9 Technology

Optional Reading: Chapter 18 & 19 in Varian

Our next topic is to think about firm behavior. Exactly as with consumer theory we then need to specify:

- What the firms **can** do (describe the feasible set, possible production plans)
- How firms evaluate the different options.

Unlike consumer theory we will not really need to think too deeply about the second question, but will simply assume that firms try to make large profits as possible. Given that we accept that firms behavior should be in the interest of the shareholders of a firm it is actually hard to think of a reasonable alternative to the assumption of profit maximization. Instead it is the description of the *feasible options* for the firm that is a bit more involved compared with consumer theory.

In consumer theory, the description of what a consumer to do is fairly straightforward: the consumer has a given budget and can spend it on various goods. When thinking about a firm on the other hand it is less obvoious how to describe the processes that generate Ford Explorers, Computers, Can't Believe it's Not Butter and other goods. Now:

- Economists think of good production as a process where a number of **inputs** or **factors of production** are somehow transformed into one or more goods. For example you may think of the production of a Big Mac as a process where beef, bread, lettuce, unskilled labor and some capital equipment are used as inputs. To produce a car one needs robots, machinery, steel, paint, cupholders etc...
- It is then natural to think about the **technology** as a function that describes how a given combination of inputs is transformed into outputs, that is we let

  $$y = f(x)$$

  where x is the inputs, y is the output or outputs and f is the function that described the technology

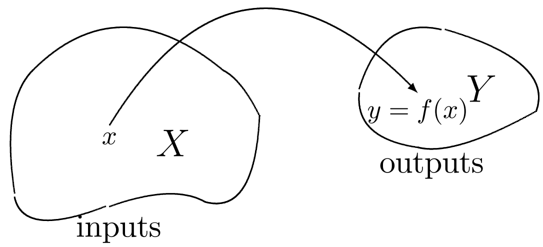

Figure 1: The Role of Technology

- $f(\cdot)$ called **production function.**
- In general a production process may involve a large number of inputs as well as a large number of output goods. This is really no problem for the theory. One just needs to have inputs given by $x = (x_1, ..., x_n)$ and outputs given by $y = (y_1, ..., y_m)$. Much of what we will do will extend to this setup, but I will *not* consider the possibility of "joint production" (=more than one output) in this course.

  To understand why the discussion about "joint production" above is at all relevant: note that when producing white sugar one gets molasses as well, when producing gas one gets several thicker oil products.... (before cars, the gasoline was burned/poured into the ocean because it was just a useless by-product when producing other things).
- Simplest example:

  1 output y

  1 input x

We will use this simplest case later on (when the point is to understand something different), but we'll concentrate on the case with one output and two inputs, although most things would be almost as easy with n inputs. Again, the reason for this restriction is that it allows pictures to be drawn. Now, with

$$y = f(x_1, x_2)$$

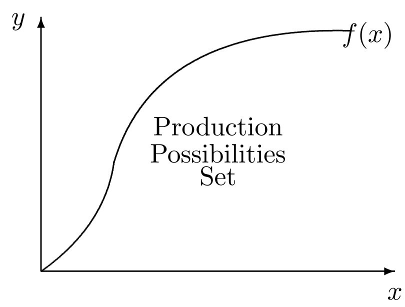

Figure 2: A Single Input and a Single Output

we can apply the same graphical ideas as when we worked with consumer choices. That is, rather than to look at the three-dimensional surface that describes the function (depicted in Figure 3) we may instead project the picture down to a two-dimensional figure where we look at "the topographical map" of the function f by using the level curves as in Figure (4). A level curve to a production function is mathematically the same thing as a level curve to a utility function, but they are customarily referred to as "isoquants" in production theory.

