Bad semidefinite programs:

they all look the same

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A semidefinite program (SDP)

$$(P(c)) \quad \begin{array}{ll} Sup_x & \sum_{i=1}^m c_i x_i \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad \begin{array}{ll} Inf_Y & B \bullet Y \\ s.t. & Y \succeq 0 \\ & A_i \bullet Y = c_i \ \forall i \end{array} \quad (D(c))$$

where

- A_i, B, Y are symmetric matrices, the x_i and c_i scalars.
- $A \preceq B$ means that $B - A$ is positive semidefinite.
- $A \prec B$ means that $B - A$ is positive definite.
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

Def: (Duffin, Jeroslow, Karlovitz, for semi-infinite LPs, '83)

If Sup = Min for any c with $(P(c))$ bounded, we say that (P), the feasible set of $(P(c))$ has **uniform LP-duality**. (attainment in primal is *not* required!)

Theorem: (Known, follows from more general results for convex programs).

If there is a feasible $\bar{x}$ for (P) such that

$$B - \sum_{i=1}^{m} \bar{x}_i A_i \succ 0 \;\; (\text{“strict feasibility''}, \text{ or “Slater} - \text{condition''}),$$

then (P) has uniform LP-duality.

Ramana, '95: We can **always** have uniform LP-duality (without a Slater-point)! Cost: adding polynomially many new variables, and psd constraints to the dual.

A badly behaved semidefinite system (no unif. LP-duality)

$$x_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -x_1 \\ -x_1 & 0 \end{bmatrix} \succeq 0.$$

Not strictly feasible. When we seek $\sup 2x_1$, the dual is

$$\begin{array}{llll} \inf & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet Y & st. \; Y \succeq 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet Y = 2 & \Leftrightarrow \\ \inf & y_{11} & st. \begin{bmatrix} y_{11} & 1 \\ 1 & y_{22} \end{bmatrix} \succeq 0 & \end{array}$$

Primal max $= 0$, dual inf $= 0$ $(y_{11} \to 0,\ y_{22} \to +\infty)$, but not attained!

And another:

$$x_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} -x_1 & 0 & -x_2 \\ 0 & 1 - 2x_2 & 0 \\ -x_2 & 0 & 0 \end{bmatrix} \succeq 0$$

When we seek $\sup 2x_2$, the value of primal is 0, value of dual is 1, and both are attained!

But: a well behaved semidefinite system (has unif. LP-duality), which is not strictly feasible

$$x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For all objectives, the dual attains, and min of dual = sup of primal.

The wider context:

Constraint Qualifications (CQs)

- Goal: finding a "good" sufficient condition for uniform LP duality to hold.
- Very active area in the 60s, 70s mainly for convex programs: Mangasarian-Fromowitz, Arrow-Hurwitz-Uzawa, Guignard, Gould-Tolle, ...

Constraint Qualifications (CQs) cont'd

- "Good" CQ means:
  1. Weak – preferably necessary and sufficient.
  2. Simple – easy to verify computationally.
- The most frequently used CQ is still the "Slater-type" CQ: It has (2), but not (1).
- Lack of a better CQ for SDP is one of the reasons that the statement "SDP is poly-time solvable" is only true modulo *many* technical assumptions.

Comparing a "bad" and a "good" system

$$\text{(bad) } x_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{(good) } x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

After staring at the bad systems long enough, they start looking similar.

Theorem on "bad" SDPs Suppose w.l.o.g. that in the feasible semidefinite system

$$(P) \quad \sum_i x_i A_i \preceq B$$

the maximum rank psd slack ($Z = B - \sum_i x_i A_i$) is

$$Z = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \text{ with } 0 \leq r \leq n.$$

Then (P) does not have uniform LP-duality $\Leftrightarrow$ there is a Y of the form

$$Y = y_0 B - \sum_i y_i A_i = \begin{bmatrix} \overbrace{Y_{11}}^{r} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}, \text{ with}$$

$$Y_{22} \succeq 0, \text{ and } \mathcal{R}(Y_{12}) \not\subseteq \mathcal{R}(Y_{22}).$$

$$Z = \overbrace{\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}}^{\text{max. rank slack}}, \quad Y = \overbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}}^{\text{certificate of the bad behaviour}}$$

In the example

$$x_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

these matrices are

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ Indeed, $\mathcal{R}([1]) \not\subseteq \mathcal{R}([0])$.

Complexity of "Is (P) well behaved?"

"No" answer has poly-size certificate in the real number model of computing:

- Show Z maximum rank slack (checking that Z has maximal rank is nontrivial!).
- Show Y: checking that Y has the required form is trivial.

