

4 Borrowing and Saving

Instead of rushing immediately into traditional demand theory we now move to an application. The point of this is to show that the model of consumer behavior just set up is a very rich model in the sense that it can help us understand lots of interesting real world phenomena, such as intertemporal choice, choice under uncertainty (insurance) etc.

Imagine a consumer who lives for two periods (college, and then the boring life after college for example). Let

c_1 be the consumption in period 1 (today)

c_2 be the consumption in period 2 (tomorrow)

Also, let

m_1 be the income in period 1

m_2 be the income in period 2

Furthermore, let's make the assumption that the consumer can borrow and lend at interest rate r (this seems unrealistic from a "real world" perspective, but it is i) easy to relax this assumption, ii) one actually has to think about rather sophisticated informational models in order for a difference in borrowing and lending rates not to be a puzzle).

Now, exactly as in our basic "apple & bananas-model" we want to formulate the appropriate consumer choice problem, so we need to describe:

1. The set of feasible consumption plans (budget set)
2. How consumer evaluates different consumption plans.

4.1 The Budget Set

Actually, I've given you all the information you need to describe the budget set, although that is not obvious. First of all, just define an accounting variable

$$s = m_1 - c_1.$$

We allow s to be either positive or negative:

- If $s > 0$ it is the amount the consumer saves
- If $s < 0$ it is the amount the consumer borrows (negative savings).

Now

$$m_2 + (1 + r) s$$

is what is available for consumption in the second period. Now, we are looking at a simple world where the consumer cares only about herself. Hence, since the consumer knows she'll die after the period *any reasonable preferences will have the implication that the consumer consumes everything she has in the second period*. Thus,

$$\begin{aligned} c_2 &= m_2 + (1 + r) s \\ \Downarrow \text{ /def: } s &= m_1 - c_1 / \\ c_2 &= m_2 + (1 + r) (m_1 - c_1) \\ \Updownarrow \\ c_1 + \frac{1}{1 + r} c_2 &= m_1 + \frac{1}{1 + r} m_2 \end{aligned}$$

Important about this is:

1. From the point of view of the consumer, $m_1 + \frac{1}{1+r}m_2$ is just a number. Choice variables are c_1, c_2 and this means that the budget constraint has *exactly the same form* as the budget constraint in our standard model:

$$\underbrace{1}_{\text{"}p_1\text{"}} c_1 + \underbrace{\frac{1}{1+r}}_{\text{"}p_2\text{"}} c_2 = \underbrace{m_1 + \frac{1}{1+r} m_2}_{\text{"}m\text{"}}$$

2. $c_1 + \frac{1}{1+r}c_2$ is called the *present value* of the consumption plan (c_1, c_2) and $m_1 + \frac{1}{1+r}m_2$ the *present value* of the income of the consumer given earnings (m_1, m_2) .
3. Also note that

$$1 + r = \frac{1}{\frac{1}{1+r}} = \frac{\text{price of consumption today}}{\text{price of consumption tomorrow}}$$

4. Finally, if somebody worries about the omission of “ordinary” prices of the consumption goods-don’t worry. You may think of all these things as taken care of by the interest rate r . It is clear that p_1 can always be normalized to 1. If $p_2 \neq 1$, then second period consumption would be

$$c_2 = \frac{m_2 + (1+r)(m_1 - c_1)}{p_2}$$

$$\Downarrow$$

$$c_1 + \frac{p_2}{1+r}c_2 = m_1 + \frac{1}{1+r}m_2.$$

The apparent asymmetry between present value of consumption and income simply reflects that second period income (m_2) now is measured in dollars rather than in units of the consumption good. Letting r^* be the “real interest rate” and measuring second period income in units of the consumption good we get exactly the form we had initially.

4.2 The Choice Problem

We assume that the consumer has preferences represented by

$$U(c_1, c_2) = u(c_1) + \delta u(c_2),$$

where δ is called the *discount factor* and $u(c)$ is some “instantaneous” or “flow” utility function. We make the usual assumption that $\delta < 1$, which means that the consumer is impatient (i.e., if $r = 0$, then the consumer would want to consume more today than tomorrow. In a little while I will make a specific functional form assumption on u , but for now we just assume that it is some *concave function*, which simply means that the slope of $u(c)$ is decreasing in c as in Figure 1. The choice problem is then

$$\begin{aligned} & \max_{c_1, c_2} u(c_1) + \delta u(c_2) \\ \text{subj to. } & c_1 + \frac{1}{1+r}c_2 = m_1 + \frac{1}{1+r}m_2 \equiv M \end{aligned}$$

NOTE: Before even doing anything it is evident that it doesn’t matter WHEN the consumer gets her income as long as present value is constant. All today or all tomorrow is the same.

