CHAPTER 3
THE COMMON FACTOR MODEL IN THE POPULATION

From
Exploratory Factor Analysis
Ledyard R Tucker
and
Robert C. MacCallum

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3.0. Introduction

In Chapter 1 we presented a conceptual, non-mathematical view of common factor theory. The purpose of the present chapter is to develop a detailed mathematical framework for this approach. Such a framework will serve the dual purpose of (a) providing an explicit representation of common factor theory as a mathematical model, so that its properties and implications can be understood; and (b) providing a basis for the solution of the problem, to be addressed in Chapter 7, of fitting the model to observed data.

The common factor model will be developed in this chapter in the context of a population. That is, we will express the model in terms of population parameters rather than sample statistics. Issues of sampling and parameter estimation will be treated in later chapters. So, for now, the reader should conceive of the case where the theoretical population of observations is available. The development of the model and the discussion of related topics and issues will require the use of rather extensive mathematical notation. In general, we will adhere to a notational system where population parameters are represented by Greek letters and italicized English letters. Sample statistics (though none are used in this chapter) will be presented by normal (un-italicized) English letters. A glossary of notation is presented for the reader's use in Appendix A. The approach taken in this mathematical presentation employs linear algebra, or matrix algebra. Readers must be comfortable with matrix terminology and manipulations in order to follow easily the developments to be presented. A basic review of matrix operations is presented in Appendix B. In addition, we will make use of a number of operations and basic theorems involving linear transformations of attributes. These operations and theorems are reviewed in Appendix C.

3.1. Algebraic Representation of the Common Factor Model

To begin, let us recall the basic concepts of common factor theory as presented in Chapter 1. For a given battery of surface attributes, there will exist a set of factors. These factors will be of three different types of common, specific, and error of measurement. The common factors are those that affect more than one attribute in the set, and specific and error of measurement factors each affect a single attribute. Each surface attribute is, to a degree, linearly dependent on these underlying factors. The variation in a surface attribute is accounted for in part by its dependence on common, specific, and error of measurement factors. The covariation between any two surface attributes is accounted for in part by their joint dependence on one or more of the same common factors. Since common factor theory cannot be expected to represent the real world precisely, it is not expected that the variation and covariation of the surface attributes will be exactly explained by the underlying common, specific, and error of measurement factors.

To begin to express this theory in formal mathematical terms, let us first define n as the number of surface attributes to be observed. We next define a row vector $\underline{C}$, of order *n*, as a generalized row vector of measures on the surface attributes. This generalized row vector can be thought of as containing measures on the surface attributes for any individual sampled from the population.

A critical point made in Chapter 1, and reiterated above, is that the common factor model is not expected to account precisely for the variation and covariation of the surface attributes in the population. Alternatively, it could be stated that each surface attribute can be conceived of as being composed of two parts: one part that is accounted for by common factor theory, and the remainder, which is not accounted for by the theory. In conjunction with this conception of the surface attributes, we define two more generalized row vectors. Vector $\underline{D}$, of order *n*, contains that part of vector $\underline{C}$ which is accounted for by common factor theory. Vector $\underline{\ddot{D}}$, also of order *n*, contains that part of vector $\underline{C}$ which is not accounted for by common factor theory. Given these definitions, the relationship among the three generalized vectors is given by:

$$\underline{C} = \underline{D} + \underline{\ddot{D}} \qquad (3.1)$$

This in turn implies that

$$\underline{\ddot{D}} = \underline{C} - \underline{D} \qquad (3.2)$$

Eq. (3.2) simply shows that the non-systematic portion of the surface attributes can be defined as the difference between the observed attributes and the systematic portion. In almost every discussion of factor analytic theory, no distinction is drawn between entries in $\underline{C}$ and $\underline{D}$. That is, the model is presented as exactly representing the surface attributes in the population. However, it is well understood that this will not be the case in the real world. The model, in fact, should be thought of as exactly representing the systematic portion of the surface attributes, as given in $\underline{D}$, and only imperfectly representing the surface attributes themselves, as given in $\underline{C}$. Therefore, we feel it is important to maintain explicitly the distinction between the surface attributes and the systematic portion of the surface attributes. This distinction will be seen to have important philosophical and practical consequences. For the purpose of maintaining efficient terminology, we will continue to refer to the attributes represented in $\underline{C}$ as the surface attributes, and we will refer to the attributes in $\underline{D}$, which represent that portion of the surface attributes that are accounted for by the common factor model, as <u>modeled attributes</u>. In addition, we will refer to entries in $\ddot{\underline{z}}$ as <u>errors of fit</u>; again, these values represent that portion of the surface attributes not accounted for by the model.

As discussed earlier, factor analytic theory postulates the existence of underlying internal attributes, or factors. Let $\underline{x}$ represent a generalized row vector of measures on the factors. The vector $\underline{x}$ will be of order *m*, where *m* is the total number of factors of all three types. Thus, this vector contains the (unobservable) measures for an individual on the common, specific, and error of measurement factors. Next, let us define a matrix W containing weights which represent the effects of the factors on the modeled attribute. This matrix will be of order $n \times m$, where the rows represent the modeled attributes and the columns represent the factors. An entry w_{jk} is a weight, analogous to a regression weight, representing the effects of factor *k* on modeled attribute *j*. Given these definitions, the common factor model can be represented simply as

$$\underline{z} = \underline{x}W' \qquad (2.2)$$

This equation simply defines each modeled attribute as a linear combination of the measures on the factors. This representation of the model can be stated in an expanded fashion by incorporating the distinction among common, specific and error of measurement factors. The vector $\underline{x}$ containing measures on the factors can be conceived of as containing measures on the types of factors. Let *r* be the number of common factors, and recall that *r* should be much less than *n*, the number of attributes. We then can define $\underline{x}_c$ as a row vector of order *r* containing measures on the common factors. In similar fashion, let $\underline{x}_s$ contain measures on the specific factors and let $\underline{x}_e$ contain measures on the error of measurement factors. Both $\underline{x}_s$ and $\underline{x}_e$ will be of order *n*, since there is one specific factor and one error of measurement factor for each attribute. For the present, the measures on all three types of factors will be defined as being given in standardized form. As a result of the way in which the three types of factors have been defined, the measures on the factors can be seen to have some interesting properties. In particular, since the specific factors represent systematic portions of each modeled attribute which are unique to each attribute, the specific factors must be uncorrelated with each other. Similarly, since the error of measurement factors represent transient aspects of each modeled attribute, they also must be uncorrelated with each other. Finally, the definitions of these two types of factors also require that they be uncorrelated with each other and with the common factors. The vectors of measures on the three types of factors can be adjoined horizontally to form the super-vector $\underline{x}$, as follows:

$$\underline{x} = [\underline{x}_c, \underline{x}_s, \underline{x}_e] \qquad (2.3)$$

The order of $\underline{x}$, defined as *m* above, now can be seen more clearly. The vector $\underline{x}$ will contain measures on the *r* common factors, the *n* specific factors, and the *n* error of measurement factors. Thus, the total number of factors, *m*, is given by

$$m = r + 2n \tag{3.5}$$

Just as the vector of measures on the factors can be seen as containing separate sections corresponding to the three types of factors, so also can the factor weight matrix H be partitioned. We can define weight matrices for each of the three types of factors. Let F be a matrix of order $n \times r$, where the rows represent the modeled attributes, the columns represent the common factors, and each entry $"_{jk}$ represents the weight for common factor *k* on modeled attribute j. Let B be a matrix of order $n \times n$, where the rows represent the modeled attributes, the columns represent the specific factors, and each diagonal entry 0_j represents the weight for specific factor *j* on modeled attribute *j*. Note that matrix B will be diagonal, since each specific factor affects only one modeled attribute. Similarly, let I be an $n \times n$ matrix, with rows representing modeled attributes, columns representing error of measurement factors, and diagonal entries $\%_j$ representing the weight for error of measurement factor *j* on modeled attribute *j*. This matrix of error of measurement factor weights must also be diagonal, since each error of measurement factor affects only one modeled attribute. These three matrices of factor weights then can be adjoined horizontally to form the full matrix H of factor weights:

$$H = [F, B, I] \tag{3.6}$$

Thus, H can be seen to be a super-matrix containing the three separate matrices of factor weights representing the three types of factors.

The representations of $\underline{B}$ and H given in Eqs. (3.4) and (3.6) respectively can be substituted into Eq. (3.3) to provide an expanded representation of the common factor model, as follows:

$$\underline{D} = [\underline{B}_{"}, \underline{B}_0, \underline{B}_{\&}] \begin{bmatrix} F' \\ B' \\ I' \end{bmatrix} = \underline{B}_{"} F' + \underline{B}_0 B' + \underline{B}_{\&} I' \tag{3.7}$$

This representation of the model shows more clearly that each modeled attribute is defined as a linear combination of the common, specific, and error of measurement factors. Furthermore, given that B and I are diagonal, it can be seen that each modeled attribute is represented as a linear combination of *r* common factors, one specific factor, and one error of measurement factor.

An interesting aspect of this representation is that error of measurement factors contribute to the modeled attributes. Recognition of this fact should help the reader distinguish between errors of measurement, which give rise to error of measurement factors, and errors of fit, which were defined in Eq. (3.2). Errors of measurement are represented explicitly in the model and contribute to the modeled attributes. Errors of fit represent the lack of fit of the model to the real world, or the lack of correspondence between the modeled attributes and the surface attributes.

As defined to this point, the common factor model represents the underlying structure of each modeled attribute. Recall from Chapter 1 that the general objective of factor analysis is to account for the variation and covariation of the attributes. To achieve this, it is necessary to express the model in terms of the variances and covariances of the modeled attributes. This can be accomplished fairly easily by making use of theorems presented in Appendix C. We are dealing with a situation where we have two sets of attributes (the modeled attributes and the factors) which are related via a linear transformation as defined in Eq. (3.3). In this case it is possible to define a functional relationship between the covariances of the two sets of attributes, as shown in Appendix C. Let us define Σ_{yy} as a population covariance matrix for the modeled attributes. The matrix Σ_{yy} will be of order $n \times n$, with diagonal entries σ_{jj} representing the population variance of modeled attribute j, and off-diagonal entries σ_{hj} representing the population covariance of modeled attributes h and j. Let us next define Σ_{xx} as a population covariance matrix for the factors. Matrix Σ_{xx} will be of order $m \times m$ and will contain entries σ_{kl} representing the population variances (on the diagonal) and covariances (off the diagonal) of the factors. The relationship between Σ_{yy} and Σ_{xx} can be obtained by direct application of Corollary 3 of Theorem 4 in Appendix C. This yields

$$\Sigma_{yy} = W \Sigma_{xx} W' \qquad (3.8)$$

This equation represents the common factor model in terms of covariances of modeled attributes and factors. It states that the variances and covariances of the modeled attributes are a function of the factor weights and the variance and covariances of the factors.

