

8 Equilibrium Analysis in a Pure Exchange Model

So far we have only discussed decision theory. That is, we have looked at consumer choice problems on the form

$$\begin{aligned} & \max_{x_1, x_2} u(x_1, x_2) \\ \text{s.t. } & p_1 x_1 + p_2 x_2 \leq m. \end{aligned}$$

The consumer choice problem generates **demand functions** $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$, which are to be interpreted as giving the optimal consumption bundle for any configuration of prices and income.

However, the determination of prices (and arguably income as well) is central in economics and should be treated as *endogenous*. Here, the basic idea in neoclassical economics is that prices ought to be set so as to “clear the market”. This idea, which goes back to Adam Smith, is intuitive, and also surprisingly powerful. Roughly speaking, if given a price the demand for a good exceeds supply, then the price is “too low” and must be adjusted upwards. If instead supply exceeds demand the price is “too high” and ought to be adjusted downwards to restore equilibrium.

Hence, only prices such that supply=demand are “stable” and we will simply postulate that equilibrium prices is set this way. Thus, we will not explicitly model **how** prices are set or any dynamic adjustments (the “invisible hand” or “auctioneer”).

In economics we make a distinction between **partial equilibrium** and **general equilibrium** analysis. Partial equilibrium analysis is roughly about “studying a market in isolation” and while being very useful it has some drawbacks, in particular when studying markets that constitutes a large share of the economy. The supply/demand graphs from 101 are examples of partial equilibrium and I may do some partial equilibrium exercises later (need “firms” first).

General equilibrium differs from partial equilibrium analysis in that the economy is studied as a closed system, with all prices being determined simultaneously. If we would insist on “realism” the general equilibrium approach would lead to very complex models. However,

important insights about “competitive markets” can be gained in the simplest possible model with two goods and two agents acting competitively in an economy with no production.

8.1 The 2×2 Model of Pure Exchange

We will study a model where,

- 2 agents, A, B
- 2 goods x_1, x_2
- *endowments* $e^A = (e_1^A, e_2^A)$ and $e^B = (e_1^B, e_2^B)$ are the quantities of good one and two the agents have before any trade
- I will denote consumption bundles $x^A = (x_1^A, x_2^A)$ and $x^B = (x_1^B, x_2^B)$ for Mr. A and Mrs. B .

The first thing to note here is that I have **not** specified any particular “dollar incomes” m^A, m^B . The reason is that what we will do is to let agents trade (barter) with each other and if one of the agents would have some green pieces of paper with presidents on them, then the other agent wouldn’t want to give up anything he/she cares about for those green pieces of paper unless there was a **consumption value** of those green pieces of paper. It is hard to make the case that bills have a consumption value and what that means is that this simple (static) environment doesn’t allow money to be valued (indeed, economic theory that *explains* why people use and value money is a relatively recent phenomenon and the theory is still very crude and very far from being of much value for the Fed).

The bottom line of the discussion above is that the “income” of the consumer will be taken as the *value of the endowment*. Hence, the relevant maximization problem given prices (p_1, p_2) for consumer A is

$$\begin{aligned} & \max_{x_1, x_2} u^A(x_1, x_2) \\ \text{s.t } & p_1 x_1 + p_2 x_2 \leq p_1 e_1^A + p_2 e_2^A \end{aligned}$$

and similarly for B . Now, we've spent weeks on this and the only thing to watch out for is that *the "income" depends on prices*. This was actually true for the three applications we looked at as well (intertemporal choice, uncertainty and labor supply) but I did not stress it so much. However, the relevant demand given any price p_1, p_2 can still be derived from the demand functions we derived when we viewed m as independent from p_1 and p_2 . That is let $x_1^A(p_1, p_2, m)$ and $x_2^A(p_1, p_2, m)$ be the demand functions derived from the standard problem with utility function $u^A(x_1, x_2)$, then the demand for good 1 and 2 from the problem above will be given by

$$x_1^A(p_1, p_2, \underbrace{p_1 e_1^A + p_2 e_2^A}_{\substack{\text{value of } e \text{ given} \\ p_1 \text{ \& } p_2}}) \text{ and}$$

$$x_2^A(p_1, p_2, \underbrace{p_1 e_1^A + p_2 e_2^A}_{\substack{\text{value of } e \text{ given} \\ p_1 \text{ \& } p_2}})$$