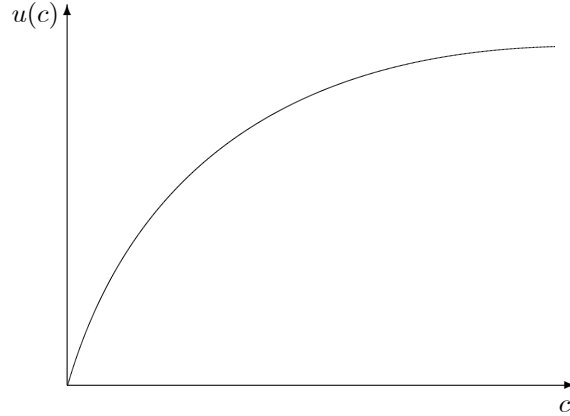


Figure 1: A Concave "Instantaneous Utility Function"

Substituting away c_1 we get

$$\max_{c_2} u\left(M - \frac{1}{1+r}c_2\right) + \delta u(c_2),$$

where I will simply ignore the boundary constraints ($c_1 \geq 0$ and $c_1 \leq m_1 + \frac{1}{1+r}m_2 = M$).

The FOC is

$$u'\left(M - \frac{1}{1+r}\right) \left(-\frac{1}{1+r}\right) + \delta u'(c_2) = 0$$

or

$$\frac{u'(c_1)}{\delta u'(c_2)} = 1 + r$$

The condition has the usual interpretation: it says that the marginal rate of substitution between consumption now and later must equal the slope of the budget line (–the relative price).

For concreteness, let $u(c) = \ln c \Rightarrow u'(c) = \frac{1}{c}$. The FOC then simplifies to

$$\frac{c_2}{\delta c_1} = 1 + r \Leftrightarrow \frac{c_2}{c_1} = \delta(1 + r),$$

so

$$c_1 > c_2 \text{ if } \delta(1 + r) < 1 \Leftrightarrow \delta < \frac{1}{1 + r}$$

$$c_1 = c_2 \text{ if } \delta = \frac{1}{1 + r}$$

$$c_1 < c_2 \text{ if } \delta > \frac{1}{1 + r}$$

You may think of δ , the subjective discount factor, as measuring how the consumer values consumption today relative consumption tomorrow. If δ is high, the consumer is patient and if δ is low the consumer is impatient. The term $\frac{1}{1+r}$ is the market price for consumption tomorrow expressed in units of consumption today, so it makes sense that if the consumer is less patient than the market, then the consumer will tend to consume more today and less tomorrow.

Actually, these comparisons depending on the relation between r and δ hold *for any concave function* u (reason $u'(c_1) > u'(c_2) \Leftrightarrow c_1 < c_2$ when the slope is decreasing). The slope of an indifference curve is

$$-\frac{u'(c_1)}{\delta u'(c_2)}$$

and since higher c_1 means lower c_2 to keep indifference $u'(c_1) \searrow$ and $u'(c_2) \nearrow$ along any

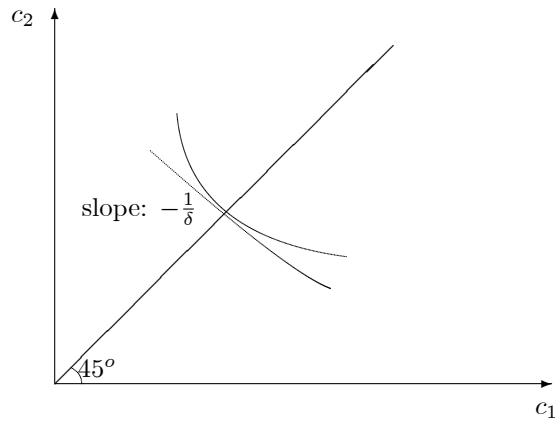


Figure 2: An Indifference Curve

particular indifference curve when increasing c_1 . Hence, the indifference curves have the nice convex shape of Figure 2.