It is informative to consider a more detailed version of Eq. (3.8), which can be obtained by incorporating the distinction among common, specific, and error of measurement factors. First, let us consider more closely the nature of Σ_{xx}. It will be shown that this matrix has a very simple and interesting form. Recall that we are defining the measures on the factors to be in standardized form. Therefore, the matrix Σ_{xx} will take the form of a correlation matrix. Recall also that there exist three different types of factors (common, specific, and error of measurement), as defined in Eq. (3.6). This implies that Σ_{xx} can be thought of as a super-matrix, with sub-matrices containing correlations within or between types of factors as follows:

$$\Sigma_{xx} = \begin{bmatrix} \Sigma_{cc} & \Sigma_{cs} & \Sigma_{ce} \\ \Sigma_{sc} & \Sigma_{ss} & \Sigma_{se} \\ \Sigma_{ec} & \Sigma_{es} & \Sigma_{ee} \end{bmatrix} \qquad (3.9)$$

For each sub-matrix in D_{BB}, the first subscript represents the factors defined by the rows of the sub-matrix, and the second subscript represents the factors defined by the columns of the sub-matrix. For example, the sub-matrix $D_{0"}$ will be of order $n \times r$ and will contain correlations between specific factors (rows) and common factors (columns). Given this representation of D_{BB}, it is very useful to consider the nature of the sub-matrices. Considering the diagonal submatrices in Eq. (3.9), note that these contain correlations among the factors within each given type (common, specific, error of measurement). Since the common factors may be correlated with each other, let us define

$$F = D_{""} \qquad (3.10)$$

Matrix F will be of order $r \times r$ and will contain intercorrelations among the common factors. It was noted earlier that specific factors will be mutually uncorrelated, as will error of measurement factors. Therefore, it can be stated that

$$D_{00} = I \qquad (3.11)$$

and

$$D_{\%\%} = I \qquad (3.12)$$

The identity matrices shown in Eqs. (3.11) and (3.12) will be of order n. Considering the off-diagonal sections of D_{BB} shown in Eq. (3.9), it can be seen that these submatrices will contain all zeroes. Each of these sub-matrices represents a matrix of intercorrelations among factors of two different types (common and specific, common and error of measurement, specific and error of measurement), and, as was explained earlier in this section, factors of different types are, by definition, uncorrelated in the population. Given these properties of the sub-matrices in Eq. (3.9), we can rewrite that equation as

$$D_{BB} = \begin{bmatrix} F & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (3.13)$$

A very important representation of the common factor model can now be obtained by substitution of the super-matrix form of H, from Eq. (3.6), and D_{BB}, from Eq. (3.13), into the model as shown in Eq. (3.8). This yields

$$D_{DD} = \left[F, B, I \right] \begin{bmatrix} F & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F' \\ B' \\ I' \end{bmatrix} \qquad (3.14)$$

$$= \mathbf{F}\mathbf{\Phi}\mathbf{F}' + \mathbf{B}^2 + \mathbf{E}^2 \qquad (3.15)$$

This equation shows that the population covariances for the modeled attributes are functions of the weights and intercorrelatlons for the common factors, the squared specific factor weights, and the squared error of measurement factor weights. When the common factor model is written in this form, the mathematical representation of the model can be seen to be consistent with the basic principles of common factor theory discussed in Chapter 1. In particular, it was stated that the common factors alone account for the relationships among the modeled attributes. This is revealed in Eq. (3.15) by the fact that the off-diagonal elements of $\mathbf{\Sigma}_{yy}$ are a function of only the parameters in the term $\mathbf{F}\mathbf{\Phi}\mathbf{F}'$. Thus, the common factor weights and intercorrelations combine to account for the relationships among the modeled attributes. Another principle of common factor theory is that the variation in the modeled attributes is explained by all three types of factors. In mathematical terms, this is seen in Eq. (3.15) in that the diagonal elements of $\mathbf{\Sigma}_{yy}$ are a function of the common factor parameters in $\mathbf{F}\mathbf{\Phi}\mathbf{F}'$, as well as the diagonal elements in $\mathbf{B}^2$ and $\mathbf{E}^2$. Thus, since $\mathbf{B}^2$ and $\mathbf{E}^2$ are diagonal, the specific and error of measurement factors can be seen as contributing only to the variances of the modeled attributes (i.e., the diagonal elements of $\mathbf{\Sigma}_{yy}$), and not the covariances.

This view implies that it should be possible to define separate components of the variance of each modeled attribute; i.e., portions due to common, specific, and error of measurement factors. This is, in fact, quite simple and will be seen to provide useful information. Let us define a diagonal matrix $\mathbf{H}^2$ as

$$\mathbf{H}^2 = \mathrm{Diag}(\mathbf{F}\mathbf{\Phi}\mathbf{F}') \qquad (3.16)$$

Matrix $\boldsymbol{H}^2$ will be diagonal and of order $n \times n$, with diagonal elements h_j^2 representing the amount of variance in modeled attribute *j* that is accounted for by the common factors. This will be referred to as the common variance of modeled attribute *j*. It can be seen that

$$\mathrm{Diag}(\mathbf{\Sigma}_{yy}) = \mathbf{H}^2 + \mathbf{B}^2 + \mathbf{E}^2 \qquad (3.17)$$

In terms of the variance of a given modeled attribute *j*, this implies that

$$\sigma_{jj} = h_j^2 + b_j^2 + e_j^2 \qquad (3.18)$$

Thus, the variance of each modeled attribute is made up of a common portion, a specific portion, and an error of measurement portion. It is interesting to convert these variance components into proportions by dividing each side of Eq. (3.18) by σ_{jj}. This yields

$$\sigma_{jj}/\sigma_{jj} = h_j^2/\sigma_{jj} + b_j^2/\sigma_{jj} + e_j^2/\sigma_{jj} = 1 \qquad (3.19)$$

If we define

$$\tilde{h}_j^2 = h_j^2 / \sigma_{jj} \qquad (3.20)$$

$$\tilde{b}_j^2 = b_j^2 / \sigma_{jj} \qquad (3.21)$$

$$\tilde{e}_j^2 = e_j^2 / \sigma_{jj} \qquad (3.22)$$

we then can note that for any modeled attribute, we obtain

$$\tilde{h}_j^2 + \tilde{b}_j^2 + \tilde{e}_j^2 = 1 \qquad (3.23)$$

These three proportions of variance are important characteristics of the modeled attributes. The first, $\tilde{h}_j^2$, represents the proportion of variance in modeled attribute *j* that is due to the common factors. This quantity is called the communality of modeled attribute *j*. The second, $\tilde{b}_j^2$, represents the proportion of variance in modeled attribute *j* that is due to the specific factor for that attribute. This quantity is called the specificity for modeled attribute *j*. The third, $\tilde{e}_j^2$, represents the proportion of variance in modeled attribute *j* that is due to the error factor for that attribute This quantity is called the error of measurement variance for modeled attribute *j*. Note that we can define diagonal matrices $\tilde{\mathbf{H}}^2$, $\tilde{\mathbf{B}}^2$, and $\tilde{\mathbf{E}}^2$, containing these proportions of variance for the *n* modeled attributes. Let us define a diagonal matrix $(\mathbf{D}_{\Sigma})$ containing the variances of the modeled attributes. That is,

$$(\mathbf{D}_{\Sigma}) = \mathrm{Diag}(\boldsymbol{\Sigma}) \qquad (3.24)$$

Following Eqs. (3.20)-(3.22), we then can define matrices $\tilde{\mathbf{H}}^2$, $\tilde{\mathbf{B}}^2$, and $\tilde{\mathbf{E}}^2$ as follows:

$$\tilde{\mathbf{H}}^2 = \mathbf{H}^2 (\mathbf{D}_{\Sigma})^{-1} \qquad (3.25)$$

$$\tilde{\mathbf{B}}^2 = \mathbf{B}^2 (\mathbf{D}_{\Sigma})^{-1} \qquad (3.26)$$

$$\tilde{\mathbf{E}}^2 = \mathbf{E}^2 (\mathbf{D}_{\Sigma})^{-1} \qquad (3.27)$$

According to Eq. (3.23), the sum of these matrices would be an identity matrix:

$$\tilde{\mathbf{H}}^2 + \tilde{\mathbf{B}}^2 + \tilde{\mathbf{E}}^2 = \mathbf{I} \qquad (3.28)$$

There are two interesting variance coefficients which can be derived from the three types of variance defined in the common factor model. The first is the reliability of a modeled attribute. Reliability is defined in mental test theory as unity minus the error of measurement variance for standardized measures. In the present context, this can be written

$$r_j = 1 - \tilde{e}_j^2 \qquad (3.29)$$

where r_j is the reliability of modeled attribute j. Based on Eq. (3.23), we can rewrite Eq. (3.29) as

$$r_j = \tilde{h}_j^2 + \tilde{s}_j^2 \qquad (3.30)$$

This equation implies that, in factor analytic terms, reliability is the sum of the variances due to the common factors and the specific factor, or, in other words, the sum of the communality and specificity. Since reliability is defined as true score variance, this result means that true score variance can be represented as the sum of the two types of systematic variance in the common factor model: that due to the common factors and that due to the specific factor.

An alternative combination of components of variance is to sum the specific and error portions for each modeled attribute. This sum would yield a portion which is called unique variance; i.e., that due to factors (both systematic and error of measurement) which affect only one attribute. In unstandardized form this would be written

$$U^2 = S^2 + E^2 \qquad (3.31)$$

where U^2 is a diagonal matrix with entries u_j^2 representing the variance in modeled attribute j due to specific and error of measurement factors. In terms of the variance expressed as proportions, we would write

$$\tilde{U}^2 = \tilde{S}^2 + \tilde{E}^2 \qquad (3.32)$$

where $\tilde{U}^2$ is a diagonal matrix with entries $\tilde{u}_j^2$ representing the proportion of variance in modeled attribute j due to specific and error of measurement factors. This proportion is called the uniqueness of modeled attribute j, and is defined as the sum of the specificity and error of measurement variance of that attribute. The relation between the uniquenesses of the modeled attributes, defined in Eq. (3.32), and the unique variances, defined in Eq. (3.31), can be obtained by substitution from Eqs. (3.26) and (3.27) into Eq. (3.32). This yields the following:

$$\tilde{U}^2 = U^2 [D_{\Sigma}]^{-1} \qquad (3.33)$$

This relation follows those given in Eqs. (3.25)-(3.27) for other variance components and merely represents the conversion of unique variances into proportions called uniqueness. A final interesting relation is obtained by substituting from Eq. (3.32) into Eq. (3.28), yielding

$$\hat{L}^2 + \hat{Y}^2 = 1 \qquad (3.33)$$

This equation simply implies that the communality plus the uniqueness for each modeled attribute will equal unity. In other words, the variance of a modeled attribute can be viewed as arising from two sources: (a) the common factors, or influences which are common to other attributes in the battery; and (b) the unique factors, or influences which are unique to the given attribute.

A most important equation is produced by substituting from Eq. (3.31) into the expression for the common factor model given in Eq. (3. 15). This yields

$$D_{DD} = F F F' + Y^2 \qquad (3.34)$$

Alternatively, we could write

$$D_{DD} - Y^2 = F F F' \qquad (3.35)$$

In either form, these equations provide a very fundamental expression of the common factor model. They show the functional relationship between the population variances and covariances of the modeled attributes, given in D_{DD}, and critical parameters of the model contained in F, the matrix of common factor weights, F, the matrix of common factor intercorrelations, and Y^2, the diagonal matrix of unique variances. As will be seen in subsequent chapters, this expression of the model provides a basis for some procedures for estimating model parameters. Thus, Eq. (3.35) should be viewed as a fundamental equation of common factor theory, expressing the common factor model in terms of the variance and covariances of the modeled attributes. It is important to understand, however, that this version of the model is a derived statement, i.e., derived from the model as given in Eq. (3.3), which defined the structure of the modeled attributes themselves. That is, based on the model defined in Eq. (3.3), the model in Eq. (3.35), defining the structure of the covariances of the modeled attributes as a function of correlated common factors, is derived. Correlated common factors are also called <u>oblique</u> common factors, and the model represented by Eq. (3.35) is called the <u>oblique common factor model</u>.

It is interesting and useful to recognize that the model given by Eq. (3.35) could have been derived based on definitions of <u>B</u> and H slightly different than those given in Eqs. (3.4) and (3.6), respectively. This alternative approach involves combining the specific and error of measurement factor portions of the model into a unique factor portion at the outset. That is, rather than define <u>B</u> as containing scores on common, specific, and error of measurement factors, as in Eq. (3.4), let us define it as containing scores on the r common factors and n unique factors. There would be a unique factor for each attribute, representing that portion of the attribute not accounted for by the common factors. To differentiate this representation from that given above, let us define a vector $\underline{B}_{".}$ whose contents are given by

$$\underline{B}_{".} = [\underline{B}_{"}, \underline{B}_{.}] \qquad (3.37)$$

where $\underline{B}_{"}$ and $\underline{B}_{.}$ are vectors of scores on common and unique factors, respectively. In a similar fashion, we could define a factor weight matrix $H_{".}$ whose contents are given by

$$H_{".} = [F, Y] \qquad (3.38)$$

where F is the n x r matrix of common factor weights and Y is an n x n diagonal matrix of unique factor weights. Following the form of Eq. (3.3), the common factor model could then be written as

$$\begin{aligned} \underline{z} &= \underline{B}_{".} H'_{".} \\ &= [\underline{B}_{"}, \underline{B}_{.}] \begin{bmatrix} F' \\ Y' \end{bmatrix} \\ &= \underline{B}_{"} F' + \underline{B}_{.} Y' \end{aligned} \qquad (3.39)$$

This model represents the modeled attributes as linear combinations of common and unique factors. The model then could be stated in terms of covariances of modeled attributes, following the form of Eq. (3.8). This requires that we first define a covariance matrix $D_{B_{".}B_{".}}$ for the common and unique factors. This matrix would take the form

$$D_{B_{".}B_{".}} = \begin{bmatrix} D_{""} & D_{".} \\ D_{."} & D_{..} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & M \end{bmatrix} \qquad (3.40)$$

since we define all factors as being in standardized form and since unique factors would, by definition, be uncorrelated with each other and with the common factors. The covariance form of the model then becomes

$$\begin{aligned} D_{DD} &= H_{".} D_{B_{".}B_{".}} H'_{".} \\ &= [F, Y] \begin{bmatrix} F & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} F' \\ Y' \end{bmatrix} \\ &= F F F' + Y^2 \end{aligned} \qquad (3.41)$$

Obviously, this expression is identical to that given in Eq. (3.35), thus verifying that these alternative definitions of the factor score vector and factor weight matrix yield the same oblique common factor model.