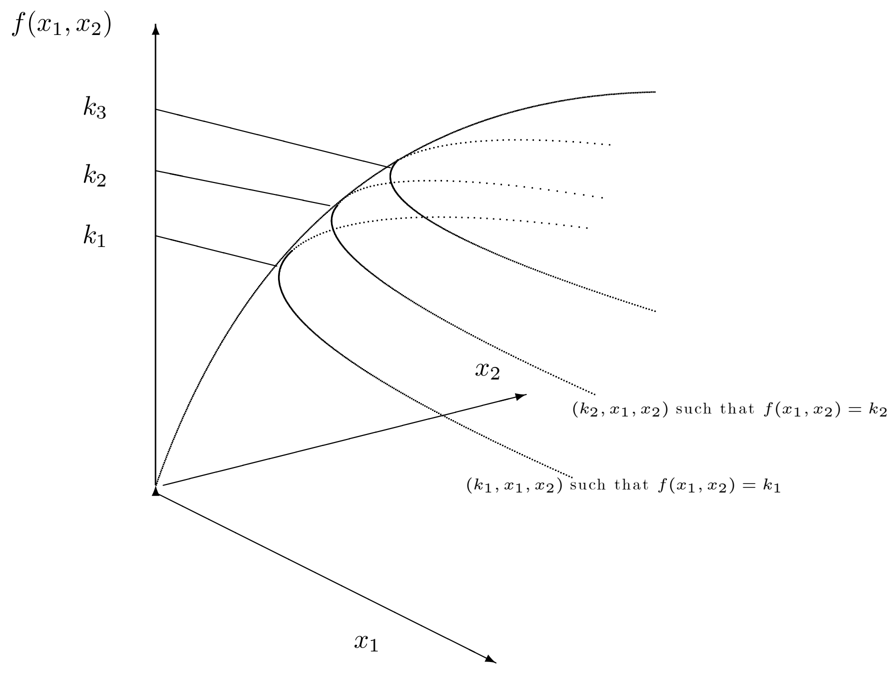

Figure 3: Production function with 2 Inputs and One Output

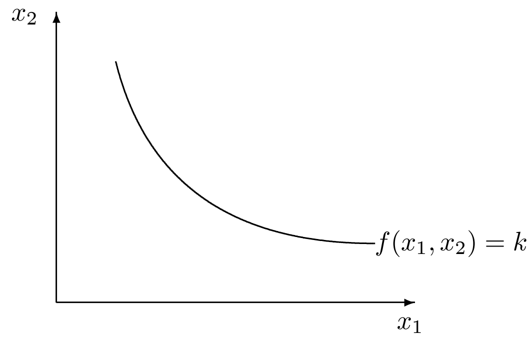

Figure 4: A Level Curve of $f(x_1, x_2)$ (Isoquant)

9.1 A Very Very Important Difference Between Production and Utility Functions

- For **utility functions** the particular choice of "happiness scale" or "utility index" is not important for the theory (ordinal utility)

- For **production functions** the scale corresponds to **output levels.** Thus, changing the scale would mean a change in outputs corresponding to each combination of inputs, which would affect profits. Thus, **THE SCALE IS FIX,** so **CARDINAL PROPERTIES OF f ARE IMPORTANT**$\Rightarrow$we can't make monotone transformations the same way as before to simplify the problems.

9.2 Terminology

9.2.1 Marginal Products

Production functions are used so much in economics that the first partial derivative has been given a specific name: economists talk about *marginal product of a factor* rather than say "the first derivative with respect to factor...". That is,

- $\frac{\partial f(x_1,x_2)}{\partial x_1}$ is the marginal product of factor 1 and

- $\frac{\partial f(x_1,x_2)}{\partial x_2}$ is the marginal product of factor 2

This language makes sense since for small Δ

$$\frac{\partial f(x_1,x_2)}{\partial x_1} \approx \frac{f(x_1+\Delta,x_2)-f(x_1,x_2)}{\Delta} = \frac{\text{change in output}}{\text{change in input}},$$

so the partial derivative says how much output changes (increases) when the input of a factor is slightly increased. While it would be correct to say "the first derivative with respect to labor input" rather than (as most economist do) "the marginal product of labor", the latter way of talking is useful since we are less likely to forget how to interpret the derivative of the production function.

9.2.2 Technical Rate of Substitution

Mathematically concept **identical** to concept of marginal rate of substitution in consumer theory. That is, the technical rate of substitution is given by the *slope of a level curve to f* (which is now called isoquant). Conceptually this slope tells us *how much extra factor 2 we need to keep output constant if factor 1 is decreased by a small unit.*

We can describe a level curve/isoquant to f as a function $x_2(x_1)$ satisfying

$$f(x_1,x_2(x_1)) = k$$

for all x_1. Total differentiation of this identity using the chain rule gives

$$\begin{aligned} \frac{\partial f(x_1,x_2(x_1))}{\partial x_1} + \frac{\partial f(x_1,x_2(x_1))}{\partial x_1}\frac{dx_2(x_1)}{dx_1} &= \frac{d}{dx_1}(k) = 0 \Rightarrow \\ \frac{dx_2(x_1)}{dx_1} &= -\frac{\frac{\partial f(x_1,x_2(x_1))}{\partial x_1}}{\frac{\partial f(x_1,x_2(x_1))}{\partial x_2}} = \text{TRS} \end{aligned}$$

9.3 Common Assumptions

Typically, the same sort of assumptions are usually made on f as the type of assumptions we made on u in consumer choice theory. The reasons for making these assumptions are also similar. In particular we will assume:

9.3.1 Positive Marginal Products (Monotonicity)

This means that $f(x_1, x_2)$ is increasing in both arguments, or

$$\frac{\partial f(x_1, x_2)}{\partial x_1} > 0$$
$$\frac{\partial f(x_1, x_2)}{\partial x2} > 0$$

Even if there would be ranges where this assumption would not hold (and it is easy to think of reasons why, like overcrowding if thousands of workers are put to work on a field), profit maximizing firms would never choose to operate in such a range.