We can then see that the claims about the relations between r and δ hold in terms of a simple graph: The point with the lower level indifference curve is that it is supposed to be a tangency with the line with slope $-\frac{1}{\delta}$, which we know is the slope of the indifference curves at the 45° line. Now, since this line is steeper than the budget line when $\delta < (1+r)$ we see that the maximum must be at a point below the diagonal.

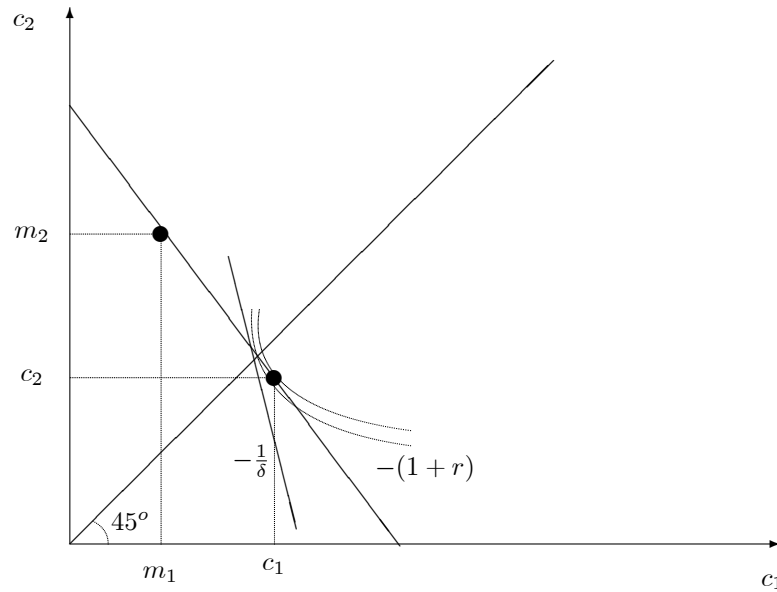


Figure 3: The Case with a Consumer Less Patient than Market

NOTE: How borrowing vs savings can be read out in the graph.

4.3 Remarks About Intertemporal choice

- In this class, mainly as an example/application.
- In Macroeconomics. This model & extensions used to analyze savings behavior, consumption smoothing etc.
- Also (with some uncertainty added on) used to understand asset pricing, relative price between bonds & stocks and lots of other things.

5 Uncertainty and Insurance

Optional Reading: Chapter 12 in Varian.

Study of choice under uncertainty can be motivated in many ways, for example:

- Understand insurance markets

- Make sense of differences in average returns between stocks (with uncertain returns) and bonds (with more or less certain return).

In fact, in modern economic research it is hard to find examples of new developments that don't have uncertainty as an important building block (you will see a little of this towards the end of course-“signaling”, “lemons problem”, “asymmetric information”...). However, the first order of business is to understand rational choice under uncertainty.

5.1 Consumer Preferences Under Uncertainty

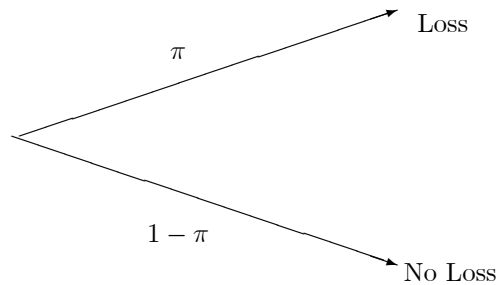


Figure 4: Possible Outcomes

For simplicity we will think of a situation with two possible *outcomes*.

- With probability π , the consumer suffers a “loss”
- With probability $1 - \pi$, the consumer suffers “no loss”.

Observe that π is a number between 0 and 1 and has nothing to do with areas of circles (I'm just conforming with Varians choice of notation). Thus, one example is $\pi = 0.5$ (fair coinflip). Also, the language “loss” versus “no loss” I use only because of the particular application in mind. It could be any situation where there is uncertainty about something the consumer cares about. Examples:

- House burns down-House does not burn down

- Loose job-keep job
- Win state lottery-don't win state lottery (or any other gamble)

However which example you prefer we will think of the “state” occurring with probability π as the “bad state” and the “state” occurring with probability $1 - \pi$ as the “good state” and we assume that for some reason the disposable income is lower in the bad than in the good state. Write

- m_b for the income in the bad state (probability π)
- m_g for the income in the good state (probability $1 - \pi$)
- \Rightarrow “Loss” = $m_g - m_b$