Definition 1 A competitive (Walrasian) equilibrium in the pure exchange model is a price vector (p_1^*, p_2^*) and consumption bundles $x^{A*} = (x_1^{A*}, x_2^{A*})$, $x^{B*} = (x_1^{B*}, x_2^{B*})$ satisfying:

1. [optimality] x^{A*} solves the consumer choice problem for A given prices p_1^*, p_2^*

x^{B*} solves the consumer choice problem for B given prices p_1^*, p_2^*

2. [Market clearing].

$$x_1^{A*} + x_1^{B*} = e_1^A + e_1^B$$

$$x_2^{A*} + x_2^{B*} = e_2^A + e_2^B$$

I will start with a graphical treatment, but we may note that once we have demand functions $x_1^A(\cdot), x_2^A(\cdot), x_1^B(\cdot)$ and $x_2^B(\cdot)$ we can *solve* for an equilibrium by solving

$$\underbrace{x_1^A(p_1, p_2, p_1 e_1^A + p_2 e_2^A) + x_1^B(p_1, p_2, p_1 e_1^B + p_2 e_2^B)}_{\text{aggregate demand for } x_1 \text{ given prices } p_1, p_2} = \underbrace{e_1^A + e_1^B}_{\text{resources of } x_1}$$

There is a similar condition for good 2 (which is redundant due to Walras Law which I will discuss later). At this point, just recall all the graphs we did in standard demand theory.

In almost all cases the consumption *changed* when the price changed, so if you pick a price at random, there is no particular reason to believe that the equilibrium condition above will hold. That is, only very particular prices will be consistent with equilibrium.

8.2 Graphical Treatment

In later discussions it will be useful to distinguish between the parts in the definition of equilibrium that has to do with *feasibility* from the part that has to do with *optimizing behavior*.

Definition 2 *An allocation (a list of consumption bundles for each agent) is feasible if*

$$\begin{aligned}x_1^A + x_1^B &\leq e_1^A + e_1^B \\x_2^A + x_2^B &\leq e_2^A + e_2^B\end{aligned}$$

It is rather clear that *in equilibrium* (that is if we add optimal behavior as well) all resources must be used meaning that the more interesting feasible allocations are those where the resource constraints hold with equality.

Graphically any feasible allocation that uses all resources ($x_1^A + x_1^B = e_1^A + e_1^B$ and $x_2^A + x_2^B = e_2^A + e_2^B$) can be conveniently described as a point in a “box” as in figure 1. In the figure, the length of each side is the total resources of each good which immediately means that if we pick any point different from e in the box total consumption of each good will be equal to the total resources.

Now, optimal behavior is determined exactly as before. Given a price vector (p_1, p_2) we have that:

- The budget set for A consists of all (x_1, x_2) such that

$$p_1x_1 + p_2x_2 \leq p_1e_1^A + p_2e_2^A,$$

which are just all points *below* a line with slope $-\frac{p_1}{p_2}$ that goes through the endowment point e (note that when we look at it from the point of view of A the endowment e is located at (e_1^A, e_2^A) from the relevant origin in the southwest corner.