9.3.2 Diminishing Marginal products

$\frac{\partial f(x_1,x_2)}{\partial x_1}$ decreasing

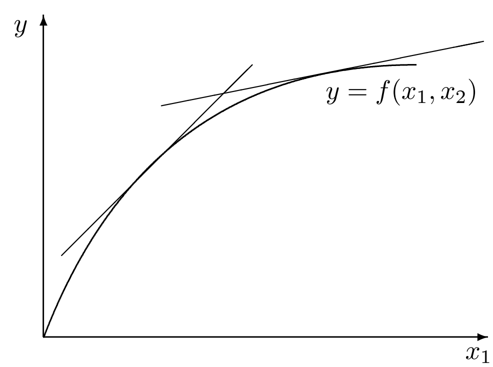

Figure 5: Diminishing Marginal Returns

OBSERVE THAT THIS IS LOOKING AT ONE FACTOR AT A TIME AND THIS HAS NOTHING TO DO WITH RETURNS TO SCALE (SEE DISCUSSION ON SCALE PROPERTIES BELOW).

9.3.3 Convexity

For any $0 \leq \lambda \leq 1$ and input combinations x, x'

$$f\left(\lambda x_1 + (1-\lambda)\, x_1', \lambda x_2 + (1-\lambda)\, x_2'\right) \geq \lambda f(x_1, x_2) + (1-\lambda) f(x_1', x_2')$$

Graphically this implies that a straight line between any two points on a given isoquant (=level curve to the production function $f(\cdot)$) is above the isoquant (check this from notes on utility theory)

9.4 Returns to Scale

The best way to understand that scaling now matters in another way than in utility theory is to think about "scale properties" of production functions. There we ask what happens to output if all inputs are doubled (or scaled by a factor $t > 1$). This will not give us any "results", but the language it gives us is useful.

9.4.1 Constant Returns to Scale

We say that f satisfies *constant returns to scale* if doubling all inputs would double the output, that is

$$2f(x_1, x_2) = f(2x_1, 2x_2)$$

or in general if for all $t > 0$

$$tf(x_1, x_2) = f(tx_1, tx_2).$$

It is often argued on philosophical grounds that constant returns is the most reasonable assumption:

- The reason why constant returns is a priori appealing is that it should always be possible to double the output by *replicating* what is done in a particular plant as long as all factors are available.

9.4.2 Increasing Returns to Scale

However, it is kind of clear that for certain production processes there are advantages of big scale production and this can come directly from laws of physics in some cases and from research/development costs in other cases. For example

- The bigger the radius of a pipeline, the more oil/unit of steel can flow through the pipeline (increasing returns due to physical constraints).
- The blueprint for a new car model costs the same irrespective of how many units are sold (increasing returns due to fix costs of R&D).

Hence, for these reasons there may be increasing returns, that is

$$tf(x_1, x_2) < f(tx_1, tx_2) \text{ for } t > 1$$

9.4.3 Decreasing Returns to Scale

$$tf(x_1, x_2) > f(tx_1, tx_2) \text{ for } t > 1$$

"Weird case" according to Varian, but he acknowledges that it may be motivated as "reduced form" of fixed factors. The reason for why he considers it weird is the replication argument. I can see other reasons than "fixed factors" for small scale production being an advantage (having to do with incentives).

9.5 Examples of Specific Functions we will Use

The functional forms that are used in production theory are more or less the same as the functions used as utility functions, so we consider for example

$$f(x_1, x_2) = \min\{ax_1, bx_2\},$$

which now is referred to as a **fixed proportions** production function. Also

$$f(x_1, x_2) = ax_1 + bx_2$$

may be used for some purposes, but the leading example is the Cobb-Douglas function

$$f(x_1, x_2) = Ax_1^a x_2^b$$

9.5.1 Scale Properties of Cobb-Douglas Function

Let $t > 1$ and consider

$$f(tx_1, tx_2) - tf(x_1, x_2)$$

We know that

1. If difference is zero, then we have constant returns to scale
2. If difference is positive, then we have increasing returns
3. If difference is negative we have decreasing returns.

Now

$$\begin{aligned} f(tx_1, tx_2) - tf(x_1, x_2) &= A(tx_1)^a (tx_2)^b - tAx_1^a x_2^b = \\ &= At^a x_1^a t^b x_2^b - tAx_1^a x_2^b = \\ &= Ax_1^a x_2^b \left(t^a t^b - t\right) = \\ &= Ax_1^a x_2^b \left(t^{a+b} - t\right) \begin{cases} > 0 & \text{if } a + b > 1 \\ = 0 & \text{if } a + b = 1 \\ < 1 & \text{if } a + b < 1 \end{cases} \end{aligned}$$

Hence, the scaling parameter A is irrelevant for scale properties, while the sum of exponents divides the parametric class into three cases.