Now, we will assume that the consumer can purchase insurance, so the consumption *need not* be m_b or m_g . We denote actual consumptions by

- c_b for the consumption in the bad state
- c_g for the consumption in the good state

In terms of how the consumer evaluates different “bundles” of (c_b, c_g) the *interesting question is how these are evaluated before the consumer knows whether she suffered a loss or not*. Just think about it, the decision on how much insurance you buy or how to invest in a risky portfolio *must be done before the uncertainty is resolved*. We will assume that the consumer evaluates different “bundles” (ex ante) according to

$$U(c_b, c_g) = \pi u(c_b) + (1 - \pi) u(c_g),$$

which is called an EXPECTED UTILITY function. What is particular about this is that the utility function is:

1. Linear in probabilities

2. The same *function* $u(c)$ is used for the “ex post evaluation” of consumption in both “states”.

There are VERY GOOD REASONS to think that the expected utility representation is a good way to model choice under uncertainty. However, this is rather deep and advanced material—you’ll have to take on faith that this is reasonable.

Note that given these preferences we can draw indifference curves in the usual way. The marginal rate of substitution will as usual be the slope of the indifference curves and this is

$$MRS(c_g, c_b) = -\frac{\frac{\partial U(c_b, c_g)}{\partial c_g}}{\frac{\partial U(c_b, c_g)}{\partial c_b}} = -\frac{(1 - \pi) u'(c_g)}{\pi u'(c_b)}$$

Now, how the indifference curves will look will clearly depend on the choice of $u(c)$ and the shape of this function turns out to have a natural interpretation.

5.1.1 Example: $u(c) = ac + b$

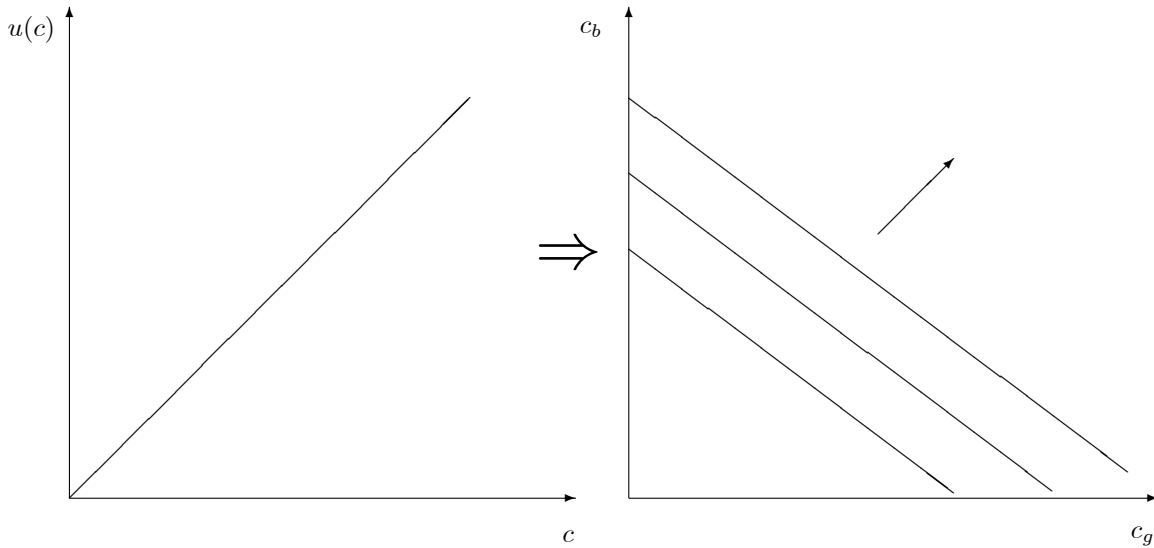


Figure 5: Indifference Curves for $u(c) = ac + b$

In this case, the slope of the indifference curve is $-\frac{1-\pi}{\pi}$, so we have the indifference curves as in Figure 5. The slope is constant and equal to the negative of the ratio of the respective

probabilities. Preferences of this sort are called *risk neutral preferences*. To see why note that

$$\pi c_b + (1 - \pi) c_b$$

is the *expected value of consumption* and that

$$\begin{aligned} & \pi (ac_b + b) + (1 - \pi) (ac_g + b) \\ &= a (\pi c_b + (1 - \pi) c_b) + b, \end{aligned}$$

so that the consumer is indifferent between all bundles (or contingent plans) that has the same expected consumption.