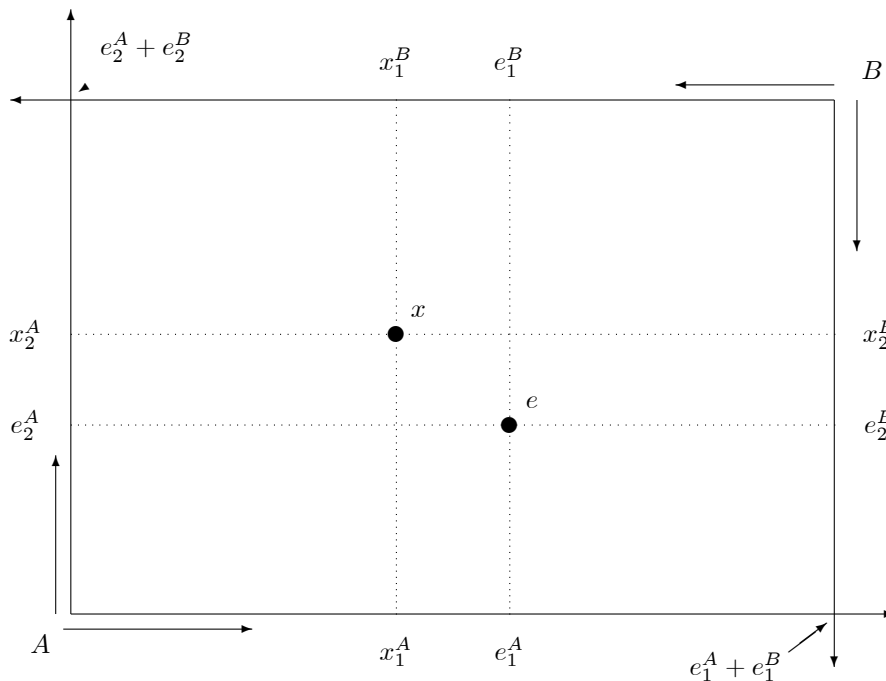


Figure 1: A Feasible Allocation in the Edgeworth box)

- The budget set for B consists of all (x_1, x_2) such that

$$p_1 x_1 + p_2 x_2 \leq p_1 e_1^B + p_2 e_2^B,$$

which are just all points *above* a line with slope $-\frac{p_1}{p_2}$ that goes through the endowment point e . That is, from the point of view of B the origin is in the northeast corner.

This is illustrated in figure 2. Observe that *there is absolutely no reason that the budget set must be in the set of feasible allocation*. In the picture this is indicated by the budget lines continuing across the edges in the box (but only for positive consumptions). The optimality requirement is then as usual graphically depicted as a tangency between the highest achievable indifference curve and the budget line.

Now, we can simply put the two pictures together in the box for some arbitrary prices (p_1, p_2) as in Figure 3. The way the picture is drawn we have that the net demand for good one of Mrs. B (i.e., what B wants to buy in addition to her endowment) exceeds the net supply of Mr. A for good 1. That is: B wants to buy more than A has to sell. Hence

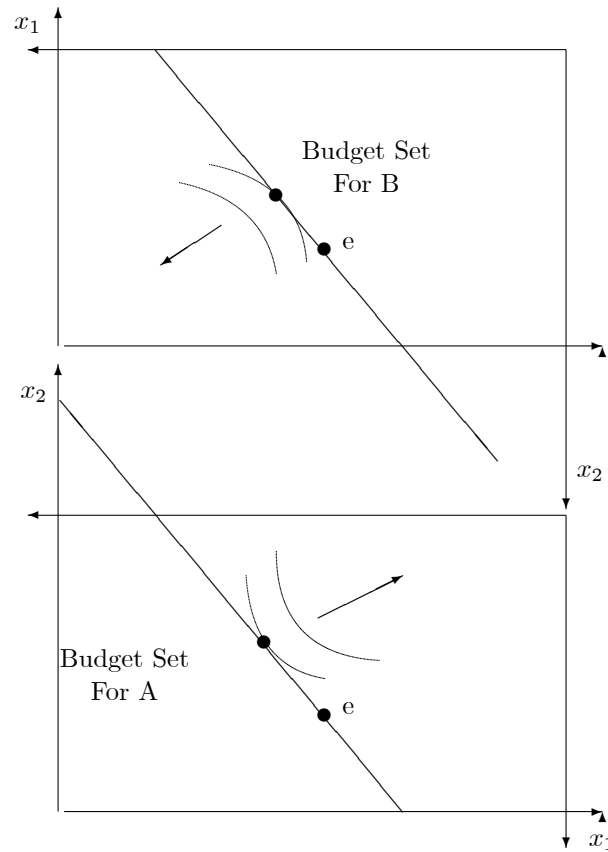


Figure 2: The Utility Maximization Conditions)

there is excess demand for good 1 : at the given prices the consumers want to consume more than is available in the market of good 1, so the market is **not** in equilibrium in Figure 3. The mirror image of this excess demand for good 1 is excess supply for good 2, but this is “automatic” given that we have excess demand for good 1 as will be discussed later.