Note that the marginal product with respect to factor 1 is

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{d}{dx_1} Ax_1^a x_2^b = Ax_2^b a x_1^{a-1},$$

which is DECREASING in x_1 for all $a < 1$. This should make clear that we can have diminishing returns in factors at the same time as constant or increasing returns to scale.

10 Profit Maximization

With few interesting exceptions, economists assume that firms exist in order to maximize profits for their owners. However, while the idea of profit maximization as the objective for firms completely dominates this does NOT mean that the theory of the firm is a "one model theory" in the same way as the basic model in consumer theory. The reason is that lots of "details" matter, such as

- Whether factors are fixed or variable
- The market structure. Competitive firms take prices as given by the market, but if there are a few firms (oligopoly) or a single firm (monopoly) this is inconsistent with rationality on behalf of the firms since a monopolist should be able to figure out that it has some market power (we'll study this later).
- Modern economic theory also consider other types of constraints on firms that in some way or the other comes from deviations in the assumption that there is perfect information about everything in the world.

10.1 Competitive Firms

We will begin by studying the traditional model of a competitive firm, which is a useful model to combine with the basic utility theory we've studied to understand several concrete policy issues. A competitive firm (as well as oligopolistic or monopolistic firms) care about its profit, which is just

Profit=Revenues -Costs

That is if

- $y = (y_1, ..., y_n)$ is the outputs produced
- $x = (x_1, ..., x_m)$ is the inputs used
- $p = (p_1, ..., p_n)$ are the output prices
- $w = (w_1, ..., w_m)$ are the input prices,

then we write

$$\Pi = \sum_{i=1}^{n} p_n y_n - \sum_{j=1}^{m} w_j x_j$$

for the profits. The case we will be most interested is with 1 output y (price p) and two inputs x_1, x_2 with prices $w_1, w_2 \Rightarrow$

$$\Pi = py - w_1 x_1 - w_2 x_2$$

10.2 Fixed and Variable Factors

Depending on context/application it is sometimes useful to have factors of production that are **not** choice variables for the firm. Such a factor of production is referred to as a *fixed factor.* One may (correctly) argue that this seems redundant since if one factor is fixed, then it is "as if" the production function didn't exist (meaning that we can just ignore the factor of production). However, to take a concrete example, while it may be the case that it is sufficiently costly to change the size of a plant to make "capital" a fixed factor for the relevant planning horizon it may still be that the price of renting capital changes, affecting profits. Also, fixed factors is a reasonable argument to motivate "as if decreasing returns to scale" as we will see. Hence, in spite of a convincing replication argument in favor of constant or increasing returns, this may simply not be relevant because of fixed factors of production.

- As a concrete example of something which looks like a (very) fixed factor, think of a nuclear power plant. Degree of utilization can be easily varied, but # of reactors is given for practical purposes. Hence, firms will have to use other factors to adjust for changes in price of electricity, labor...

- Even more fundamentally fixed are "factors" that are inventors (Edison, Bill Gates, Ford...).

- Typical disctinction: short-run-capital fixed versus long-run-all factors mobile. Distinction is both unnecessary and hookey (i.e., at some point long-term decisions have to be made and at that point firm should foresee that stuff may change in the future, so volatility of prices and factor prices are other important factors for which problem is the more relevant).

10.2.1 Problem with Fixed Factor

With one factor fixed (factor 2) the relevant profit maximization problem for a price-taking firm is

$$\begin{aligned} &\max_{x_1,y} py - w_1x_1 - w_2\overline{x}_2 \\ \text{subj to } y &\leq f(x_1,\overline{x}_2) \end{aligned}$$

where the notation $\overline{x}_2$ is there to stress that this is not a choice variable. In words-we seek a *technologically feasible* combination of y and x_1 that maximizes the profit for the firm.

10.2.2 Problem with Both factors Variable

Then problem is

$$\begin{aligned} &\max_{x_1,x_2,y} py - w_1x_1 - w_2x_2 \\ \text{subj to } y &\leq f(x_1,x_2) \end{aligned}$$

