u\left(x_1', x_2'\right) \equiv k}_{\text{just a number- } k \text{ for short}}$$

In principle, this can be solved for x_2 as a function of x_1 (as when we solved examples in class & homework). This solution is some relation $x_2 = y(x_1)$ satisfying

$$u(x_1, y(x_1)) = k$$

[I'm cheating here in that I'm ignoring some deep math....there is a question of whether $u(x_1, x_2) = k$ *can* be solved for x_2 as a function of x_1]

Once you've taken on faith that we can solve for a function $y(x_1)$ we have that the condition above is an *identity (*holds for all values of x_1). Hence, we can differentiate the identity with respect to x_1 to get

$$\begin{aligned} \frac{\partial u(x_1, y(x_1))}{\partial x_1} + \frac{\partial u(x_1, y(x_1))}{\partial x_2}\frac{dy(x_1)}{dx_1} &= \frac{d}{dx_1}k = 0 \\ &\Updownarrow \\ \underbrace{\frac{dy(x_1)}{dx_1}}_{\text{Slope}} &= -\frac{\frac{\partial u(x_1,y(x_1))}{\partial x_1}}{\frac{\partial u(x_1,y(x_1))}{\partial x_2}} = \text{MRS}\left(x_1, \underbrace{y(x_1)}_{=x_2}\right) \end{aligned}$$

Now $y(x_1)$ is *constructed* so that for each x_1 you plug in you get the same utility level (it is an indifference curve). So the interpretation of the slope of $y(x_1)$ is that it tells you *how much of good 2 you need to take away if you increase the consumption of good 1 slightly in order to keep the consumer indifferent.*

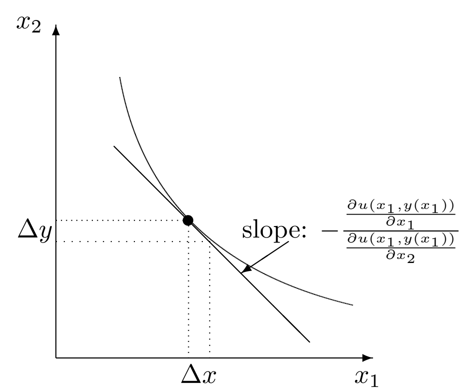

Figure 2: The Marginal Rate of Substitution

For discrete changes, the marginal rate of substitution will give you a slightly wrong answer to the question, but for sufficiently small changes the error will be negligible. So the optimality condition can be interpreted as (after multiplying the optimality condition by -1)

$$\mathrm{MRS}\left(x_1^*, x_2^*\right) = \underbrace{-\frac{\frac{\partial u\left(x_1^*,x_2^*\right)}{\partial x_1}}{\frac{\partial u\left(x_1^*,x_2^*\right)}{\partial x_2}}}_{\substack{\text{rate at which the consumer} \\ \text{is willing to exchange goods}}} = \underbrace{-\frac{p_1}{p_2}}_{\substack{\text{rate at which the market} \\ \text{is willing to exchange goods}}}$$

3.2.1 Example: Cobb-Douglas Utility

$u\left(x_1, x_2\right) = x_1^a x_2^b$. Utility maximization problem:

$$\begin{aligned} & \max_{x_1,x_2} x_1^a x_2^b \\ \text{subj to } p_1 x_1 + p_2 x_2 & \leq m \end{aligned}$$

Budget constraint must bind$\Rightarrow$solve out x_2 from constraint to get problem.

$$\max_{0 \leq x_1 \leq \frac{m}{p_1}} x_1^a \left(\frac{m - p_1 x_1}{p_2}\right)^b$$

Assuming that the solution is not at the boundary points, the first order condition needs to be satisfied. That is

$$a x_1^{a-1} \left(\frac{m - p_1 x_1}{p_2}\right)^b + x_1^a b \left(\frac{m - p_1 x_1}{p_2}\right)^{b-1} \left(-\frac{p_1}{p_2}\right) = 0.$$