3.2. The Common Factor Model with Uncorrelated Common Factors

The common factor model as presented in the previous section represents the case where the common factors were allowed to be intercorrelated. Recall that the correlations among the common factors are contained in matrix Φ, defined in Eq. (3.10) and playing a prominent role in the model given by Eq. (3.35). Let us now consider a special case of this model, where the common factors are defined as being mutually uncorrelated. It will be shown that such a solution can be obtained from a solution with correlated common factors.

Suppose that a solution has been obtained in terms of correlated common factors, and that matrices F and Φ are given. Let T be any non-singular matrix of order $r \times r$ such that

$$\Phi = TT' \qquad (3.42)$$

As will be seen as a result of procedures to be discussed in Chapter 9, it is routinely the case that, for a given Φ, an infinite number of such matrices T will exist. Substituting from Eq. (3.42) into Eq. (3.36) yields

$$\Sigma_{zz} - \Psi^2 = FTT'F' \qquad (3.43)$$

If we define a matrix A as

$$A = FT \qquad (3.44)$$

then Eq. (3.43) can be rewritten as

$$\Sigma_{zz} - \Psi^2 = AA' \qquad (3.45)$$

if we then rewrite Eq. (3.45) as

$$\Sigma_{zz} - \Psi^2 = AIA' \qquad (3.46)$$

It becomes clear that we have "transformed" the original solution containing correlated common factors into a new solution containing uncorrelated, or orthogonal, common factors. In the orthogonal solution, matrix A is an $n \times r$ matrix of common factor weights; entries a_{jk} represent the weight for modeled attribute *j* on factor *k*. The Φ matrix in the model as given in Eq. (3.36) has become an identity matrix of order *r*, indicating that the factors represented by the A matrix are uncorrelated. Eq. (3.45) can be thought of as representing the orthogonal common factor model. Thus, such an orthogonal solution can be obtained by transformation of the original oblique solution by defining a non-singular matrix T which satisfies Eq. (3.42), and then obtaining the matrix of factor weights via Eq. (3.44).

Let us now consider the opposite case. That is, instead of being given an oblique solution and transforming it into an orthogonal solution, we are given an orthogonal solution, represented by a matrix A which satisfies Eq. (3.45). It would be straightforward to transform that solution into an oblique solution. Let $\mathbf{T}$ be any non-singular matrix with elements in each row scaled so that the sum of squares in each row is unity. That is,

$$diag(\mathbf{T}\mathbf{T}') = \mathbf{I} \tag{3.47}$$

An infinite number of such matrices will exist. By solving Eq. (3.44), it then can be seen that weights for transformed oblique factors can be obtained by

$$\mathbf{B} = \mathbf{A}\mathbf{T}' \tag{3.48}$$

Correlations among the transformed factors would be obtained via Eq. (3,42). An interesting special case of this transformation process would arise if a matrix $\mathbf{T}$ were defined such that $\mathbf{T}\mathbf{T}'=\mathbf{I}$. In such a case, the transformed solution would also contain orthogonal factors, and the transformation would have converted one orthogonal solution into another. Thus, an orthogonal solution can be transformed into an oblique solution or another orthogonal solution by defining a matrix **T** having the appropriate properties.

As will be seen in subsequent chapters, the orthogonal common factor model and the capability of transforming orthogonal solution into oblique solutions, or other orthogonal solutions, form the basis for much factor analysis methodology. Most factor analytic procedures work by first seeking an orthogonal solution which fits a given covariance matrix as well as possible. This initial orthogonal solution represents an arbitrary solution. Then a transformation is determined which transforms this initial solution, by the equations given above, into a new solution such that the resulting factor weights and intercorrelations are substantively interpretable. For example, the three-factor oblique solution presented in Chapter 1 for the nine mental tests was in fact obtained by transforming an initial orthogonal solution. This process is commonly referred to as the rotation, or transformation problem, and is discussed in depth in Chapter 9.

It is interesting to note some simple properties of orthogonal common factor solutions. For instance, it can be seen that for orthogonal factors, Eq. (3.16) can be re-written as

$$\mathbf{H}^2 = diag(\mathbf{A}\mathbf{A}') \tag{3.49}$$

This implies that the common portion of the variance of modeled attribute *j* can be defined as

$$h_j^2 = \sum_{k=1}^{r} a_{jk}^2 \tag{3.50}$$

Thus, the common portion of the variance of modeled attribute j can be obtained by summing the squared entries in row *j* of $\mathbf{A}$. This definition for this quantity is the one commonly used in the factor analytic literature. The definition given in Eq. (3.46) is more general in that it applies to both correlated and uncorrelated common factors. Note that summing the squares of the entries in a row of a correlated factor matrix, F, does not necessarily yield the communality of the modeled attribute. Given an initial orthogonal solution, the common variances for the modeled attributes can be obtained via Eq. (3.50), and the unique variances could in turn be obtained by

$$U^2 = \text{Diag}(D_{DD}) - H^2 \qquad (3.51)$$

which is implied by Eqs. (3.17) and (3.31). Thus, the common and unique variances of the modeled attributes can be obtained from the factor weights of an initial orthogonal solution. These values will not be affected by transformation of that solution.

Another simple property of an orthogonal factor solution involves the relationship between the off-diagonal entries in D_{DD} and the factor weights in E. Eq. (3.45) implies that the following relation will hold, for h not equal to *j*:

$$a_{hj} = \sum_{k=1}^{r} a_{hk} a_{jk} \qquad (3.52)$$

According to this equation, the covariance between modeled attributes *h* and *j* can be obtained by summing the products of the factor weights in rows *h* and *j* of E. This provides a simple method for determining the relations between modeled attributes, given an orthogonal factor solution.

It is important to recognize that the quantities defined in Eqs. (3.50) and (3.52) are not altered when the orthogonal solution is transformed. That is, the representation by the common factors of the variances and covariances of the modeled attributes is not altered when those factors are transformed according to the procedure described earlier in this section. This can be seen by noting that the elements of the matrix ($D_{DD} - U^2$) are not affected by the transformation defined in Eqs. (3.42)-(3.45), nor by the transformation given in Eqs. (3.47)-(3.48).

3.3. A Geometric Representation of the Common Factor Model

It is very interesting and useful to recognize that there exists a geometric representation of the common factor model, which corresponds to the algebraic representation presented in the previous two sections. This geometric framework will be described here. It will serve to aid in the understanding of the common factor model. It will also provide the basis for defining several additional types of information which are quite useful in representing a complete common factor solution.

Suppose we are given an orthogonal common factor solution characterized by a matrix E of factor weights. Consider a geometric representation of this solution defined by a Euclidean space, where the axes in the space correspond to the common factors. Thus, the space will be of dimensionality *r*. Let each modeled attribute be represented by a vector in the space, with the coordinates of the end-points of each factor weights for the corresponding modeled attribute. Thus, there will be n vectors in the space. Figure 3.1 provides an illustration of such a space for a case of two factors, designated A_1 and A_2, and two modeled attributes, designated V_h and V_j. The reader should keep in mind that this illustration is over-simplified in the sense that real world cases would involve more attributes (vectors), and very often more factors (dimensions). Nevertheless, this illustration will serve to demonstrate a number of important points. Note that the axes are orthogonal, meaning that the common factors are uncorrelated. The coordinates of the end-points of the two attribute vectors represent the factor weights for those attributes. These values would correspond to the entries in rows *h* and *j* of the factor weight matrix Ω. Such a matrix, along with other information referred to below, is shown in Table 3.1.

It is interesting to consider some simple geometric properties of these modeled attribute vectors. For instance, the squared of such a vector could be obtained by

$$h_j^2 = \sum_{k=1}^{r} \omega_{jk}^2 \qquad (3.53)$$

This equation can be recognized as equivalent to Eq. (3.50), which defines the common portion of the variance of each modeled attribute. Thus, it is seen that the common variance of each modeled attribute corresponds to the squared length of the corresponding vector. For the two attributes represented in Figure 3.1, these values are obtained as follows:

$$\begin{aligned} h_h^2 &= \omega_{h1}^2 + \omega_{h2}^2 \\ &= (.3)^2 + (.5)^2 \\ &= .34 \end{aligned}$$

$$\begin{aligned} h_j^2 &= \omega_{j1}^2 + \omega_{j2}^2 \\ &= (.8)^2 + (.4)^2 \\ &= .80 \end{aligned}$$

Recall that if the modeled attributes are taken to be standardized, then these values are communalities, representing the proportion of variance in each modeled attribute due to the common factors. For the unstandardized case, these values are actual variances due to the common factors. An obvious interpretation of these values is that modeled attributes which are more strongly associated with the common factors will be represented by longer vectors in the geometric model. Another interesting aspect of the attribute vectors is the scalar product. For each pair of vectors the scalar product is defined as

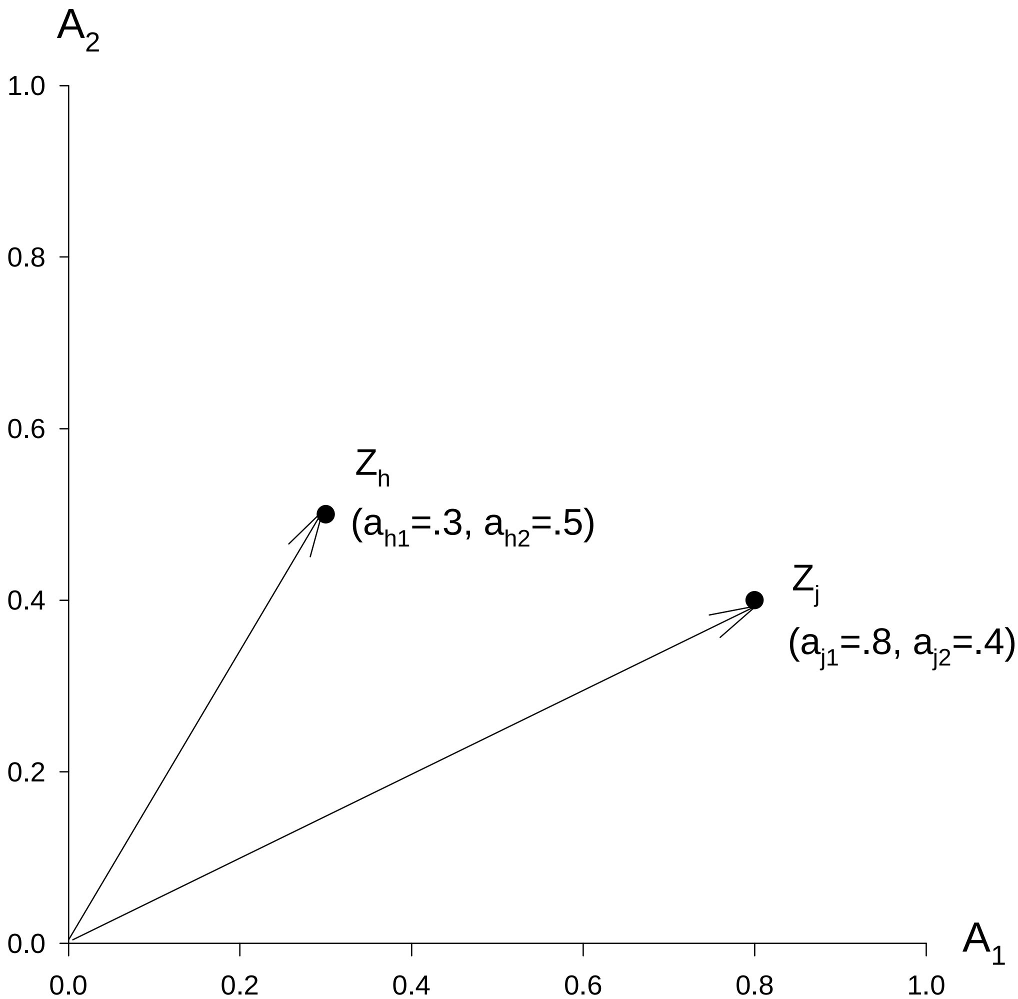

Figure 3.1: Geometric representation of two attribute vectors in a two-factor space

Table 3.1
Matrices for Illustration of Transformation of Two-factor Solution

Matrix A

$$
\begin{array}{c} Z_1 \\ \cdot \\ Z_h \\ \\ Z_j \\ \\ Z_n \end{array}
\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{bmatrix} & \\ & \\ .30 & .50 \\ .80 & .40 \\ & \\ & \\ & \end{bmatrix}
\quad (\text{columns } A_1, A_2)
$$