10.3 Profit Maximization with a Single Variable Factor

This is the simplest problem to solve, so we start with it. We seek solutions to

$$\begin{aligned} &\max_{x_1,y} py - w_1x_1 - w_2\overline{x}_2 \\ \text{subj to } y &\leq f(x_1,\overline{x}_2) \end{aligned}$$

where p, w_1 and w_2 are all strictly positive numbers. Clearly, the constraint must bind since otherwise there is stuff the firm could sell with no additional factor costs, so we can plug in the constraint to get

$$\max_{x_1} pf(x_1,\overline{x}_2) - w_1x_1 - w_2\overline{x}_2$$

Assuming that there is an interior solution, it must then be that the FOC holds,

$$p\frac{\partial f(x_1, \overline{x}_2)}{\partial x_1} - w_1 = 0$$

In a sense we are already done with this. We find that the optimal solution to the profit maximization problem with a single factor is to set *the value of the marginal product=price of the variable factor.* For intuitive understanding we note that the additional output that is generated from a small increase Δx_1 in the factor is given approximately by

$$\Delta y = f(x_1 + \Delta x_1, \overline{x}_2) - f(x_1, \overline{x}_2) \approx \frac{\partial f(x_1, \overline{x}_2)}{\partial x_1}\Delta x_1$$

so the additional revenue is

$$p\Delta y \approx p\frac{\partial f(x_1, \overline{x}_2)}{\partial x_1}\Delta x_1$$

and the additional costs are

$$w_1 \Delta x_1,$$

so the change in profits are

$$\Delta\Pi \approx p\frac{\partial f(x_1, \overline{x}_2)}{\partial x_1}\Delta x_1 - w_1\Delta x_1 = \left(p\frac{\partial f(x_1, \overline{x}_2)}{\partial x_1} - w_1\right)\Delta x_1$$

Nothing above (except words "additional" and "increase") required that Δx_1 is positive and the error of the approximation can be made arbitrarily small by considering small enough changes, so:

1. If value of marginal product>price of factor, then profits would increase by a slight increase in use of factor

2. If value of marginal product<price of factor, then profits would increase by a slight decrease in use of factor

Hence, there is nothing mysterious with the optimality condition and we could arrive at it using common sense reasoning as long as we know that derivatives are good approximations of the original non-linear function. The more careful reader can also note that the verbal reasoning above is identical to the one used to motivate first order conditions in general.

10.4 Graphical Representation

Note that the "profit function" is a function of choice variables as well as parameters (exogenous variables):

$$\pi(\underbrace{y, x_1}_{\text{choices}}, \underbrace{\overline{x}_2, p, w_1, w_2}_{\substack{\text{Beyond control}\\ \text{of firm}}}) = py - w_1x_1 - w_2\overline{x}_2$$

Consider combinations of (y, x_1) such that profit is constant

$$\pi = py - w_1x_1 - w_2\overline{x}_2$$

In this situation it is important to be able to recognize how many "free" variables there are (answer 2). We can solve for y as a function of x_1 only (the rest is just parameters and not choice variables) to get

$$y = \frac{\pi}{p} + \frac{w_1}{p}x_1 + \frac{w_2}{p}\overline{x}_2,$$

which defines a family of lines with slope $\frac{w_1}{p}$ as depicted in Figure 6.

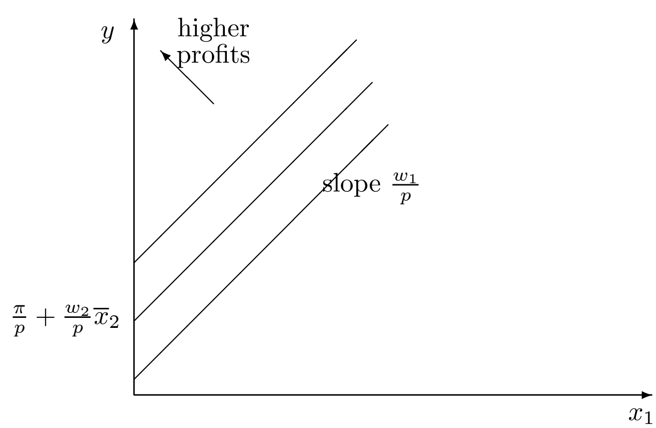

Figure 6: Combinations of y and x_1 that yields constant profits-Isoprofit lines

Note that the lines derived are done so without consideration of what is technologically possible. However, combining with the technological constraint just means that we are only to look for combinations of y and x_1 that are in the production possibilities set, which with diminishing marginal returns and one fixed and a variable factor gives us a picture like Figure 7. Clearly, a solution must occur at a tangency between the straight line depicting constant profit (called isoprofit) and the production possibilities set. If a potential solution is not at a tangency (draw!) then you should see that there is a direction to move *in the set of technologically feasible options* that gives higher profits than the assumed solution.