So, how will an equilibrium look like in the box?

1. Allocation must be feasible \Rightarrow graphically this means that both agents choose “same point” in the Edgeworth box.
2. Both agents must choose the best bundle given the prices \Rightarrow the equilibrium must be such that both agents have a tangency between price line and indifference curve at equilibrium allocation.

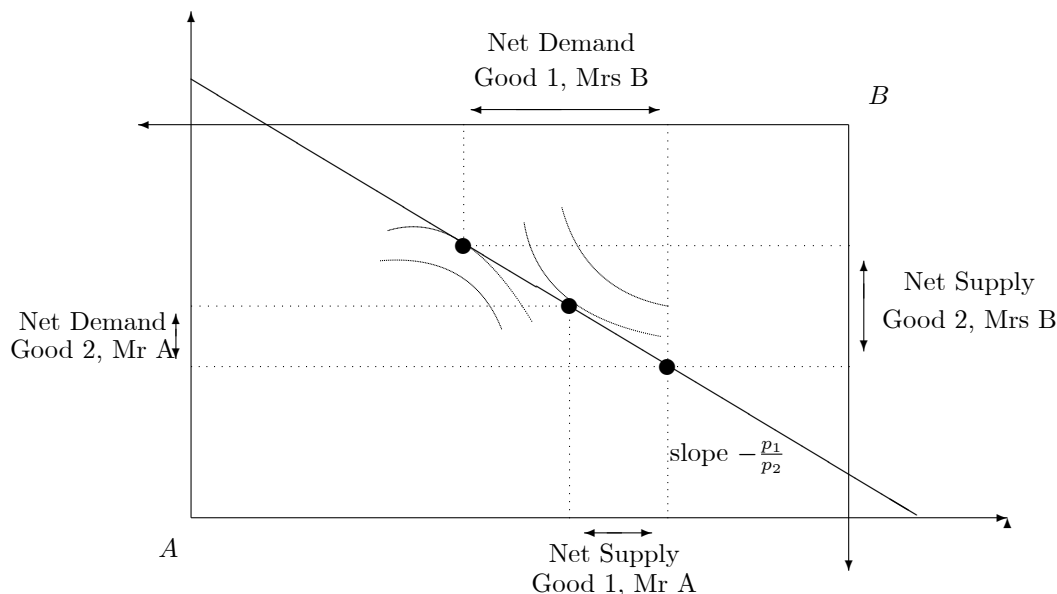


Figure 3: Example of Prices NOT Consistent with Equilibrium)

An equilibrium can thus be depicted as in Figure 4 as a budget line that goes through the endowment which is such that both agents have a tangency with the price line at the same point.

8.3 Self Interest Leads to Good Allocations in the Competitive Model

Some examination of this picture reveals a rather remarkable property of competitive (Walrasian) equilibria. Given the equilibrium allocation x^* all bundles that are better for A are those to the northeast of the indifference curve intersecting x^* . Similarly, the bundles that are better for B are those to the southwest of the indifference curve intersecting x^* . This means that it is impossible to make one person better off without making the other person worse off.

This important feature (which is true under much more general circumstances) is emphasized in Figure 5 where the only difference from Figure 4 is that I've taken away all indifference curves not going through x^* .