At this point "only" some algebra remains. This can be done in a number of different ways, but here is one calculation

$$\begin{aligned} a x_1^{a-1} \left(\frac{m - p_1 x_1}{p_2}\right)^b + x_1^a b \left(\frac{m - p_1 x_1}{p_2}\right)^{b-1} \left(-\frac{p_1}{p_2}\right) &= 0 \\ \Big/ \text{multiply with } \frac{1}{x_1^\alpha} \left(\frac{p_2}{m - p_1 x_1}\right)^b > 0 \Big/ & \Updownarrow \\ \frac{a}{x_1} - \left(\frac{b p_2}{m - p_1 x_1}\right) \frac{p_1}{p_2} &= 0 \\ & \Updownarrow \\ a\left(m - p_1 x_1\right) - b p_1 x_1 &= 0 \\ & \Leftrightarrow x_1 p_1 \left(a + b\right) = am \\ & \Leftrightarrow x_1 = \frac{a}{a+b} \frac{m}{p_1} \end{aligned}$$

Now we have the candidate solution for good one. Plugging back into budget constraint gives

$$\begin{aligned} m &= p_1x_1 + p_2x_2 = p_1 \frac{a}{a+b}\frac{m}{p_1} + p_2x_2 \\ &\Updownarrow \\ m - \frac{a}{a+b}m &= m\left(\frac{a+b-a}{a+b}\right) = \frac{b}{a+b}m = p_2x_2 \\ &\Leftrightarrow x_2 = \frac{b}{a+b}\frac{m}{p_2} \end{aligned}$$

Hence, the candidate solution is

$$\begin{aligned} x_1^* &= \frac{a}{a+b}\frac{m}{p_1} \\ x_2^* &= \frac{b}{a+b}\frac{m}{p_2} \end{aligned}$$

Now,

- We know that a solution must either satisfy the first order condition (the point (x_1^*, x_2^*) is the only point on budget line which does) or be at a boundary point.
- In this example, any bundle with either $x_1 = 0$ or $x_2 = 0$ gives utility $u(x_1, x_2) = 0$, whereas $u(x_1^*, x_2^*) = (x_1^*)^a (x_2^*)^b > 0$ since $x_1^* > 0$ and $x_2^* > 0$.
- Combining these two facts we know that the bundle $(x_1^*, x_2^*) = \left(\frac{a}{a+b}\frac{m}{p_1}, \frac{b}{a+b}\frac{m}{p_2}\right)$ is indeed the solution to the consumer choice problem.

3.2.2 Final Remark About Ordinality and Monotone Transformations

Let

$$c = \frac{a}{a+b} \Rightarrow 1 - c = 1 - \frac{a}{a+b} = \frac{a+b-a}{a+b} = \frac{b}{a+b}$$

Plug in c instead of a and $1-c$ instead of b above and you immediately get that

$$\begin{aligned} x_1^* &= \frac{cm}{p_1} = \frac{a}{a+b}\frac{m}{p_1} \\ x_2^* &= \frac{(1-c)\,m}{p_2} = \frac{b}{a+b}\frac{m}{p_2}, \end{aligned}$$

so the solution does not change. This is because

$$f(u) = u^{\frac{1}{a+b}}$$

is an increasing function of u and

$$\left(x_1^a x_2^b\right)^{\frac{1}{a+b}} = x_1^{\frac{a}{a+b}} x_2^{\frac{b}{a+b}} = x_1^c x_2^{1-c}$$

therefore is a monotone transformation. Hence there is no added flexibility in preferences by allowing $a + b \neq 1$ (and this simplifies calculations). Thus, I will typically use utility function

$$u(x_1, x_2) = x_1^a x_2^{1-a}$$

Moreover, those of you who are comfortable with logaritms may note that

$$\ln\left(x_1^a x_2^{1-a}\right) = a \ln x_1 + (1-a) \ln x_2$$

and you are free to use that transformations whenever you like. The FOC then immediately becomes

$$\frac{a}{x_1} + \frac{1-a}{\left(\frac{m-p_1x_1}{p_2}\right)}\left(-\frac{p_1}{p_2}\right) = 0,$$

so this saves some work.