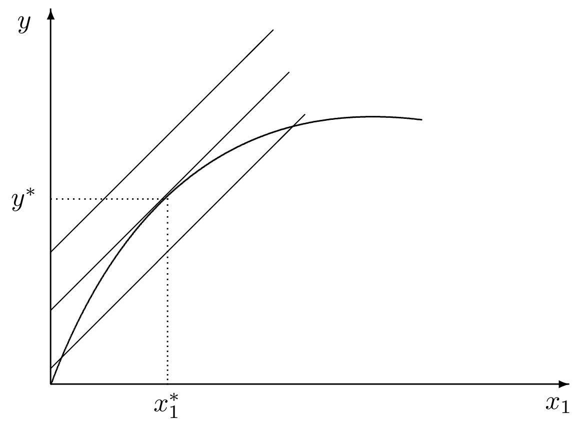

Figure 7: The Profit Maximizing Solution

Clearly (you should convince yourself of this) this is **exactly the same condition as the one we derived by simply taking the derivative of the profit function.** However, this graph is useful to gain understanding in how the solution changes when **exogenous variables change.**

10.5 Comparative Statics

There are three obvious exogenous variables to investigate:

1. w_1−price of variable factor

2. w_2 -price of fixed factor

3. p -price of output.

One could also ask what happens as $\overline{x}_2$ changes, but this is more complicated and we will not pursue it (the reason why this is harder is that the whole production possibilities frontier changes).

Figure 8 depicts the effect of what is labeled as a decrease in the price of factor 1, but the effect is the same if the price goes up. As can be seen from picture and should be clear this leads to an increase in the use of x_1 and an increase in output y under the assumption of diminishing marginal returns.

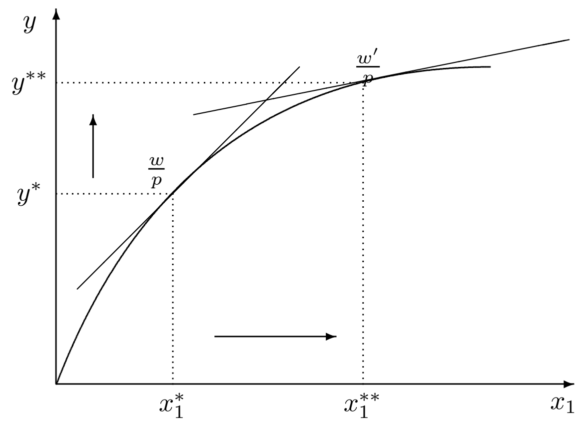

Figure 8: Effect of Decreased Wage (or Increase in Output Price)

Remains to check is the effect from a change in w_2. **This has no effect on the optimal decision,** but do affect profits in the obvious way. However, the important thing to realize that the common reasoning that if capital (fixed factor) becomes more expensive one should "cut the losses" by using less labor is nothing but fuzzy not even logic in this context. One **could** think of more elaborate stories where it would have an impact. For example with limited liability ownership an increase in the price of a fixed factor could make default an optimal strategy, but this and any other story takes us quite far away from the basic competitive framework.

10.6 Profit Maximization with both Factors Variable

The problem is

$$\max_{x_1,x_2,y} py - w_1x_1 - w_2x_2$$
$$\text{subj to } y \leq f(x_1, x_2)$$

or equivalently

$$\max_{x_1,x_2} pf(x_1,x_2) - w_1x_1 - w_2x_2$$

- This is a multivariate optimization problem (more than a single choice variable).

- This is no problem, but needs to be explained.

10.7 Math: Optimization with 2 or n Choice Variables

Recall first what we know about the univariate case where we want to solve

$$\max_{a\leq x\leq b} f(x).$$

As you *should* know by now there are three possibilities for the solution. It may:

- Be interior, in which case the first order condition $f'(x^*) = 0$ must be satisfied.

- or $x^* = a$

- or $x^* = b$

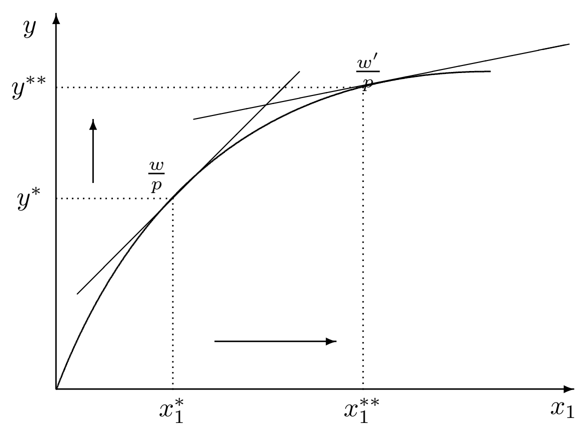

Figure 9: Possible Solutions for Univariate Max Problem

The profit maximization problem with two factors can however not be reduced to a univariate problem, but takes the form

$$\max_{x_1,x_2} g(x_1,x_2)$$

after the technological constraint has been plugged in. However, we can solve this almost exactly as we solve a univariate problem. The most useful way to understand this is to observe that:

Claim If (x_1^*, x_2^*) solves $\max_{x_1,x_2} g(x_1, x_2)$, then

1. x_1^* solves $\max_{x_1} g(x_1, x_2^*)$ (univariate problem with x_2 fixed at x_2^*)
2. x_2^* solves $\max_{x_2} g(x_1^*, x_2)$ (univariate problem with x_1 fixed at x_1^*)

WHY?