The $\in \text{co} - NP$ already follows from an application of Ramana's dual.

How about characterizing **well behaved** semidefinite systems, and a certificate?

Theorem Let $Z = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, with $0 \leq r \leq n$ be as before.

Then (P) is well behaved $\Leftrightarrow$

(1) $\exists\, V$ s.t. $V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$, and $A_i \bullet V = B \bullet V = 0\ \forall i$.

(2) If $Y = y_0 B - \sum_i y_i A_i$, and $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & 0 \end{bmatrix}$, then $Y_{12} = 0$.

In the example

$$x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have

$$(1) \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and (2) is trivial too.

Complexity of "Is (P) well behaved?", cont'd

"Yes" answer has poly-size certificate:

- Show V: checking that V has required shape and $A_i \bullet V = B \bullet V = 0 \; \forall i$ is trivial.
- Checking (2) amounts to checking the equality of two *subspaces*.

A more general framework: conic LPs

$$
(P)c)) \qquad \begin{array}{ll} \sup & \langle c, x \rangle \\ s.t. & Ax \leq_K b \end{array} \qquad\qquad \begin{array}{ll} \inf & \langle b, y \rangle \\ s.t. & y \geq_{K^*} \\ & A^* y = c \end{array} \qquad (D(c))
$$

where

- K be a closed, convex cone, $(x \in K,\ \lambda \geq 0 \ \Rightarrow\ \lambda x \in K)$.
- $K^* = \{y \mid \langle y, s \rangle \geq 0\ \forall s \in K\}$ the *dual* of K.
- A a linear map, A^* its adjoint (transpose).

Theorem Suppose that K belongs to the class of nice cones, i.e. it satisfies $K^* + F^{\perp}$ is closed for all F faces of K, and in (P), the feasible set of $(P(c))$ the most interior slack is

$$z = b - Ax.$$

Then (P) does *not* have uniform LP-duality $\Leftrightarrow$

$$\exists\, y = v_0 b - Av \in \operatorname{cl} \operatorname{dir}\,(z, K) \setminus \operatorname{dir}\,(z, K).$$

$\square$

Here $\operatorname{dir}\,(x, K) := \{\, y \,|\, x + \alpha y \in K \text{ for some } \alpha > 0 \,\}$: the *feasible directions* at x in K.

Corollary K is polyhedral **or** Slater condition holds $\Rightarrow$ $\operatorname{dir}\,(x, K)$ is closed $\Rightarrow$ (P) does have uniform LP-duality

How general is our class of cones?

- Nice cones include: polyhedral cones, the semidefinite cone, the second order cone, p-cones, geometric cones, ... essentially all cones interesting from an optimization viewpoint!
- For all these, our main theorem gives a very simple characterization, when the conic system is badly behaved.

The background: the classical part

Theorem: (Duffin, Jeroslow, Karlovitz, 1983):

The feasible conic system

$$(P) \;\; Ax \leq_K b$$

yields uniform LP duality $\Leftrightarrow$

The set

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} K^* \\ \mathcal{R}_+ \end{pmatrix} \tag{1}$$

is closed.

The background: the new part

Theorem: (P 2003):

The feasible conic system

$$(P) \;\; Ax \leq_K b$$

over a nice cone K yields uniform LP duality $\Leftrightarrow$

The set

$$\begin{pmatrix} A^* \\ b^* \end{pmatrix} K^* \tag{2}$$

is closed.

The background: the new part

Theorem: (P 2001): Let K be a nice cone, M a linear map, $x \in \text{ri}(\mathcal{R}(M) \cap K)$ (nonneg. orthant: max # of nonzeros; semidef. cone: max. rank).

Then

$$M^* K^* \text{ is not closed} \qquad \Leftrightarrow$$

$$\exists y \in \mathcal{R}(M) \cap (\text{cl dir } (x, K) \setminus \text{dir } (x, K).$$

Conclusion

- A very simple, and *exact* characterization of badly behaved
  - semidefinite systems,
  - second order conic systems,
  - ...
- The obvious reasons for the failure of strong duality are the *only* reasons!

Papers

1. Closedness result: Part I $\rightarrow$ see at `www.or.unc.edu/ pataki`.
2. Bad SDP and other system's characterization (this talk): Part II $\rightarrow$ coming soon.
3. Complexity implications, and connection with Ramana's dual: Part III $\rightarrow$ coming soon.

Further work

- We characterized when the pair (A, b) is "bad" for some objective c: gave a "canonical form" for semidefinite and second order systems.
- How about classifying all badly behaved (A, b, c) depending on the *type* of bad behavior? E.g. dual attains, but there is a duality gap; dual does not attain; etc.