Matrix T

$$
\begin{array}{c} \\ T_1 \\ T_2 \end{array}
\begin{array}{cc} A_1 & A_2 \\ \end{array}
\begin{bmatrix} .96 & .28 \\ .60 & .80 \end{bmatrix}
$$

Matrix Φ=TT'

$$
\begin{array}{c} T_1 \\ T_2 \end{array}
\begin{bmatrix} 1.00 & .80 \\ .80 & 1.00 \end{bmatrix}
\quad (\text{columns } T_1, T_2)
$$

Matrix T^{-1}

$$
\begin{array}{c} A_1 \\ A_2 \end{array}
\begin{bmatrix} 1.33 & -.47 \\ -1.00 & 1.60 \end{bmatrix}
\quad (\text{columns } T_1, T_2)
$$

Matrix B=AT^{-1}

$$
\begin{array}{c} Z_1 \\ \cdot \\ Z_j \\ \cdot \\ \cdot \\ Z_n \end{array}
\begin{bmatrix} & \\ & \\ .66 & .26 \\ & \\ & \\ & \end{bmatrix}
\quad (\text{columns } T_1, T_2)
$$

Matrix Q=AT'

$$
\begin{array}{c} Z_1 \\ \cdot \\ Z_j \\ \cdot \\ \cdot \\ Z_n \end{array}
\begin{bmatrix} & \\ & \\ .88 & .80 \\ & \\ & \\ & \end{bmatrix}
\quad (\text{columns } T_1, T_2)
$$

Matrix D=(Diag(T'$^{-1}$T^{-1})$^{1/2}$

$$
\begin{array}{c} T_1 \\ T_2 \end{array}
\begin{bmatrix} .60 & .00 \\ .00 & .60 \end{bmatrix}
\quad (\text{columns } F_1, F_2)
$$

Matrix F=DT'$^{-1}$

$$
\begin{array}{c} F_1 \\ F_2 \end{array}
\begin{bmatrix} .80 & -.60 \\ -.28 & .96 \end{bmatrix}
\quad (\text{columns } A_1, A_2)
$$

Matrix G=AF'

$$
\begin{array}{c} Z_1 \\ \cdot \\ Z_j \\ \cdot \\ \cdot \\ Z_n \end{array}
\begin{bmatrix} & \\ & \\ .40 & .16 \\ & \\ & \\ & \end{bmatrix}
\quad (\text{columns } F_1, F_2)
$$

$$c_{hj} = \sum_{m=1}^{r} a_{hm} a_{jm} \qquad (3.54)$$

This equation can be recognized as being equivalent to Eq. (3.52), which expressed the covariance between modeled attributes h and j as a function of the factor weights. Thus, it is seen that the scalar product of two attribute vectors in the factor space corresponds to the covariance (or correlation, for standardized modeled attributes) between the two modeled attributes. For the example shown in Figure 3. 1, this value is obtained by

$$\begin{aligned} c_{hj} &= a_{h1}a_{j1} + a_{h2}a_{j2} \\ &= (.8)(.3) + (.4)(.5) \\ &= .44 \end{aligned}$$

Again, for the case of standardized modeled attributes, this value is a correlation. For the unstandardized case, it is a covariance.

Let us now consider the geometric representation of the transformation process defined in the previous section. Recall that the transformation of an orthogonal solution, such as that shown in Figure 3.1, to another solution (oblique or orthogonal) is based on first defining a non-singular matrix T of order $r \times r$, satisfying Eq. (3.47). It is possible to represent such a matrix directly in the geometric model being developed here. Consider the columns of T as corresponding to the original factors, and let each row of T be represented by a vector in the factor space, where the coordinates of the end-points of the vectors are given by the entries in T. Each entry t_{kl} represents the coordinate for vector k on original factor l. These vectors will be called <u>trait vectors</u>, or simply <u>traits</u>. They correspond to transformed factors, and these terms will be used interchangeably. Conceptually, the objective in factor analysis is to determine trait vectors which represent substantively meaningful internal attributes. The matrix T, whose row correspond to these trait vectors, will be referred to as a trait matrix. For illustrative purposes, such a matrix is show in Table 3.1. Figure 3.2 shows these trait vectors inserted into the factor space. The two vectors, designated T_1 and T_2 correspond to the rows of T. It is worth noting that, because the trait matrix satisfies Eq. (3.47), the trait vectors will be unit length vectors. Such vectors are referred to as normalized; similarly, it is said that the rows of T are normalized.

An interesting aspect of any pair of trait vectors is their scalar product. The scalar products of all pairs of trait vectors can be obtained by taking the matrix product TT'. Eq. (3.42) shows that the resulting scalar products will correspond to correlations among the transformed factors, or traits. For the traits represented in the illustration, the resulting matrix Φ is given in Table 3.1. The off-diagonal entry of .80 represents the correlation between the two transformed common factors, or traits. Since the vectors are unit length, this value also corresponds to the

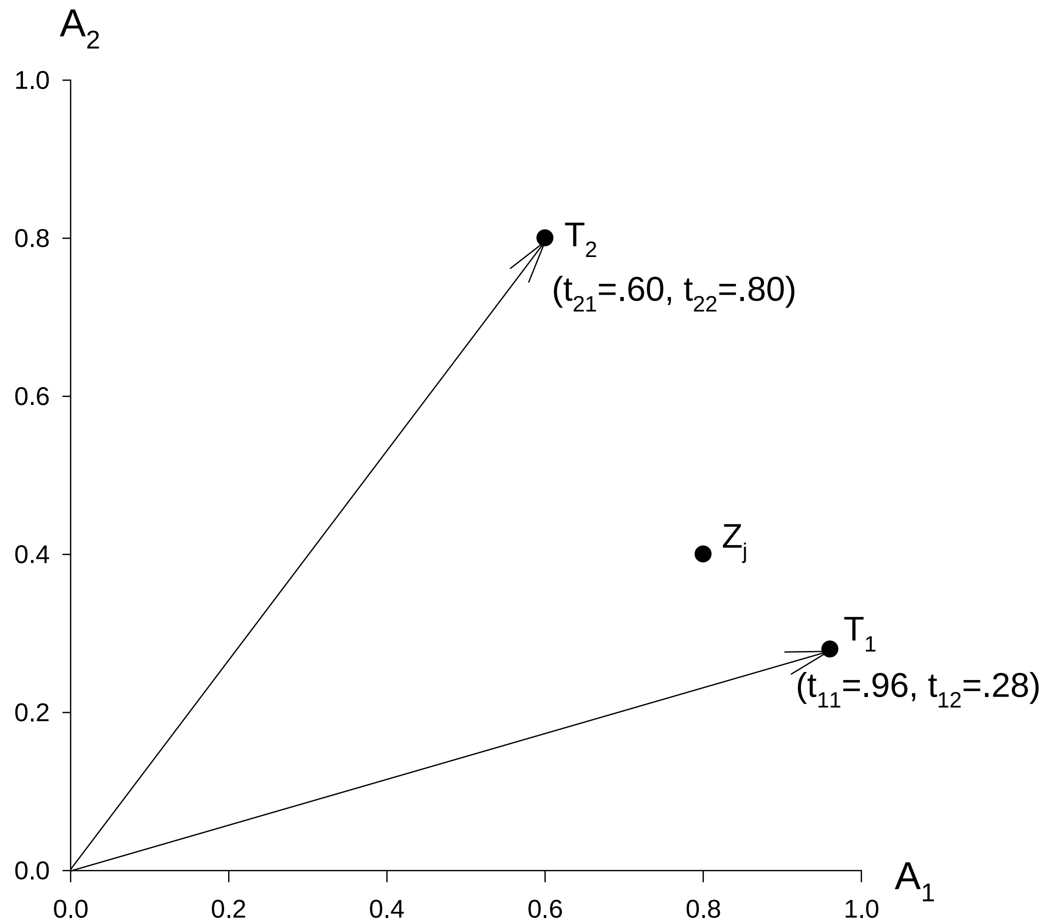

Figure 3.2: Geometric representation of two attribute vectors in a two-factor space

cosine of the angle between the two trait vectors in Figure 3.2.

We will next consider how to represent the relationships between the trait vectors and the modeled attributes. As we will show, this can be approached in a number of different ways. One approach has, in effect, already been defined. Eq. (3.48), repeated here for convenience, defines a transformation of factor weights from the initial orthogonal factors to the transformed oblique factors, or traits:

F œ EX ʷ

The resulting matrix F contains weights for the transformed factors on the modeled attributes. In the geometric model, these weights correspond to the Cartesian projections of the attribute vectors on the trait vectors. Each such weight "45 is analogous to a partial regression coefficient, representing the unique effect on modeled attribute *j* of factor *k*. The computation of these weights for one of the attributes is given in Table 3.1 for attribute *j*, and the projections are shown in Figure 3.3.

An alternative representation of the relationships of the traits to the modeled attributes can be obtained by determining scalar product, of the attribute vectors and the trait vectors. Algebraically, the scalar products would be given by

U œ EX ʷ Ð$Þ&&Ñ

Matrix U will have *n* rows representing the modeled attributes and *r* columns representing the traits. Each entry *q*45 will represent a scalar product, or covariance, between a modeled attribute and a trait. (Again, note that for the case of standardized modeled attributes, these values will be correlations between modeled attributes and traits). Geometrically, these values correspond to perpendicular projections of modeled attribute vectors on trait vectors. The computation of these values for one of the attribute vectors is shown in Table 3.1. and the corresponding projections are shown in Figure 3.3. These projections are analogous to simple measures of covariance (or correlation) between a modeled attribute and a trait, ignoring the fact that other traits are present.

Continuing further with the issue of evaluating relationships between modeled attributes and traits, it has been found very useful to define a second type of vector associated with the trait vectors. This second type of vector is based on the notion of a hyperplane. Consider a solution containing *r* correlated factors. For any given factor, the remaining factors can be conceived of as defining a space of (*r*-1) dimensions. This subspace is referred to as a hyperplane. For each factor, there exists a corresponding hyperplane made up of the other (*r*-1) factors. For each such hyperplane, let us define a unit-length vector orthogonal to that hyperplane. These vectors are called normals to hyperplanes, or more simply, just normals. These normals are equivalent to what Harman (1 967) and others call "reference vectors". Note that for each trait there will be a

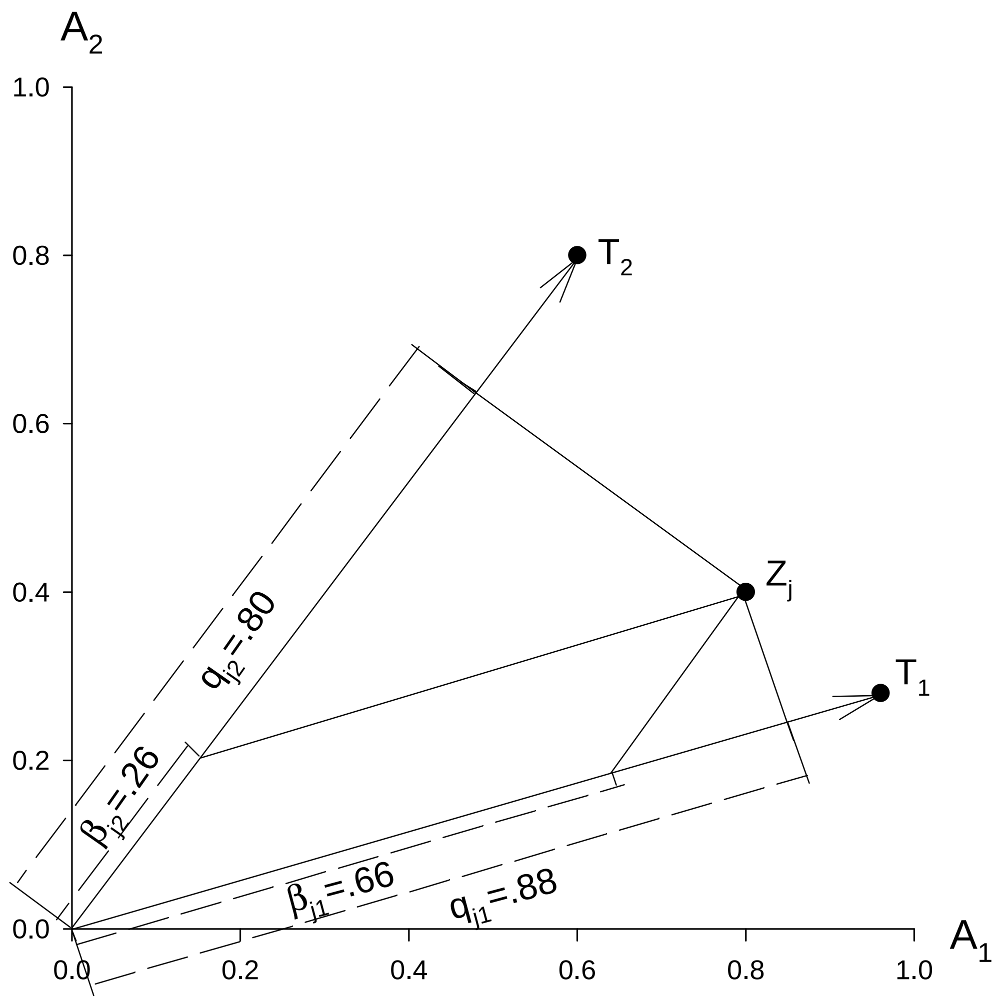

Figure 3.3: Geometric representation of projections of one attribute vector on two trait vectors

corresponding normal, defined as a vector orthogonal to the corresponding hyperplane. These normals have been used extensively in representing oblique common factor solutions.