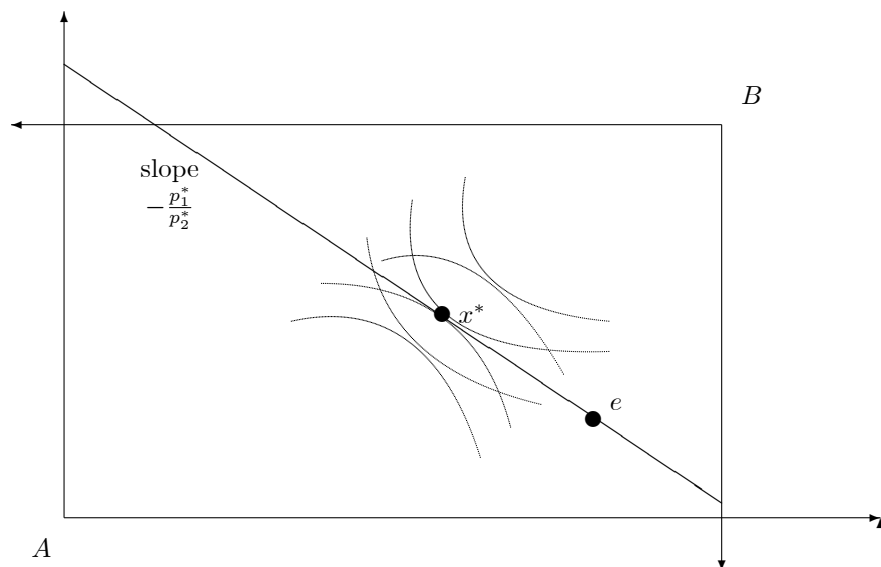


Figure 4: An Equilibrium in the Edgeworth Box)

In the language of an economist, we just argued that the equilibrium outcome is *Pareto efficient* or simply *efficient*:

Definition 3 *An allocation is Pareto efficient if it is feasible and if there is no other feasible allocation that makes both agents better off.*

Pareto efficiency is **the** concept of efficiency in economics. Clearly, allocations that are not Pareto efficient are undesirable. Then, there is a way to make all agents in the economy better off and if everyone is happier then that is clearly a better use of the resources.

Note that there is an infinite number of Pareto optimal allocations even in the simply 2×2 pure exchange model. To see this note that for **any** point such that there is a tangency between the indifference curves of the agents it is impossible to increase the happiness of one agent without making the other less happy. One can thus trace out the set of Pareto optimal allocations in the Edgeworth box as the set of tangencies as in Figure 6. The curve that connects all the Pareto optima is sometimes called the *contract curve*.

Important to note is:

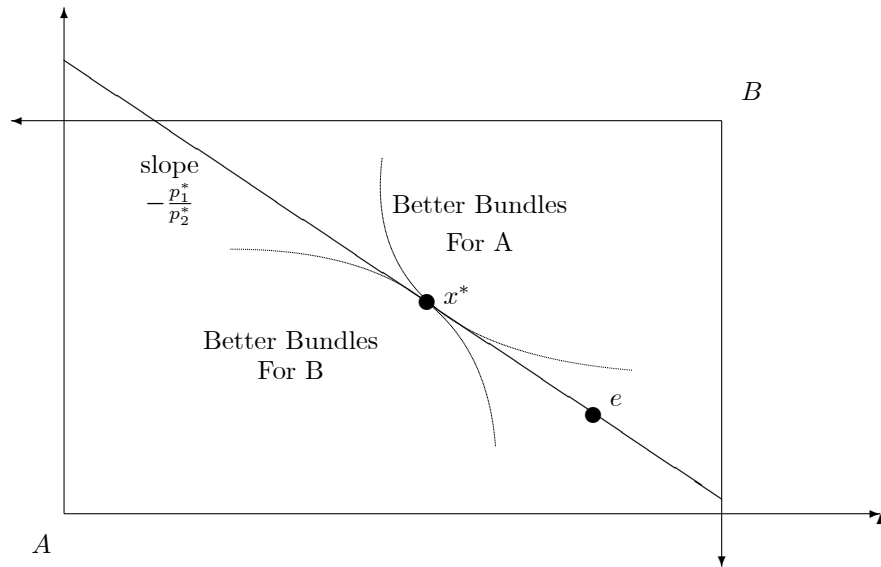


Figure 5: An Equilibrium is Pareto Efficient)

1. Efficiency has nothing to do with distribution of resources. There are efficient allocations where one agent consumes everything or almost everything and one may disagree that such an allocation is “unfair”. However, as long as the consumers only care about their own consumptions this isn’t a Pareto inefficiency.
2. Equilibria depend on the initial distribution of resources, the notion of efficiency does not.