5.1.2 Example: $u(c) = \sqrt{c}$

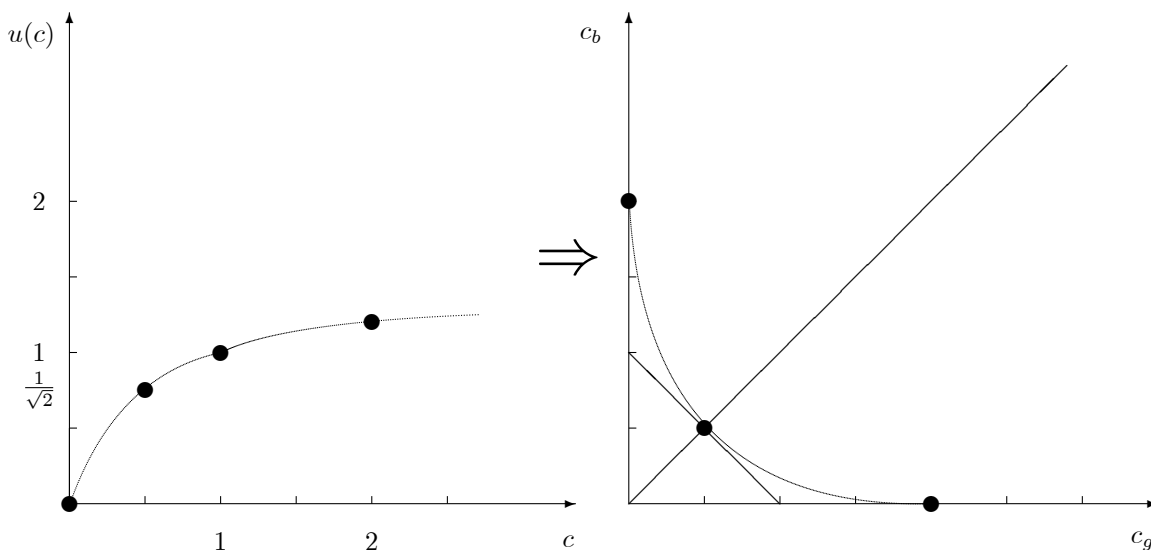


Figure 6: Indifference Curves for $u(c) = \sqrt{c}$

In this case, the slope of the indifference curve is

$$-\frac{(1 - \pi)\sqrt{c_g}}{\pi\sqrt{c_b}}$$

To actually depict them, let $\pi = \frac{1}{2}$ and consider the curve through $(1, 1)$.

- $\frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{1} = 1$, gives $k = 1$ for the curve going through $(1, 1)$

- $\frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{0} = 1 \Rightarrow (4, 0)$ and $(0, 4)$ on curve

Thus, the indifference curves look like in Figure 6.

Hence, the example generates nice convex preferences, which is true for all functions $u(c)$ that have a slope which is decreasing in c (that is concave functions u)

5.2 Interpretation of Convex Preferences in Applications with Uncertainty

In the canonical model when we think of convex preferences we motivate it by saying that it may be natural for agents to “prefer a little bit of each” rather than going for extremes. In this particular application we can put a bit more flesh and bones on the story.

We say that consumers with the kind of preferences graphed in the last example (or any other $u(c)$ with decreasing slope) are *risk averse*—they rather “mix their portfolio” than put all the eggs in the same basket. To see this in a little bit of a different way inspect Figure 7

THOUGHT EXPERIMENT:

Take all your savings, would you

- rather flip a coin and double savings if heads and lose everything if tails?
- keep your savings for sure?

If answer is b) \Rightarrow “risk averse”. (a. would be “risk loving” and indifference “risk neutral”).

Economists usually assume agents are risk averse (otherwise there wouldn’t be any rationale for a market for insurance) or risk neutral. In particular large firms are often modelled as risk neutral (idea—lot’s of independent risks \Rightarrow firm can pool risks and thereby eliminate them).