Let us define a matrix J which contains these vectors as rows. Thus, the *r* columns of J correspond to the original orthogonal factors and the *r* rows represent the normals. An entry f_{56} represents the coordinate of the end-point of normal J_5 on original factor A_6. It is useful to consider the relationship between these normals and the traits, represented by the rows of X. Note that each normal is defined as orthogonal to all but one trait. Therefore, if we obtain the scalar products of the normals with the traits, as given by

$$X J' = H \qquad (3.56)$$

the resulting product matrix H, of order $r \times r$, must be diagonal. This will be the case because each normal is orthogonal to all but one trait, and vice versa. This interesting relationship can be employed to determine a method to obtain the matrix J of normals. Since the rows of J are normalized vectors, this implies

$$\mathrm{diag}(J J') = M \qquad (3.57)$$

Solving Eq. (3.56) for J' yields

$$J' = X^{-1} H \qquad (3.58)$$

Substituting from Eq. (3.58) into Eq. (3.57) yields

$$\mathrm{diag}(H X'^{-1} X^{-1} H) = M \qquad (3.59)$$

Solving Eq. (3.59) for H gives us

$$\begin{aligned} H &= [\mathrm{diag}(X'^{-1} X^{-1})]^{-\frac{1}{2}} \\ &= [\mathrm{diag}(X X')^{-1}]^{-\frac{1}{2}} \\ &= [\mathrm{diag}\, F^{-1}]^{-\frac{1}{2}} \end{aligned} \qquad (3.60)$$

This result is important because it provides a method to obtain matrix H, the diagonal matrix containing relations between traits and normals. The diagonal elements of H actually are equivalent to constants which serve to normalize the columns of X^{-1}, as shown in Eq. (3.58). Given the trait matrix X, H can be obtained according to Eq. (3.60). It can then be used to obtain J' according to Eq. (3.58). Matrix J then contains the normals as rows. These normals can be conceived of as an additional set of vectors inserted into the factor space. The calculation of H and J' is illustrated in Table 3.1, and the geometric representation of the normals, designated J_1, and J_2, is shown in Figure 3.4. Note in Figure 3.4 that normal J_1, is orthogonal to all traits other than X_1; i.e., it is orthogonal to X_2 in this limited illustration. Also, normal J_2 is orthogonal to all

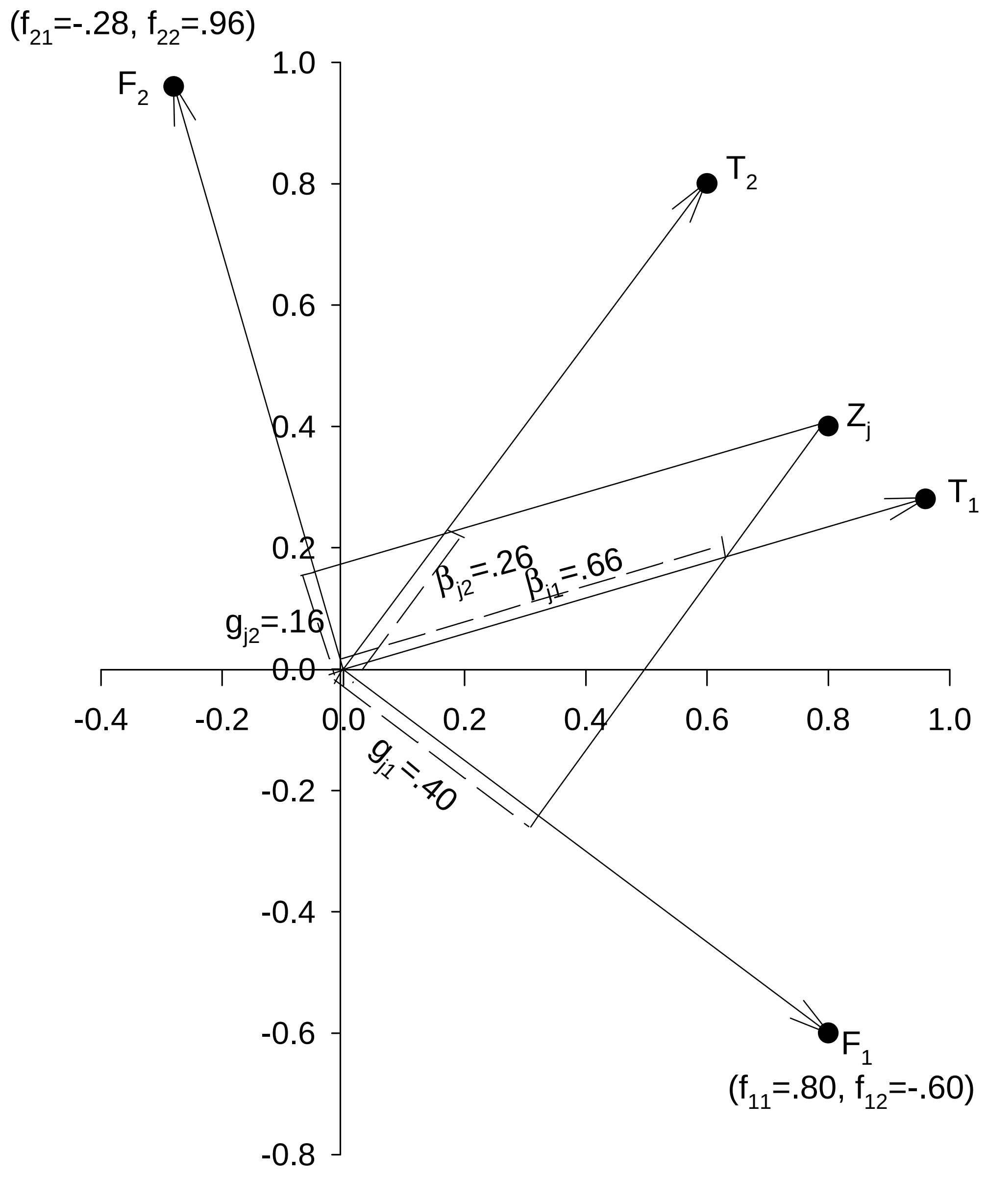

Figure 3.4: Geometric representation of projections of one attribute vector on two normals

traits other than X_2; i.e., it is orthogonal to X_1.

The utility of these normals can be seen when we consider the relationship of the modeled attributes to the normals. Let us define the scalar products of the modeled attributes with the normals as

$$K = EJ' \tag{3.61}$$

Matrix K will be of order $n \times r$, with rows representing modeled attributes and columns corresponding to the normals. Each entry g_{45} can be interpreted as a measure of a partialled relationship. Such an entry represents the partial covariance (or correlation in the standardized case) between modeled attribute *j* and trait *k*, with the effects of the other (*r*-1) traits partialled out. This interpretation is based on the definition of normals as being orthogonal to hyperplanes. As such, each normal represents a partialled trait; i.e., that portion of a given trait which remains after the effects of the other (*r*-1) traits are partialled out. The computation of these values is illustrated for a single attribute in Table 3.1. Geometrically, these values correspond to perpendicular projections of attribute vectors on normals. These projections for a single attribute are shown in Figure 3.4.

A very interesting relationship exists between the factor weights in matrix F and the projections on the normals, given in matrix K. Substitution from Eq. (3.58) into Eq. (3.61) produces

$$K = EX'^{-1}H \tag{3.62}$$

Then, by substituting into this equation from Eq. (3.48), we obtain

$$K = FH \tag{3.63}$$

According to this equation, the columns of K and F will be proportional. The constants of proportionality will be the diagonal elements of H. Thus, the factor weights for a given factor will be proportional to the projections on the normals for the corresponding factor. An examination of Figure 3.4 reveals the geometric basis for this relationship. Note that the projection of a given modeled attribute on a normal in this example is obtained by "continuing" the projection from the attribute, through the trait, to the normal. The Cartesian projections of the attribute vectors on the trait vectors were illustrated originally in Figure 3.3. These same projection are shown also in Figure 3.4, which illustrates how those projections are related to the projections of the attribute vectors on the normals. This gives rise to the proportionality defined in Eq. (3.63)

To summarize the developments presented in this section, begin with the assumption that an orthogonal solution characterized by matrix E has been obtained. Given such a solution, a transformation then can be carried out by defining a trait matrix X, the rows of which represent the traits, or transformed factors. One then can obtain, by equations presented above, matrices F, containing the correlations of the factors; F, containing the factor weights; U, containing the covariance of the modeled attributes with the factors; and K, containing the partialled covariances of the modeled attributes with the factors. In the geometric representation of the model, the modeled attributes are represented by vectors in a space whose axes correspond to the initial orthogonal factors. The transformed factors are represented by trait vectors, defined by the rows of matrix X. The correlations among the trait vectors, contained in matrix F, corresponds to cosines of angles between the trait vectors. The factor weights in matrix F correspond to Cartesian projections of the attribute vectors on the trait vectors. The covariances of the modeled attributes, with the trait vectors, contained in matrix U, correspond to perpendicular projections of the attribute vectors on the trait vectors. Finally, the partialled covariance of the modeled attributes with the trait vectors, contained in matrix K, correspond to projections of the attribute vectors on normals to the hyperplane defined by each trait vector. These matrices define an oblique common factor solution.

It is important to recognize the properties of such a solution when the traits themselves are orthogonal. That is, consider the case in which

$$XX^{w} = M \qquad (3.64)$$

Under this condition, X^{w} is equivalent to X . As a result, matrix F as defined in Eq. (3.48) will be equivalent to matrix U as defined in Eq. (3.55). In addition, the matrix H as defined in Eq. (3.60) will be an identity matrix. As a result, Eq. (3.63) shows that F and K will be equivalent. The implication of these relationships is that when the traits are orthogonal, there will be no distinction among matrices F, U, and K. All of the types of information contained in these matrices will be equivalent in this case. In geometric terms, this phenomenon is revealed by the fact that when the traits are orthogonal, there will be no distinction between Cartesian projections (as given in F) and perpendicular projections (as given in U) of the attribute vectors on the trait vectors. Furthermore, each normal will be equivalent to the corresponding trait vector, since the trait vectors themselves are orthogonal to the hyperplanes. Thus, the projections on the normals (given in K) will be equivalent to the projections on the trait vectors. This all demonstrates that an orthogonal solution is, in some senses, simpler than an oblique solution, since there are no trait correlations to consider and since the types of relationships between attributes and traits are all equivalent. However, as will be discussed in Chapter 9, this relative simplicity should not be considered to represent an advantage of orthogonal solutions over oblique solutions. The critical issue is to determine traits which correspond to substantively meaningful internal attributes, whether or not they are or orthogonal.

3.4. Effects of Factor Transformations on Factor Scores

A recurring theme in the previous three sections has been the transformation of factors. Mathematical and geometric frameworks have been described for representing the transformation of common factor weight matrices. It is very important that the reader recall the role of these factor weights in the common factor model. According to the model as defined in Eq. (3.7) or Eq. (3.39), these weights are weights which are applied to scores on the common factors. The resulting linear combination of common factor scores, combined with specific factor and error of measurement factor terms, yield the modeled attributes. Let us focus at this point on the common factor contribution to the modeled attributes. Given the nature of the model, it is clear that any transformation of the common factor weights must also imply a corresponding transformation of the common factor scores. To put this more simply, if some operation is carried out to alter the common factor weights in model expressed in Eq. (3.7) or Eq. (3.39), then the common factor scores must also be altered in order that the model still produce the same modeled attributes. In this section we will present a framework for representing this phenomenon, and we will consider a number of related issues of both theoretical and practical importance.