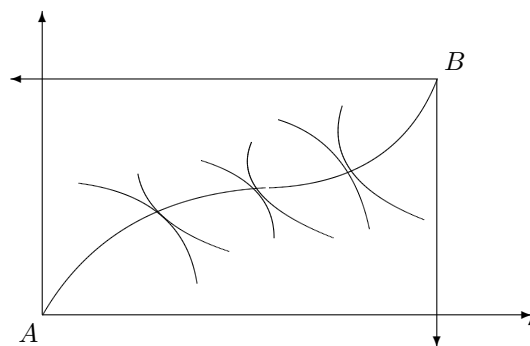


Figure 6: The contract Curve-All Efficient Allocations)

3. Despite potential issues about “fairness” the result that competitive equilibria are efficient may be thought of as a “greed is good” type of result. Indeed it is the basic reason for why economists are often very sceptical towards market interventions. Leaving the market alone (under the competitive assumptions which are loosely based on ideas of many firms and many consumers) we have reasons to believe that the market outcome is at least approximately efficient. Messing with the market we may help some individuals or groups, but, as we’ll see with more concrete examples of interventionist policies, efficiency is typically lost.
4. Later in the course we will analyze and discuss reasons for why the market may not produce Pareto efficient outcomes. In spite of the seeming generality of the result that equilibria are efficient (we have only considered the simplest exchange model, but it holds also when we have arbitrary numbers of goods and/or agents and production by firms...) there are lots of reasons why the market could produce inefficient equilibrium outcomes (public goods, externalities, informational issues, monopoly power....).

8.4 Walras Law

Several ways to state it, but for our purposes the only thing we are interested in is the following:

Claim If (p_1, p_2) is such that supply=demand in the market for good 1, then supply=demand also in the market for good 2.

This comes directly from the fact that the budget constraint holds with equality for every agent *for any prices*. For simplicity of notation, let

$$\begin{aligned}
 m^A(p) &= p_1 e_1^A + p_2 e_2^A \\
 m^B(p) &= p_1 e_1^B + p_2 e_2^B
 \end{aligned}$$

We know (because of optimization) that

$$\begin{aligned} p_1 x_1^A(p_1, p_2, m^A(p)) + p_2 x_2^A(p_1, p_2, m^A(p)) &= m^A(p) = p_1 e_1^A + p_2 e_2^A \\ p_1 x_1^B(p_1, p_2, m^B(p)) + p_2 x_2^B(p_1, p_2, m^B(p)) &= m^B(p) = p_1 e_1^B + p_2 e_2^B \end{aligned}$$

Summing we get (write out sums if you don't like “ \sum ” signs)

$$p_1 \left(\sum_{J=A,B} [x_1^J(p_1, p_2, m^J(p)) - e_1^J] \right) + p_2 \left(\sum_{J=A,B} [x_2^J(p_1, p_2, m^J(p)) - e_2^J] \right) = 0$$

Since $p_1 > 0$ and $p_2 > 0$ it follows that if

$$\sum_{J=A,B} [x_1^J(p_1, p_2, m^J(p)) - e_1^J] = 0 \text{ (market for good 1 clears)}$$

then the equality above guarantees that

$$\sum_{J=A,B} [x_2^J(p_1, p_2, m^J(p)) - e_2^J] = 0 \text{ (market for good 2 clears)}$$

The economics behind these summations are actually straightforward. We begin by observing that agents will use their full budgets, which means that the value of the optimal demand given any price equals the value of the endowment for both agents. Summing over the agents, the value of the optimal demand for A +the value for the optimal demand for B must equal the value of the sum of the endowments. This means, *regardless of whether the price is an equilibrium price or not*, that the value of the excess demand/supply for good 1+the value of the excess demand/supply for good 2 must be identical to zero, *regardless of whether the prices clear the market or not*.