5.3 Affordable Consumption Plans in the Presence of Insurance

Now suppose that consumer can buy insurance:

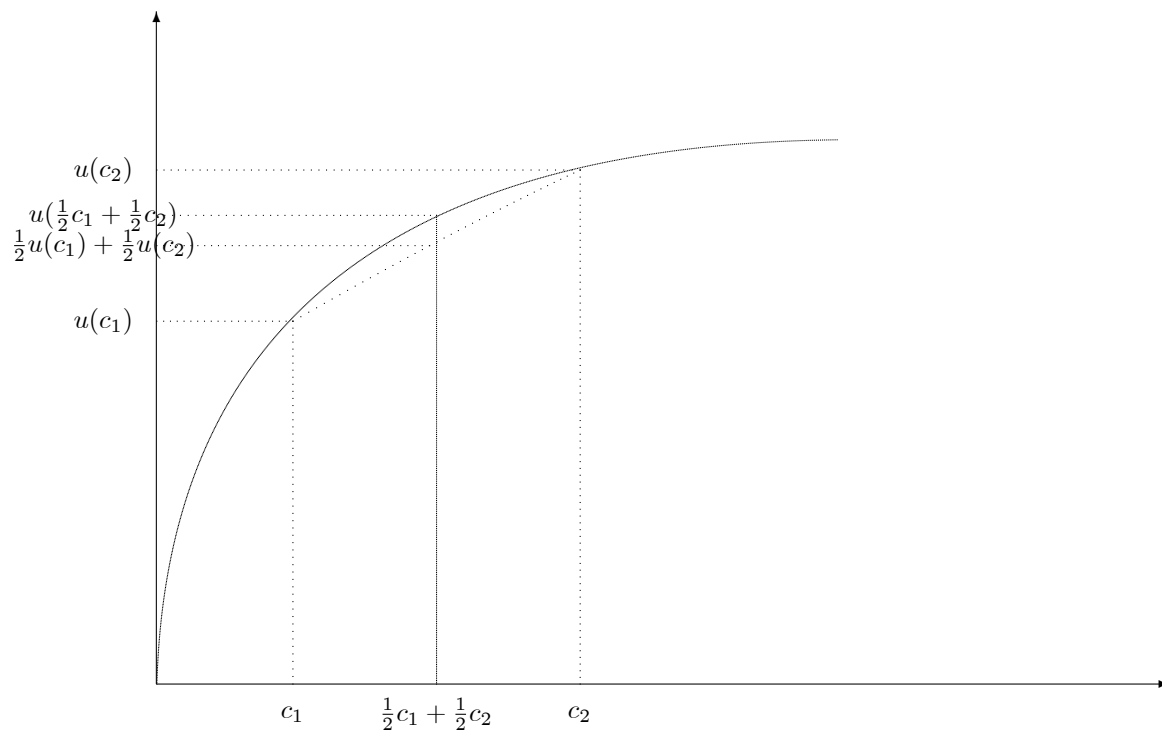


Figure 7: Comparing Utility of Expected Value with Expected Utility of Gamble

- Let z be quantity insurance (units paid to consumer in the event of a loss)
- Let p be the price per unit of insurance

\Rightarrow

- $c_b = m_b + z - pz$ is the consumption in bad state
- $c_g = m_g - pz$ is the consumption in good state

Now, we can eliminate z from this to get

$$\begin{aligned}
 c_b &= m_b + z(1 - p) = \\
 &= m_b + \frac{(m_g - c_g)}{p}(1 - p)
 \end{aligned}$$

or, equivalently

$$c_g + \frac{p}{1-p}c_b = m_g + \frac{p}{1-p}m_b$$

or

$$(1-p)c_g + pc_b = (1-p)m_g + pm_b$$

AS IN THE INTERTEMPORAL PROBLEM-LIKE A STANDARD “APPLE & BANANA” SETUP WITH c_g instead of x_1 , c_b instead of x_2 , prices $\left(1, \frac{p}{1-p}\right)$ and income $m_g + \frac{p}{1-p}m_b$. The budget set is depicted in Figure 8