To begin, let us consider Eqs. (3.7) and (3.39). In these expressions of the model, the common factor contributions to the modeled attribute scores in $\underline{D}$ are contained in the product $\underline{B}_{"} F'$. Let us define a vector $\underline{D}_{-}$ as containing these contributions. That is

$$\underline{D}_{-} = \underline{B}_{"} F' \qquad (3.65)$$

As described in sections 3.2 and 3.3, let us consider F to be a transformed matrix of factor weights. Suppose that an initial orthogonal weight matrix E had been obtained, and that F is the result of transforming E according to Eq. (3.48). Substituting from Eq. (3.48) into Eq. (3.65) yields

$$\underline{D}_{-} = \underline{B}_{"} X'^{\bullet "} E' \qquad (3.66)$$

Let us define a vector $\underline{B}_{!}$ as follows:

$$\underline{B}_{!} = \underline{B}_{"} X'^{\bullet "} \qquad (3.67)$$

Substituting from this equation into Eq. (3.66) yields

$$\underline{D}_{-} = \underline{B}_{!} E' \qquad (3.68)$$

A comparison of Eqs. (3.65) and (3.68) indicates that these provide two different representations of the $\underline{D}_{-}$ scores, based on two different factor weight matrices. Thus, it can be seen that $\underline{B}_{"}$

contains the common factor scores corresponding to the weights given in F, and $\underline{B}_0$ contains the common factor scores corresponding to the weights given in E. Furthermore, Eq. (3.67) represents the transformation between the two sets of common factor scores.

It is interesting to consider also the covariance among the factor scores. The covariance among the factor scores in $\underline{B}_1$ are given by D_{11}, which is defined in Eq. (3. 10) as matrix F, the intercorrelation matrix for the common factors. In the present section, we will employ an additional level of subscripts and will designate this matrix as $D_{B_1B_1}$. Eq. (3.67) defines a linear transformation relating $\underline{B}_1$ to $\underline{B}_0$. Based on such a transformation, the relation between the covariance matrices for these two sets of common factor scores can be obtained by employing Corollary 3 of Theorem 4 in Appendix C. This yields the following:

$$D_{B_0B_0} = X^{-1} D_{B_1B_1} X'^{-1} \qquad (3.69)$$

Since $D_{B_0B_0}$ is designated as F, and since F is identified in Eq. (3.42) as being equal to XX', Eq. (3.69) can be rewritten as follows:

$$D_{B_0B_0} = X^{-1} X X' X'^{-1} \qquad (3.70)$$

This shows that the covariance matrix for the common factor scores corresponding to the orthogonal weight matrix E will be an identity matrix; i.e., these common factor scores will be uncorrelated.

A geometric representation of these relations is of interest. Let us consider a geometric framework of a different nature than that employed in section 3.3. In particular, let us define a factor score space in terms of the factor scores given in $\underline{B}_0$. The axes in this space represent the common factors, and each individual will be represented by a point in the space, with the coordinates of each individual's point being given by the common factor score vector, $\underline{B}_0$, for that individual. Equivalently, each individual can be thought of as being represented by a vector in the factor score space, with the coordinates of the endpoint of each individual's vector being defined by the scores for that individual on the common factors. Note that, according to Eq. (3.70) the distribution of points in the space will be such that the scores on the original axes will be standardized and uncorrelated. Let us next consider the representation of the scores on the transformed factors, given in $\underline{B}_1$. Employing Eq. (3.67), we can obtain the following relation:

$$\underline{B}_1 = \underline{B}_0 X' \qquad (3.71)$$

This equation indicates that the factors scores in $\underline{B}_1$ are scalar products of the scores in $\underline{B}_0$ with the trait vectors in X. Since the trait vectors are of unit length, these scalar products correspond to projections of the $\underline{B}_0$ vectors on the trait vectors. The correlations among the factor scores in

$\underline{B}_{"}$, which are given in F, are cosines of angles between the trait vectors. A point of considerable importance is that the relations among the factors are defined in this factor score space.

An interesting special case of the transformation problem is the transformation from one uncorrelated solution to a second such solution. Let us examine the impact of such a transformation on the factor scores and their interrelationships, employing the mathematical framework developed above. In this case, X will be defined such that

$$XX^{w} œ M \qquad (3.72)$$

This implies that

$$X^{w} œ X^{\bullet "} \qquad (3.73)$$

Let $E_{"}$ be the weight matrix for the first uncorrelated solution and $E_{\#}$ be the weight matrix for the second such solution. By Eq. (3.48)

$$E_{\#} œ E_{"} X^{w} \qquad (3.74)$$

Let the factor scores corresponding to $E_{"}$ be given by $\underline{B}_{!"}$, then by using Eqs. (3.68), (3.73), and (3.74), we can obtain the following:

$$\underline{D}_{-} œ \underline{B}_{!"} E_{"}^{w} œ \underline{B}_{!"} X^{w} E_{\#}^{w} \qquad (3.75)$$

Let the scores on the second solution be given by $\underline{B}_{!2}$, which according to Eq. (3.71), would be given by

$$\underline{B}_{!2} œ \underline{B}_{!"} X^{w} \qquad (3.76)$$

Employing Eqs. (3.75) and (3.76), we obtain the following:

$$\underline{D}_{-} œ \underline{B}_{!"} E_{"}^{w} œ \underline{B}_{!2} E_{\#}^{w} \qquad (3.77)$$

This equation simply shows that both sets of common factor scores and weights produce the same $\underline{D}_{-}$ values. It also can be shown that the covariance matrices for the two sets of factor scores have the expected form. Making use of the linear transformation defined in Eq. (3.74) along with Corollary 3 of Theorem 4 in Appendix C, we obtain the following:

$$D_{B_{+2}B_{+\#}} œ X D_{B_{+"}B_{+1}} X^{w} \qquad (3.78)$$

Since the initial solution $E_{"}$ is defined as orthogonal, we can write

$$D_{B_{+"}B_{+1}} œ M \qquad (3.79)$$

Substituting from Eqs. (3.79) and (3.72) into Eq. (3.78), we obtain

$$D_{B_{+2}B_{+\#}} = XX' = M \qquad (3.81)$$

Thus, both solutions are characterized by factor scores which are standardized and uncorrelated.

We wish to consider next a very interesting issue which often gives rise to substantial confusion in empirical applications of factor analysis. The issue concerns the nature and meaning of product matrices obtained by pre-multiplying factor weight matrices by their respective transposes. A common misconception is that the relations among the common factors are represented in such matrices. Let us first consider this issue in the context of the special case just described--the transformation of one orthogonal solution E_1 into a second such solution E_2. Let us define product matrices T_1 and T_2 as follows:

$$T_1 = E_1' E_1 \qquad (3.82)$$

$$T_2 = E_2' E_2 \qquad (3.83)$$

Substituting from Eq. (3.74) into Eq. (3.82) yields

$$T_2 = X T_1 X' \qquad (3.84)$$

The misconception involves the issue of whether the relations among the common factors are indicated by the entries in T_1 and T_2 . Suppose T_1 is diagonal and the diagonal entries are unequal (this is a property of a particular type of solution, called a principal factors solution, to be described in Chapter 7). According to Eq. (3.83), T_2 generally will not be a diagonal matrix. That is, even though the transformed solution represented by E_2 is an orthogonal solution, the product matrix T_2 generally will not be diagonal. This is a point which troubles many practitioners of factor analysis. How can E_2 be an orthogonal factor matrix when its columns are not "orthogonal"? The resolution of this apparent paradox was given earlier in this section: the orthoganality of the factors is defined in the factor score space, and is not defined by product matrices such as T_2. That is, the factor scores for the factors defined by E_2 are uncorrelated, as shown in Eq. (3.80), even though matrix T_2 is not diagonal.

Let us consider this issue in the more general context of transformations from uncorrelated factors to correlated factors. Eq. (3.71) defines the impact of such a transformation on the factor scores. The area of concern here is with the product matrices, which will be designated as follows:

$$T_{!} = E'E \qquad (3.85)$$

$$T_1 = F'F \qquad (3.85)$$

Employing Eq. (3.48), we can rewrite Eq. (3.85) as follows:

$$T_1 = X'^{-1}E'EX^{-1} \qquad (3.86)$$

Substituting from Eq. (3.84) into Eq. (3.86) yields

$$T_1 = X'^{-1}T_0X^{-1} \qquad (3.87)$$

This equation gives the transformation of T_0 to T_1. The inverse transformation is implied. The critical aspect of Eq. (3.87) is that it shows that there is no necessary, direct relation of T_1 to F. That is, matrix F defines the relations among the common factors, and the relations are not indicated in any way in matrix T_1.

An illustration of the distinction between T_1 and F involves a type of solution called an independent cluster solution. In this case, each modeled attribute has a non-zero weight on one and only one factor in B. In such a case, T_1 would be diagonal since the sums of products between columns of **B** would be zero. However, matrix F would not necessarily be diagonal. The independent clusters of attributes might be intercorrelated so that F would have non-zero off-diagonal entries.

To complete the discussion of this issue, let us consider four possible situations defined by combining the possibilities of F being diagonal or not diagonal. When both are diagonal, the solution would be a principal factors solution; this type of solution was mentioned earlier in this section and will be discussed in detail in Chapter 7. The other cases can be thought of as arising from various possible transformations of such a solution. The case where F is diagonal and T_1 is not diagonal would be produced by an orthogonal transformation of a principal factor solution. This is a legitimate view since any factor solution could be transformed into a principal factors solution. The case where F is not diagonal and T_1 is diagonal would be produced by certain special transformations from a principal factors solution, including a transformation to an independent cluster solution. Finally, the case where F is not diagonal and T_1 is not diagonal represents the more general case of transformation to a general oblique solution.

This completes our discussion of the effects of factor transformations on factors scores. Critical points to keep in mind are that transformations of factor weights imply corresponding transformations of factor scores., and that the relations among the factors are defined in the factor score space. It is a misconception to consider the relations among the factors as being defined by the factor weights themselves.

3.5. Correspondence Between the Model and the Real World

An issue emphasized in Chapter 1 and in the first section of the present chapter involves the correspondence between the common factor model and the real world. It is recognized that we do not expect the model to provide an exact and complete accounting for the variance and covariances of the surface attributes. This fact is represented in the mathematical framework by differentiating between the surface attributes (vector $\underline{C}$) and that portion of the surface attributes that is consistent with the common factor model (the "modeled attributes" in vector $\underline{D}$). According to the mathematical representation of the model defined in this chapter, the model does account for the variances and covariances of the modeled attributes. That is, the covariance matrix D_{DD} is a function of the parameters of the model, as shown in Eq. (3.35).

It is important to represent explicitly the fact that, though the model accounts for the variances and covariance of the modeled attributes, it will not necessarily do the same for the surface attributes themselves. This can be seen by defining a population covariance matrix D_{CC} for the surface attributes. Matrix D_{CC} will be of order n , n, with entries $\mathbf{5}_{24}$ representing the population covariance for surface attributes h and j. It is especially important to consider the relation between D_{CC} and D_{DD}. This can be done by making use of the fact, as defined in Eq. (3.1), that the surface attributes in $\underline{C}$ are actually sums of the component portions in $\underline{\mathbf{z}}$ and $\underline{\ddot{\mathbf{z}}}$. Appendix C treats the case where one vector of measures is defined as the sum of two other vectors of measures, and shows the relationship of the covariances of the summed measures to those of the component portions. By applying the relationship shown in Theorem 5 of Appendix C, we can write, for the present case, the following:

$$D_{CC} = D_{DD} + D_{D\ddot{D}} + D_{\ddot{D}D} + D_{\ddot{D}\ddot{D}} \tag{3.88}$$

From this equation it can be seen clearly that the population covariance matrix for the surface attributes is not, in general, equivalent to the population covariance matrix for the modeled portions of the attributes. More specifically, if we define a matrix $?_D$ as

$$?_D = D_{D\ddot{D}} + D_{\ddot{D}D} + D_{\ddot{D}\ddot{D}} \tag{3.89}$$

then Eq. (3.88) can be re-written as

$$D_{CC} = D_{DD} + ?_D \tag{3.90}$$

The matrix $?_D$ then represents that portion of the population covariances of the surface attributes that cannot be accounted for by the common factor model. It can then be seen that only when all entries in $?_D$ are zero will D_{CC} be equivalent to D_{DD}. In that situation, the model will exactly represent the population covariances for the surface attributes. But if some entries in $?_D$ are not zero, then D_{CC} will not be equivalent to D_{DD}, and the model will not hold exactly in the population. As the entries in $?_D$ deviate further from zero, the correspondence between the model and the population becomes weaker. This lack of correspondence between the model and the real world is a very important concept which is often overlooked by neglecting the distinction between C and D. We will refer to such lack of correspondence as model error, meaning that the representation of the real world by the model is in error to some degree. It must be understood that model error is present in the population and is completely separate from lack of fit arising from sampling. That is, even if there is no model error, meaning that the model holds exactly in the population, there still will be sampling error. That is, the model probably would not fit a sample covariance matrix exactly. This issue will be dealt with further in Chapters 4 and 5.