If x_1^* does not solve $\max_{x_1} g(x_1, x_2^*)$, then that **means**

$$g(x_1', x_2^*) > g(x_1^*, x_2^*)$$

for some $x_1' \Rightarrow (x_1^*, x_2^*)$ does not solve $\max_{x_1,x_2} g(x_1, x_2)$. The argument for x_2 is symmetric, which establishes the claim.

SO WHAT?

$\max_{x_1} g(x_1, x_2^*)$ is a problem we know how to handle. I.e., if x_1^* is a solution and the solution is interior, then the derivative with respect to x_1 must be zero and similarly for x_2^*, so we conclude that any interior solution must satisfy the first order conditions

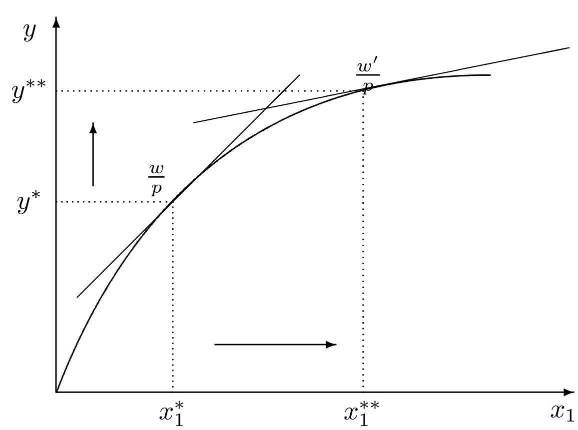

Figure 10: First Order Condition in Multivariate Case

$$\begin{aligned}
\frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_1} &= 0 \\
\frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_2} &= 0,
\end{aligned}$$

which means that we PROCEED EXACTLY AS IN THE UNIVARIATE CASE EXCEPT THAT WE NOW NEED TO DIFFERENTIATE THE OBJECTIVE FUNCTION TWICE AND SET BOTH PARTIALS EQUAL TO ZERO.

10.7.1 n Choice Variables

The interested can note that there is nothing special about 2. With n variables we have that if $x^* = (x_1^*, ..., x_n^*)$ solves $\max_x g(x)$, then x_1^* must solve $\max_{x_1} g(x_1, x_2^*, ..., x_n^*)$, x_2^* must solve $\max_{x_2} g(x_1^*, x_2, x_3^*, ..., x_n^*)$ ...The argument is identical. This in turn means that any interior solution is characterized by

$$\frac{\partial g\left(x^*\right)}{\partial x_i} = 0 \text{ for } i = 1, ..., n \ (n \text{ conditions})$$

10.8 Solving the Profit Maximization Problem

We seek solutions to

$$\max_{x_1, x_2} pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$

and we know from discussion above that this means that the first order condition**s** must be satisfied if the solution is interior (which is the more interesting case). The first order conditions are

$$\begin{aligned}
p\frac{\partial f\left(x_1, x_2\right)}{\partial x_1} - w_1 &= 0 \text{ (derivative w.r.t } x_1) \\
p\frac{\partial f\left(x_1, x_2\right)}{\partial x_2} - w_2 &= 0 \text{ (derivative w.r.t } x_2)
\end{aligned}$$

or

$$\begin{aligned}
p\frac{\partial f\left(x_1, x_2\right)}{\partial x_1} &= w_1 \\
p\frac{\partial f\left(x_1, x_2\right)}{\partial x_2} &= w_2
\end{aligned}$$

Interpretation: Conditions say that the factors should be set in such a way so that the value of the marginal product for each factor equals the price of the factor. This is exactly the same as when only one factor was variable. However, it is by no means obvious that there exist such a solution now when we are free to vary both factors. Indeed, it turns out that only if there are decreasing or constant returns to scale can we find a solution (and in the case with constant returns there will be an infinity of solutions). With increasing returns the profit maximization problem is ill-defined (contrast this with the case with a fix and a variable factor where the problem is OK as long as there is diminishing marginal returns).

10.9 Profit Maximization With Constant Returns to Scale

Constant Returns to Scale has a very striking implication about equilibrium profits, which in a way is one of the essential insights about the mechanics of competitive markets:

Claim *If the technology exhibits constant returns to scale, then profits must be zero in equilibrium.*

Thus, in spite of all firms trying to maximize their profits, no firms earn more profits than if they wouldn't be in the market at all. While counterintuitive this is a very important and central prediction of competitive equilibrium theory.

Verbal Argument:

1. Suppose that profits are greater than zero in equilibrium.

2. Doubling inputs$\Rightarrow$doubling the output

3. Price given$\Rightarrow$ doubling output means doubling revenue

4. Costs also doubling so

   $$\text{profit}=\text{revenue-costs}$$

   also doubles.