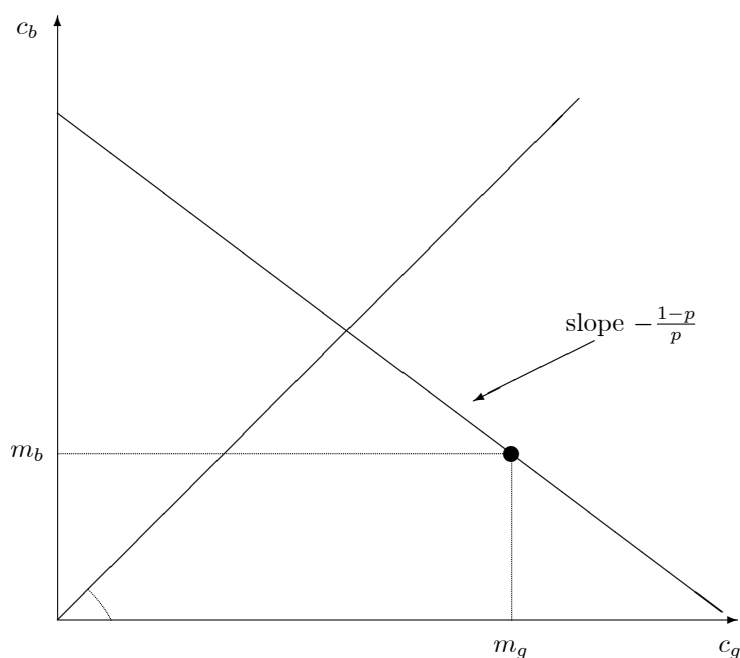


Figure 8: The Budget Set

5.4 The Choice Problem

Graphically, given convex preferences, the solution will be characterized in the usual way as a nice tangency between (the highest possible) indifference curve and the budget set. Note that, at the diagonal line (the “certainty line”) the slope of the indifference curve is

$$-\frac{1-\pi}{\pi}$$

to be compared with the slope of the budget line

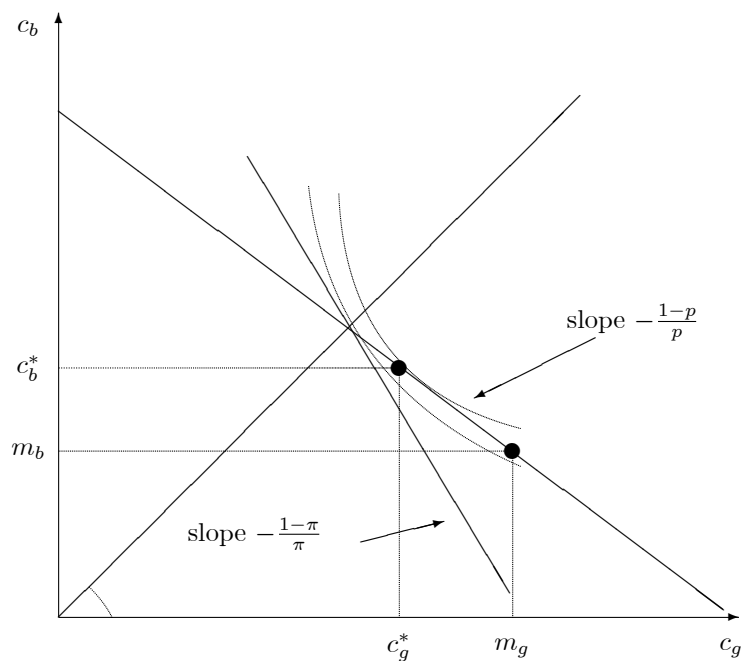


Figure 9: Graphical Solution with $p > \pi$

$$-\frac{1-p}{p},$$

so if $p > \pi$, then $\frac{1-p}{p} < \frac{1-\pi}{\pi}$, meaning that *at the diagonal* the indifference curves must be steeper than the budget line meaning that the tangency must occur somewhere below the diagonal as in Figure 9.

Characterizing the solution using calculus it is convenient to keep z as the choice variable (although you may use c_g or c_b if you want).

$$\max_z \pi u(m_b + z - pz) + (1 - \pi) u(m_g - pz)$$

FOC is

$$\pi u' \left(\underbrace{m_b + z - pz}_{c_b} \right) (1 - p) + (1 - \pi) u' \left(\underbrace{m_g - pz}_{c_g} \right) (-p) = 0$$

$$\Downarrow$$

$$\frac{1-p}{p} = \frac{(1-\pi)u'(c_g^*)}{\pi u'(c_b^*)},$$

which we sort of knew already from the picture since this just says that the slope of the indifference curve must equal the slope of the budget line. However note that:

1. If $p = \pi$, then

the consumer pays $pz = \pi z$

the insurance company gives the consumer z with probability π

the insurance company gives the consumer 0 with probability $1 - \pi$

\Rightarrow Expected profit for insurance company $pz - \pi z = 0$

This is called a “fair premium” since the insurance company breaks even on average. Note that the solution in this case is $c_g^* = c_b^*$, since $u'(c)$ is a decreasing function. The conclusion is clear: if a risk averse consumer can buy insurance at a fair price the consumer will fully insure.