A final step in developing the common factor model in the population now can be achieved by substituting from Eq. (3.35) into Eq. (3.90). This yields

$$D_{CC} \text{ œ } F\, F\, F^{w} \text{ € } Y^{\#} \text{ € } ?_D \qquad \text{Ð\$Þ*"Ñ}$$

This equation represents the milestone of expressing the common factor model in terms of the population covariances of the surface attributes. This expression represents these covariances as a function of the parameters in F, F, and $Y^{\#}$, plus a term representing model error. Most presentations of the model ignore the need to explicitly include model error.

The presence of model error can serve as a basis for introducing the concept of fitting the model to a covariance matrix. Considering the hypothetical case where a population covariance matrix D_{CC} is available, suppose we wish to obtain a solution for the model as represented in Eq. (3.91). Recognizing that $?_D$ represents the lack of fit of the model to D_{CC}, the objective would be to obtain a solution for the model which, in some sense, minimizes the entries in $?_D$. Such a solution would be optimal in the sense of providing the most accurate accounting of the population covariances of the surface attributes. As it will be seen in Chapter 7, there are different ways to define the optimal solution; i.e., different ways to define a criterion representing an optimal $?_D$. These alternative definitions of an optimal solution in turn lead to alternative techniques for obtaining a solution to the model, and these alternative techniques can yield different solutions. In other words, there may be no single correct solution. Rather, different definitions of an optimal solution yield different F, F, and $Y^{\#}$ matrices, and, in turn, different D_{DD} and $?_D$ matrices. Only when a solution can be found for which all entries in $?_D$ are zero can it be said that the model is consistent with the real world; in that case, there would be no model error.

Surely, however, almost all acceptable models will not fit the real world perfectly in the population. To insist on a perfect fit at all times is a route to chaos by the inclusion of many small, trivial factors. To be sure, we should avoid missing small factors which can be made large and important with special studies. All experimenters should be alert to this possibility. The distinction between trivia and important small influences is a matter for experimenter insight and judgment. However, the complexities of factor analysis make it imperative that a distinction be made. Inclusion of a number of very small factors in an analysis results in unmanageably large dimensionality of the common factor space. Great care is required of an experimenter in making the decision between possibly meaningful factors and trivia. There should be no doubt but that some trivia will exist.

3.6. The Common Factor Model for a Population Correlation Matrix

The development of the common factor model in this chapter has been carried out in the context of covariance matrices. That is, the fundamental reputation of the model given in Eq. (3.91) was stated in terms of a population covariance matrix for the surface attributes. Since many applications of factor analysis are conducted using correlation matrices rather than covariance matrices, it is important to consider how the expression of the model is affected when the data are correlations rather than covariances. This can be achieved easily by employing the simple relation between a covariance matrix and a correlation matrix. Given a population covariance matrix for the surface attributes, Σ_{yy}, let us define a diagonal matrix $[\Sigma_d]_{yy}$ containing the population variances of the surface attributes. That is,

$$[\Sigma_d]_{yy} = \mathrm{Diag}(\Sigma_{yy}) \qquad (3.92)$$

We can then define a population correlation matrix for the surface attributes, P_{yy}, as follows:

$$P_{yy} = [\Sigma_d]_{yy}^{-1/2}\,\Sigma_{yy}\,[\Sigma_d]_{yy}^{-1/2} \qquad (3.93)$$

The effect of this standardization of the surface attributes on the common factor model can then be seen by substituting from Eq. (3.91) into Eq. (3.93), yielding

$$\begin{aligned} P_{yy} &= [\Sigma_d]_{yy}^{-1/2}(F\Phi F' + \Psi^2 + \Delta_{\Sigma})[\Sigma_d]_{yy}^{-1/2} \\ &= [\Sigma_d]_{yy}^{-1/2} F\Phi F' [\Sigma_d]_{yy}^{-1/2} + [\Sigma_d]_{yy}^{-1/2}\Psi^2[\Sigma_d]_{yy}^{-1/2} \\ &\quad + [\Sigma_d]_{yy}^{-1/2}\Delta_{\Sigma}[\Sigma_d]_{yy}^{-1/2} \end{aligned} \qquad (3.94)$$

To simplify this expression, let us define the following matrices:

$$F^{*} = [\Sigma_d]_{yy}^{-1/2} F \qquad (3.95)$$

$$\Psi^{*2} = [\Sigma_d]_{yy}^{-1/2}\Psi^2[\Sigma_d]_{yy}^{-1/2} \qquad (3.96)$$

$$\Delta_P = [\mathbf{D}_{\Sigma_{yy}}]^{-1/2}\,\Delta_{\Sigma}\,[\mathbf{D}_{\Sigma_{yy}}]^{-1/2} \quad (3.97)$$

Substituting from these three equations into Eq. (3.94) yields

$$P_{yy} = F^{\ddagger}\Phi F^{\ddagger\prime} + \Psi^{\ddagger 2} + \Delta_P \quad (3.98)$$

This is an expression for the common factor model in terms of a population correlation matrix for the surface attributes. It has the same general form as Eq. (3.91), which expressed the model in terms of a population covariance matrix, but some of the terms in the equation have been rescaled. Note that, according to Eq. (3.95), the common factor weights obtained from a population covariance matrix would be rescaled in that each row of F would be divided by the population standard deviation of the corresponding surface attribute to yield the rows of $F^{\ddagger}$. Matrix $F^{\ddagger}$ would contain the common factor weights obtained from a population correlation matrix or, equivalently, from surface attributes standardized in the population. Note that the common factor intercorrelations, given in Φ, are not affected by this standardization. The unique variances, though, do undergo a rescaling, as shown in Eq. (3.96). The unique variances resulting from this rescaling represent population unique variances that would be obtained from a factor analysis of a population correlation matrix. Based on Eq. (3.16), corresponding population common variance, would be defined as follows:

$$H^{\ddagger 2} = \mathrm{Diag}(F^{\ddagger}\Phi F^{\ddagger\prime}) \quad (3.99)$$

Employing Eqs. (3.16) and (3.95), this could be rewritten as follows:

$$H^{\ddagger 2} = [\mathbf{D}_{\Sigma_{yy}}]^{-1/2}\, H^{2}\,[\mathbf{D}_{\Sigma_{yy}}]^{-1/2} \quad (3.100)$$

This shows how the population common variances are rescaled when they are obtained from a factor analysis of a population correlation matrix rather than a covariance matrix. It is important to understand the distinction between two sets of matrices: $\Psi^{\ddagger 2}$ and $H^{\ddagger 2}$ defined in Eqs. (3.96) and (3.100), versus $\tilde{\Psi}^2$ and $\tilde{H}^2$ defined in Eqs. (3.33) and (3.25). Both sets of matrices represent rescaling of Ψ^2 and H^2, which contain the population unique and common variances, respectively. However, the rescalings are slightly different. Matrices $\tilde{\Psi}^2$ and $\tilde{H}^2$ are obtained by dividing entries in Ψ^2 and H^2 by the population variances of the modeled attributes, thus yielding the population uniquenesses and communalities. On the other hand, matrices $\Psi^{\ddagger 2}$ and $H^{\ddagger 2}$ are obtained by dividing entries in Ψ^2 and H^2 by the population variances of the surface attributes. This operation merely rescales the variance components so that they represent surface attributes standardized in the population. It is interesting to note that in general $\Psi^{\ddagger 2}$ and $H^{\ddagger 2}$ will not sum to an identity matrix because some of the population variance of the standardized surface attributes will arise from error of fit of the model. Finally, to complete the discussion of

Eq. (3.98) above, the matrix $?_V$ represents model error in the population correlation matrix and is defined by Eq. (3.97) via a rescaling of $?_D$.

In sum, these developments show that the analysis of population correlation matrices rather than population covariance matrices results in a rescaling of factor weights, unique variances, and the model error term, but does not affect common factor intercorrelations. Further theoretical and practical implications of the issue of analysis of correlation vs. covariance matrices will be discussed in subsequent chapters. We will continue to present major developments in the context of covariance matrices and to consider the analysis of correlation matrices as a special case.

3.7. The Components Analysis Model

In the previous sections in this chapter we have presented the algebraic and geometric representation of the common factor model. There exists a variation of this model which is commonly used in practice. This alternative approach is called components analysis. Though components analysis is often viewed as a variety of factor analysis, and in fact is often called factor analysis in the applied literature, it is important to recognize it as a distinct model and to understand how it is different from the common factor model.

In mathematical terms, the components analysis model can be defined by eliminating specific and error of measurement factors from the common factor model. Thus, if we delete the specific and error of measurement factors from Eq. (3.7), we obtain

$$\underline{D} = \underline{B} F' \qquad (3.72)$$

In terms of covariance matrices, we could write an oblique components model bymodifying Eq. (3.35) to obtain

$$D_{DD} = F F F' \qquad (3.73)$$

In a similar fashion, Eq. (3.45) could be modified to yield an orthogonal components model as follows:

$$D_{DD} = E E' \qquad (3.74)$$

As represented in Eq. (3.72) the components model states that modeled attributes are linear combinations of *r* underlying dimensions. These dimensions will be called components and must be distinguished from common factors. Recall that a common factor represents a substantively meaningful internal attribute which affect more than one attribute in the battery and which partially explains why surface attributes are correlated. It is essential to recognize that the mathematical representation of common factors requires that specific and error of measurement factors be represented separately, since the latter represent real phenomena which influence the attributes in a manner different from the common factors. Since the components model does not provide for these specific and error of measurement factors, the components in this model cannot be considered to be conceptually the same thing as common factors. This is due to the fact that the components will represent not only common variance, but also specific and error of measurement variance. Furthermore, it will not be possible to differentiate these various influences in a components solution.

Nevertheless, components analysis has often been used in practice and viewed as equivalent to, or at least an acceptable substitute for, common factor analysis. Supporters of this position tend to base their view on several points. The relative simplicity of the components model often is considered to be an advantage over factor analysis. The absence of specific and error of measurement factors from the model simplifies the process of fitting the model to the data in that it eliminates the need to estimate the unique influences. In addition, it is often argued that components analysis solutions are often found to be quite similar to common factor solutions. Finally, the major argument used for support of the components model is that it is possible to determine exactly the measures for the individuals on the components (i.e., the entries in $\underline{B}$ in Eq. (3.101). This is not the case in the common factor model; that is, the measures for the individuals on the common factors, contained in vector $\underline{B}$ in Eq. (3.7), cannot be determined exactly. These scores are said to be "indeterminate", and can only be estimated. Component scores, on the other hand, are determinate. These arguments have led many researchers to view components analysis as holding advantages over common factor analysis.

Our view, however, is that these arguments carry very little impact. Let us consider them in turn. First, though it is true that the components model is simpler, it must be recognized that this simplicity is gained at the cost of the realism of the model. The common factor model offers a much more realistic representation of attributes in that it explicitly recognizes and incorporates the existence of specific and error factors. The fact that such influences are ignored in the components model yields a simpler but very unrealistic model. Furthermore, this simpler model yields dimensions which, as pointed out above, cannot be conceived of as representing substantively meaningful common factors. Considering the argument that components solutions often closely approximate common factor solutions, this will be the case <u>under certain conditions</u>. It can be seen easily by comparing Eqs. (3.102) and (3.103) to Eqs. (3.36) and (3.45), respectively, that as the unique variances of the attributes become smaller the two models become more similar. When unique variances are zero, the two models are equivalent. However, when substantial unique variances are present, solutions obtained from the two models may not be very similar. Thus, the argument that solutions from the two models are quite similar will not hold under all conditions. Finally, considering the issue of indeterminacy, we do not consider this as a fatal flaw in the common factor model for two reasons. First, as emphasized in Chapter 1, the objective of obtaining measures for the individuals on the common factors is not, in our view, a primary objective of factor analysis. The primary objective is to obtain an understanding of the nature of the common factors and the manner in which they influence the attributes. Given this view, the fact that the measures on the common factors are indeterminate is not a serious problem. Furthermore, as will be shown in Chapters 15 and 16, when a researcher does have a legitimate need for the scores on the common factors, there are procedures for estimating those scores, and to estimate them in such a way as to negate some of the effects of indeterminacy.

In summary, our view is that the components model is unrealistic and is not an acceptable substitute for the common factor model. Though there is an indeterminacy problem in the common factor model, this problem is not serious and can be dealt with when common factor scores are desired. This problem does not involve the central objective of factor analysis. In general then, we strongly oppose the use of components model and do not recommend it in practice. We support the use of the common factor model because it is based on a more realistic representation of the structure of the attributes and their relationships and because it offers a better chance to obtain an undergoing of the nature of the common factors and their effects on the attributes.