3.2.3 The Possibility of Corner Solutions

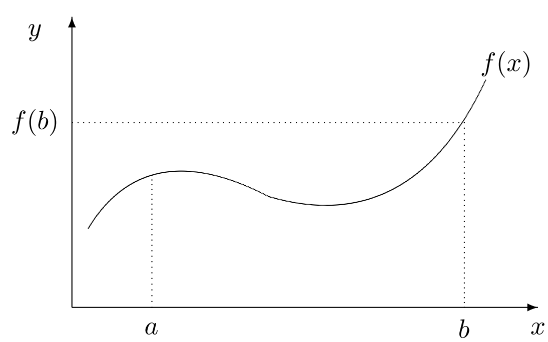

Figure 3: Derivative May be Non-Zero if Solution at the Boundary

Recall that if the highest value of the function is achieved at a boundary point, then the first order condition need not necessarily hold at the maximum. I'll skip the details, but it is straightforward to work out (and intuitive) that for our utility maximization problem:

1. If solution is at the lower end of the boundary (with x_1 being the choice variable in the reduced problem you get after plugging in the budget constraint), then that means that the slope of the indifference curve at point $(x_1, x_2) = \left(0, \frac{m}{p_2}\right)$ is flatter than the budget line.

2. If solution is at the upper end of the boundary, then that means that the slope of the indifference curve at point $(x_1, x_2) = \left(\frac{m}{p_1}, 0\right)$ is steeper than the budget line.

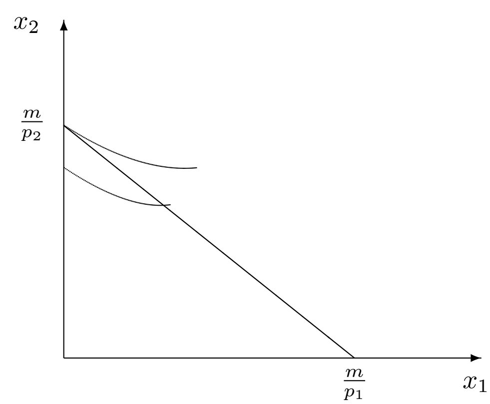

Figure 4: A Corner Solution

To see that this is a real possibility in a simple example, let $u(x_1, x_2) = \sqrt{x_1} + x_2$. Following the same steps as in the previous example this results in a utility maximization problem that may be written as

$$\max_{0 \le x_1 \le \frac{m}{p_1}} \sqrt{x_1} + \frac{m - p_1 x_1}{p_2}.$$

If there is an interior solution, we thus have that the first order condition (if $f(x) = \sqrt{x}$ then $\frac{df(x)}{dx} = \frac{1}{2\sqrt{x}}$)

$$\frac{1}{2\sqrt{x_1}} - \frac{p_1}{p_2} = 0$$

must hold for some $0 \leq x_1 \leq \frac{m}{p_1}$. Solving this condition for x_1 we get

$$x_1^* = \left(\frac{p_2}{2p_1}\right)^2.$$

For simplicity, set $p_2 = 2$ and $p_1 = 1$, in which case $x_1^* = 1$. Then, we can recover the consumption of x_2 through the budget constraint as

$$\begin{aligned} p_1 x_1^* + p_2^* x_2^* &= 1 + 2x_2^* = m \Rightarrow \\ x_2^* &= \frac{m-1}{2}. \end{aligned}$$

The point with this is that we notice that if $m < 1$, then our candidate solution for x_2^* is negative, which doesn't make any sense at all. We thus conclude that the solution must be at a corner and (it may be useful for you to plot the function $\sqrt{x_1} + \frac{1-x_1}{2}$ to see this) the solution is instead to use all income on x_1, i.e.

$$(x_1^*, x_2^*) = (m, 0).$$

solves the problem. If you have a hard time to see this you should 1) plot $\sqrt{x_1} + \frac{1-x_1}{2}$, and/or 2) plug in $(x_1, x_2) = (m, 0)$ and note that $u(m, 0) = \sqrt{m}$, 3) plug in $(x_1, x_2) = \left(0, \frac{m}{2}\right)$ and note that $u\left(0, \frac{m}{2}\right) = \frac{m}{2} < \sqrt{m}$.