2. $p > \pi \Rightarrow$ partial insurance (or no insurance).
3. $p < \pi \Rightarrow$ overinsurance.

5.5 A Simple Portfolio Problem

The same ideas as the ones for modelling an insurance problem can be applied to the choice between a risky asset and a safe bond:

1. Suppose that the safe bond earns interest r_s (there is no chance of the issuer defaulting)
2. Suppose that the risky asset earns a return r_g in the “good state” (probability $1 - \pi$) and r_b in the “bad state” (probability π), where $r_g > r_b$ to make sense of the labeling of the states.

We will abstract away from the issue of **how much will be invested in total** and simply assume that the investor has “wealth” w that he or she will invest.

If the agent can choose how much to eat in the current period and how much to invest the current analysis will still apply for whatever wealth w the agent decides not to eat. Hence it makes lots of sense to ignore the intertemporal issues.

The most straightforward way to set the problem up is to let $0 \leq x \leq w$ be the investment in the risky asset (so that $w - x$ is invested in the safe bond). Consumption is then

$$\begin{aligned} c_g &= (w - x)(1 + r_s) + x(1 + r_g) = w(1 + r_s) + x(r_g - r_s) \\ c_b &= (w - x)(1 + r_s) + x(1 + r_b) = w(1 + r_s) + x(r_b - r_s). \end{aligned}$$

For the purposes of drawing the budget set in $c_g c_b$ -space we may observe that the budget constraint reads

$$c_g + \frac{r_g - r_b}{r_s - r_b} c_b = w(1 + r_s) + w(1 + r_s) \frac{r_g - r_s}{r_s - r_b}.$$

You should try to plot this, but before doing that it is good to pass a while to think:

1. If nothing is invested in the risky asset, then $c_b = c_g = w(1 + r_s)$
2. If everything is invested in the risky asset, then $c_b = w(1 + r_b)$ and $c_g = w(1 + r_g)$.
3. $\Rightarrow r_s < r_g$ needed in order to not have a corner solution where everything is put in the safe bond.
4. $\Rightarrow r_b < r_s$ needed in order to not have a corner solution where everything is put in the risky asset.
5. That is the interesting investment problem is when $r_b < r_s < r_g$, which also implies that $\frac{r_g - r_b}{r_s - r_b}$ is a positive number.
6. It is an open question whether one should assume that it is possible for $c_g > w(1 + r_g)$ or $c_b < w(1 + r_b)$. If you allow this, you are actually assuming that the investor can *issue* a safe bond at rate r_s . It is a good exercise to draw the relevant budget set both under the assumption that borrowing is possible and when it is not.

Like in the insurance problem, it is more convenient to keep the variable x when setting up the optimization problem. Assuming that the investor can not issue a bond the problem is

$$\max_x \pi u(w(1+r_s) + x(r_b - r_s)) + (1-\pi)u(w(1+r_s) + x(r_g - r_s))$$

The first order condition is

$$\pi u'(w(1+r_s) + x(r_b - r_s))(r_b - r_s) + (1-\pi)u'(w(1+r_s) + x(r_g - r_s))(r_g - r_s) = 0$$

We will not really “solve” this problem although the solution will look like the usual tangency condition. Qualitatively, an interesting question is simply to ask “when will the investor invest at all in the risky asset. To answer this question we only need to check when to expected utility function is increasing at $x = 0$, that this when is

$$\begin{aligned} 0 &< \pi u'(w(1+r_s))(r_b - r_s) + (1-\pi)u'(w(1+r_s))(r_g - r_s) \\ &= u'(w(1+r_s))(\pi r_b + (1-\pi)r_g - r_s). \end{aligned}$$

We conclude that:

- If $\pi r_b + (1-\pi)r_g > r_s$ then some part of the wealth should be invested in the risky asset.
- In words, this means that if the *expected return* for the risky asset exceeds the expected return on the safe bond, then it cannot be optimal not to take any risk at all.
- The reasoning is that making the calculation for the first dollar it is almost as if the agent is risk neutral, even if she actually is risk averse. The idea for this is that the variability in income that one creates with the first dollar is so small that it can be ignored.