5. Since profits were assumed to be greater than zero to start with, this means that profits are bigger after the doubling, which contradicts the assumption that we started with an equilibrium since firms need to maximize profits given prices in equilibrium.

Mathematical Argument

Suppose x_1^*, x_2^* optimal solution to the profit maximization problem and p^*, w_1^*, w_2^* are the equilibrium prices of factors and the output. Suppose that

$$\pi^* = p^* f(x_1^*, x_2^*) - w_1^* x_1^* - w_2^* x_2^* > 0$$

By constant returns to scale we have that

$$\begin{aligned} p^* f(2x_1^*, 2x_2^*) - 2w_1^* x_1^* - 2w_2^* x_2^* &= 2p^* f(x_1^*, x_2^*) - 2w_1^* x_1^* - 2w_2^* x_2^* \\ &= 2\left(p^* f(x_1^*, x_2^*) - w_1^* x_1^* - w_2^* x_2^*\right) = 2\pi^* > \pi^*, \end{aligned}$$

which means that (x_1^*, x_2^*) not optimal. Clearly, profits can't be negative (no production better), so zero profits is the only remaining possibility.

10.10 Messy Example-Cobb Douglas Production Function

Problem

$$\max_{x_1, x_2} p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

If there is an interior solution, then it must satisfy the FOC

$$\begin{aligned} a p x_1^{a-1} x_2^b - w_1 &= 0 \\ b p x_1^a x_2^{b-1} - w_2 &= 0 \end{aligned}$$

or after multiplying the first equation by x_1 and the second by x_2

$$\begin{aligned} a \underbrace{p x_1^a x_2^b}_{=y} &= w_1 x_1 \\ b \underbrace{p x_1^a x_2^b}_{=y} &= w_2 x_2 \end{aligned}$$

so

$$\begin{aligned} x_1 &= \frac{apy}{w_1} \\ x_2 &= \frac{bpy}{w_2} \end{aligned}$$

Now, while this looks nice it is NOT a complete solution since y needs to be determined. However

$$y = x_1^a x_2^b = \left(\frac{apy}{w_1}\right)^a \left(\frac{bpy}{w_2}\right)^b$$

Now we may distinguish between decreasing, constant and increasing returns:

10.10.1 Decreasing Returns to Scale

$$\begin{aligned} y &= \left(\frac{apy}{w_1}\right)^a \left(\frac{bpy}{w_2}\right)^b \Rightarrow \\ y^{1-a-b} &= \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b \Rightarrow \\ y &= \left(\frac{ap}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{bp}{w_2}\right)^{\frac{b}{1-a-b}} \end{aligned}$$

Forget about the exact messy formula. What is important is

1. y is determined in terms of for the firm *exogenous* variables

   a, b describes technology

   p, w_1, w_2 are prices that are determined outside firm.

2. We write

   $$y(p, w_1, w_2) = \left(\frac{ap}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{bp}{w_2}\right)^{\frac{b}{1-a-b}}$$

   and call this *the supply function of the firm.*

3. Can plug $y(p, w_1, w_2)$ into the expression for optimal factor choices and get

   $$\begin{aligned} x_1(p, w_1, w_2) &= \frac{apy(p, w_1, w_2)}{w_1} \\ x_2(p, w_1, w_2) &= \frac{bpy(p, w_1, w_2)}{w_2}, \end{aligned}$$

   which are *the factor demand functions.*

10.10.2 Constant Returns to Scale

If there is constant returns, then $a + b = 1$. Note that then we have

$$\begin{aligned} y &= \left(\frac{apy}{w_1}\right)^a \left(\frac{bpy}{w_2}\right)^b = p^{a+b} y^{a+b} \left(\frac{a}{w_1}\right)^a \left(\frac{b}{w_2}\right)^b = \\ &= py \left(\frac{a}{w_1}\right)^a \left(\frac{b}{w_2}\right)^b \end{aligned}$$

Hence there are solutions only if

$$p \left(\frac{a}{w_1}\right)^a \left(\frac{b}{w_2}\right)^b = 1$$

and in this case **any quantity y solves the profit maximization problem. Thus output is in this case indeterminate, but the factor proportions are given by**

$$\frac{x_1}{x_2} = \frac{a}{b} \frac{w_2}{w_1}$$

10.10.3 Increasing Returns to Scale

It is a bit messy so I will not do it, but one can work out the algebra and show that you get absurd results. The economics of this is that there simply is no solution to the profit maximization problem in this case. The intuition is to show that if there is a solution, then profits must be at least zero. Doubling of the production by doubling both inputs will **more** than double the profit, so the assumed solution can be improved upon. Hence there is no solution.