8.4.1 Generalizations

The ones of you who are comfortable with the summation notation can see that there is nothing in this that depends on there being two agents. With n agents indexed by i the same reasoning gives

$$p_1 \left(\underbrace{\sum_{i=1}^n [x_1^i(p_1, p_2, m^i(p)) - e_1^i]}_{=0 \text{ by eq condition for market 1}} \right) + p_2 \left(\sum_{i=1}^n [x_2^i(p_1, p_2, m^i(p)) - e_2^i] \right) = 0$$

Now let there be more than 2 goods and let $k = 1, \dots, K$ index the goods. Then again the same reasoning means that for any $p = (p_1, \dots, p_K)$ we have

$$p_1 \left(\sum_{i=1}^n [x_1^i(p, m^i(p)) - e_1^i] \right) + p_2 \left(\sum_{i=1}^n [x_2^i(p, m^i(p)) - e_2^i] \right) + \dots + p_K \left(\sum_{i=1}^n [x_K^i(p, m^i(p)) - e_K^i] \right) = 0$$

Thus:

- With 2 goods, Walras law tells us that if we have found prices so that one market is in equilibrium (with utility maximizing behavior), then the other market is also in equilibrium.
- With K goods, Walras law tells us that if we have found prices so that $K - 1$ markets are in equilibrium, then the K^{th} market is also in equilibrium.
- If you are used to counting equations and unknowns you may be confused by this. Once the demand functions are plugged in the equilibrium conditions (for the case with 2 goods) are 2 equations in what seems to be 2 unknowns, p_1 and p_2 . However, since the budget constraints

$$p_1 x_1 + p_2 x_2 \leq p_1 e_1^J + p_2 e_2^J$$

and

$$t p_1 x_1 + t p_2 x_2 \leq t p_1 e_1^J + t p_2 e_2^J$$

are equivalent for any $t > 0$ we know that we can, for example, set $p_2 = 1$ (pick $t = \frac{1}{p_2}$).

That is, *only the relative price between the goods are determined in equilibrium*. Setting $p_2 = 1$ is thus just a matter of fixing the unit of account so that we express the price of good one in units of good 2.

- Hence, we really have 2 equations in one unknown ($\frac{p_1}{p_2}$). In general, such a system would be unsolvable. However, Walras law states that if one of the equations is solved, then so is the other. That is, the two equations must have the same set of solutions. Hence, the relative price that clears the market for good 1, also clears the market for good 2.

8.5 A Closed Form Example

Consider the case with Cobb-Douglas preferences, and let

$$U^A(x_1, x_2) = a \ln x_1 + (1 - \alpha) \ln x_2$$

$$U^B(x_1, x_2) = b \ln x_1 + (1 - b) \ln x_2,$$

also suppose that $e^A = (1, 0)$ and $e^B = (0, 1)$. The relevant demands are thus

$$\begin{aligned} x_1^A(p, m^A(p)) &= \frac{am^A(p)}{p_1} = \frac{a(p_1 \times 1 + p_2 \times 0)}{p_1} = a \\ x_1^B(p, m^B(p)) &= \frac{bm^B(p)}{p_1} = \frac{b(p_1 \times 0 + p_2 \times 1)}{p_1} = b \frac{p_2}{p_1}. \end{aligned}$$

So equilibrium requires that

$$\begin{aligned} x_1^A(p, m^A(p)) + x_1^B(p, m^B(p)) &= a + b \frac{p_2}{p_1} = 1 = e_1^A + e_1^B \Rightarrow \\ \frac{p_1^*}{p_2^*} &= \frac{b}{1 - a} \end{aligned}$$

The associated equilibrium allocation is (check!)

$$\begin{aligned} \langle x_1^A(p^*, m^A(p^*)), x_2^A(p^*, m^A(p^*)), x_1^B(p^*, m^B(p^*)), x_2^B(p^*, m^B(p^*)) \rangle \\ \langle a, b, 1 - a, 1 - b \rangle \end{aligned}$$

Notice that the more the other agent likes the good that the agent has in her endowment, the better off the agent is, simply reflecting that increased demand drives up the price, which is good for the seller of the good.

8.6 A Representative Agent Example

A common way to write down general equilibrium model that are simple is to assume that all agents in the economy are identical clones of each other. This type of models are used a lot in macroeconomics (because it allows the choice problem for the individual to be quite rich) and are called representative agent models.

Consider a world populated with lots of agents consuming only apples. Each agent lives for two periods and has an apple tree that produces e units of apples in every period. There

is a competitive market for borrowing and saving and r denotes the interest rate. All agents have identical preferences given by

$$u(c_1) + \delta u(c_2).$$

The choice problem for an individual is thus to decide how much to borrow or save. As we have seen before, we may write this either as

$$\max_{-\frac{e}{1+r} \leq s \leq e} u(e - s) + \delta u(e + s(1 + r))$$

or as

$$\begin{aligned} & \max_{c_1, c_2} u(c_1) + \delta u(c_2) \\ \text{s.t. } & c_1 + \frac{1}{1+r}c_2 \leq e + \frac{1}{1+r}e. \end{aligned}$$

For our purposes, the first expression is simpler. The first order condition is simply

$$-u'(e - s) + \delta u'(e + s(1 + r))(1 + r) = 0$$

Now, assuming that the apples are non-storable we note that:

1. In equilibrium, r^* must be such that $s^* = 0$. If not, resources will not balance. That is, if $s < 0$ then all agents borrow, so total apple consumption in the first period exceeds total apple production. Symmetrically, if $s > 0$ all agents save, so total apple consumption in the second period exceeds what is available.
2. Hence, $s^* = 0$ must solve the optimization problem, and therefore satisfy the first order condition. We conclude that

$$-u'(e) + \delta u'(e)(1 + r^*) = 0$$

or

$$r^* = \frac{1 - \delta}{\delta}.$$

This is very very simple, but it is a useful theory of how the equilibrium interest rate is determined. It simply says that the equilibrium interest rate must be determined so that people are happy to consume what is available in every period, which boils down to relation between the interest rate and the “discount factor” which measures how patient or impatient people are.

8.6.1 An Arbitrary Time Horizon

You should only read this if you have everything else in the course under complete control. No exam questions or homework problems will relate directly to this.

Suppose that the consumer lives for T periods and has preferences given by

$$u(c_0) + \delta u(c_1) + \delta^2 u(c_2) + \delta^3 u(c_3) \dots = \sum_{t=0}^T \delta^t u(c_t)$$

Also, let r_t be the interest rate on savings at time t and s_t be the savings. Maintain the assumption that the apple tree produces e units per periods. Then, the consumption at time t may be expressed as

$$c_t = e + s_{t-1}(1 + r_{t-1}) - s_t$$

and the optimal choice problem is

$$\max_{s_0, \dots, s_{T-1}} \sum_{t=0}^T \delta^t u(e + s_{t-1}(1 + r_{t-1}) - s_t)$$

This is a problem with more than one choice variable, but an optimal solution $(s_0^*, s_1^*, \dots, s_{T-1}^*)$ must be such that s_t^* solves the problem over s_t given that all other savings are set to the ones in the optimal solution. Hence, the solution must be characterized by a first order condition for each choice variable, so that

$$-\delta^t u'(e + s_{t-1}^*(1 + r_{t-1}) - s_t^*) + \delta^{t+1} u'(e + s_t^*(1 + r_t) - s_{t+1}^*)(1 + r_t) = 0$$

for every $t = 0, \dots, T - 1$. Multiplying by $\frac{1}{\delta^t}$ we get

$$u'(e + s_{t-1}^*(1 + r_{t-1}) - s_t^*) = \delta u'(e + s_t^*(1 + r_t) - s_{t+1}^*)(1 + r_t).$$

By the same reasoning as in the example with two periods, $s_0^* = s_1^* = \dots = s_{T-1}^* = 0$ for apple consumption to be equal to apple production in every period. Hence, the equilibrium interest rate at time t must satisfy

$$u'(e) = \delta u'(e)(1 + r_t^*)$$

or, once again $r_t^* = \frac{1-\delta}{\delta}$ in every period. Notice, that in the real world, r_t has been around 0.04 to 0.05 over the last 100 years on a yearly basis, which would imply that a δ around .95 is the right number to plug into a model.