3.8. The Issue of Linearity

The common factor model is commonly referred to as a linear model. It is most important to achieve an understanding of the sense in which the model is linear. Let us consider three types of relationships which are relevant to the model: relationships of modeled attributes to each other; relationships of factors to each other; and relationships of modeled attributes to factors.

As defined in Eq. (3.7), the model states that the modeled attributes are linear combinations of the factors. If, in the real world, the surface attributes are non-linear functions of the factors, these relationships cannot be explicitly represented by the linear common factor model. In such a situation, the population covariance matrix for the surface attributes, D_{CC}, could not be perfectly represented by the model. That is, given D_{CC}, it would not be possible to find a solution such that all entries in matrix $?_{\mathbf{D}}$, defined in Eq. (3.89), would be zero.

It is important to recognize, however, that the model makes no statement, either implicit or explicit, about the linearity of the relationships of the attributes to each other, or the factors to each other. In fact, the relationships can be non-linear. It is possible for the attributes to have non-linear relationships to each other, and for the factors to have non-linear relationships to each other, and for the model still to exactly represent the population covariance matrix for the surface attributes. This would simply require that the surface attributes be linear functions of the factors.

Two demonstrations will be presented to illustrate the points made here. The general procedure used in these demonstrations involves first constructing artificial data representing a population. That is, a population covariance matrix for surface attributes will be constructed such that the underlying relationships among attributes and factors have the desired properties of linearity or non-linearity. A common factor solution then will be obtained (by procedures to be described in Chapter 7) which fits the population covariance matrix as well as possible. It is not necessary for the reader to understand, at this point, the methodology for obtaining such a solution. The key issue in these demonstrations will be to determine whether or not the obtained solution fits the population covariance matrix perfectly; i.e., whether the matrix $?_D$ contains all zeroes. If this is the case, then the linear factor model holds in the population; if not, then the relationships among the surface attributes cannot be represented exactly by the linear factor model.

The first demonstration is designed to show the effects of curvelinear relations among factors and among attributes. A population distribution of measures on two common factors was defined such that the relationship between the factors was curvelinear. This distribution is shown in Figure 3.5. In this figure, the axes represent two common factors. Each individual in the population was defined as falling at one of 25 discrete points, represented by the centers of the circles in the space. The values in the circles represent relative frequencies of observations at each corresponding point. The resulting distribution represents a population distribution of measures on two factors which are related in a curvelinear fashion. Scores on both factors had a mean of .644 and a standard deviation of .302; note that these original factors are unstandardized. The correlation between these factors was -.762. Each observation draw from this distribution would provide a vector $\underline{B}$ of measures on common factors.

Next a population factor weight matrix F was defined. This matrix is presented in Table 3.2. Matrix F contains weights for the two unstandardized factors on 10 artificially defined attributes. Table 3.2 also includes the factor weight matrix which would be applied to standardized factor scores. These weights are obtained by multiplying each column of original factor weights by the standard deviation of the factor (.302). It is important to note that the weights were defined in such a way that attribute 1 is equivalent to factor 1 and attribute 10 is equivalent to factor 2. As a result, the relationship of attributes 1 and 10 will be identical to that represented in Figure 3.5. That is, in addition to there being a curvelinear relationship between the factors, there also will be a curvelinear relationship between attributes 1 and 10 (as well as other pairs of attributes).

Measures on two common factors for a hypothetical population of 100 individuals were generated so as to follow the distribution shown in Figure 3.5. Population measures on the surface attributes then were obtained according to Eq. (3.7). For simplicity, no specific or error

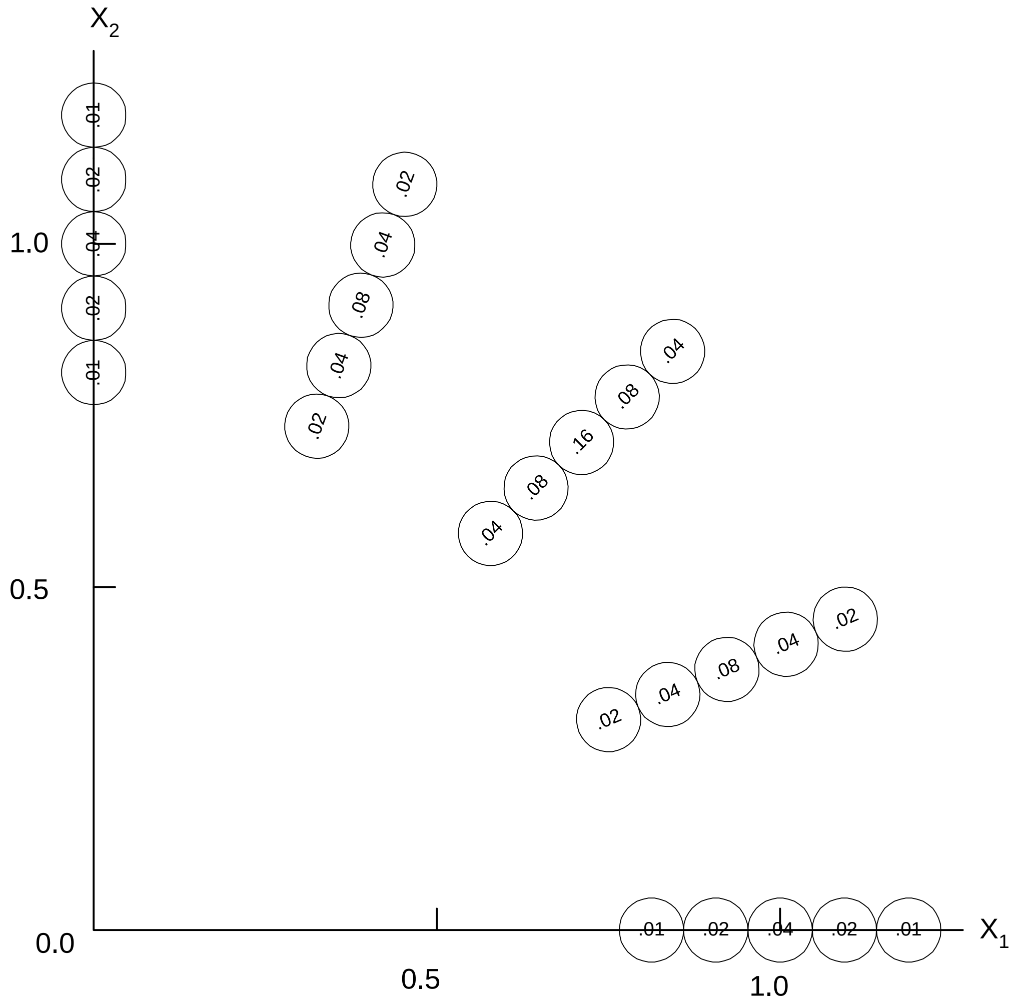

Figure 3.5: Population distribution of measures on two factors showing curvelinear relation between factors

Table 3.2

Input Factor Weight Matrix

for Demonstration of Curvelinear Factor Relations

	Unstandardized Factors		Standardized Factors	
Attribute	1	2	1	2
1	1.000	.000	.302	.000
2	1.000	.000	.302	.000
3	.960	.280	.290	.084
4	.800	.600	.241	.181
5	.800	.600	.241	.181
6	.710	.710	.214	.214
7	.600	.800	.181	.241
8	.280	.960	.084	.290
9	.000	1.000	.000	.302
10	.000	1.000	.000	.302

factors were included in the demonstration. Thus, the surface attributes were defined as linear combinations of the common factors, with the weights given in matrix F. Next, a population covariance matrix for the surface attributes was computed from the population of measures on those attributes. The resulting matrix D_{CC} is presented in Table 3.3. Note that this covariance matrix has been produced from a case where there exist curvelinear relations among factors and among attributes, but the relations of the attributes to the factors are linear.

A factor solution then was obtained (via procedures to be discussed in Chapter 7). This yielded a solution containing two factors. The resulting orthogonal factor weight matrix is shown in Table 3.4. Considering this as matrix E, it was found that the resulting matrix D_{DD} produced via Eq. (3.45), recalling that unique variances are zero in this demonstration, was identical to the population covariance matrix D_{CC} in Table 3.3. In other words, the matrix $?_D$, whose entries represent lack of correspondence between the model and the population, contained all zeroes. Furthermore, it was found that the orthogonal solution represented by matrix E in Table 3.4 could be transformed to match the original input factor weights given in Table 3.2. Note that since the obtained factor weights represent standardized factors, the transformation of the obtained solution should yield the input weights for standardized factors. Following the procedures in section 3.3 above, this transformation was achieved by passing the first trait vector and the second trait vector through the last attribute vector. The resulting trait matrix X is shown in Table 3.4. Based on this trait matrix, we then obtained the intercorrelation matrix F and the weight matrix F for the transformed factors by Eqs. (3.42) and (3.44), respectively. The resulting matrices are shown in Table 3.4 and can be seen to be identical to the corresponding input information. These findings of perfect fit and exact recovery of the input factors indicate that the linear factor model holds exactly in this population, despite the presence of curvelinear relations among factors and among attributes. The fact that the relations of the attributes to the factors are linear allows for this perfect correspondence between the factor model and the population.

The second demonstration involves the case of curvelinear relations of attributes to factors. This demonstration makes use of "binary" attributes; i.e., attributes which can take on only two possible values. The most common example of such an attribute is a test item, where responses to the item are scored as correct (1) or incorrect (0). Such items are very commonly used, and are normally considered to be surface measures of underlying common factors. For instance, a single addition item on a test could be considered as a binary measure of an underlying continuous internal attribute called numerical facility. An interesting aspect of this view is that relations of binary surface attributes to common factors must be considered to be curvelinear. This is due to the nature of the binary attributes. A linear relationship of such an attribute to a common factor would necessarily result in values of the binary attribute outside the

Table 3.3

Population Covariance Matrix for 10 Attributes

for Demonstration of Curvelinear Factor Relations

Attribute	1	2	3	4	5	6	7	8	9	10
1	.091									
2	.091	.091								
3	.068	.068	.054							
4	.031	.031	.030	.024						
5	.031	.031	.030	.024	.024					
6	.015	.015	.019	.022	.022	.022				
7	-.001	-.001	.008	.018	.018	.022	.024			
8	-.041	-.041	-.020	.008	.008	.019	.030	.054		
9	-.069	-.069	-.041	-.001	-.001	.015	.031	.068	.091	
10	-.069	-.069	-.041	-.001	-.001	.015	.031	.068	.091	.091

Table 3.4

Obtain Factor Solution for Demonstration of Curvelinear Factor Relation

Weight Matrix A

	1	2
1	.301	-.021
2	.301	-.021
3	.229	.039
4	.111	.110
5	.111	.110
6	.061	.135
7	.008	.156
8	-.123	.197
9	-.216	.211
10	-.216	.211

Trait Matrix T

	1	2
1	.997	-.071
2	-.071	.699

Factor Intercorrelation Matrix Φ=TT’

	1	2
1	1.000	-.762
2	-.762	1.000

Tranformed Weight Matrix $B=AT^{-1}$

	1	2
1	.302	.000
2	.302	.000
3	.290	.084
4	.241	.181
5	.241	.181
6	.214	.214
7	.181	.214
8	.084	.290
9	.000	.302
10	.000	.302

permissible values (e.g., 0 and 1). Thus, the view that a binary surface attribute is a measure of an underlying continuous factor implies that the relationship of the attribute to the factor must be curvelinear.

To demonstrate the effects of such relationships, an artificial population covariance matrix was constructed for 10 binary items. The items were defined as varying in difficulty, where difficulty is defined as the proportion of the population which provides the correct response to the item. Thus, lower values of this index represent more difficult items. Item difficulties for the ten items are shown in Table 3.5. Each of these items was considered to be a measure of a single common factor. Thus, the theoretical weights for the factor on the items were all unity. This implies that the theoretical correlations among these 10 items would all be unity. Given that these items would form a perfect Guttman scale (i.e., if an individual produces the correct response to a given item, the individual will also produce the correct response to all easier items), it is possible to obtain the actual population correlation matrix for the items. This matrix, shown in Table 3.5, contains phi-coefficients representing the actual relationship for each pair of binary items.

The correlation matrix in Table 3.5 represents a standardized version of a covariance matrix D_{CC} for surface attributes. Factor analysis procedures were applied to this matrix to determine whether a solution could be obtained which fit the covariances among these items perfectly. Despite the fact that the items are defined as being measures of a single factor, a one-factor solution does not provide perfect fit to D_{CC}. In fact, D_{CC} could not be fit perfectly using any small number of factors. For illustrative purposes, a two-factor orthogonal solution is shown in Table 3.6. This solution fits fairly well; that is, the elements of the resulting $?_D$ matrix are not large. However, the important point is that it does not fit perfectly. The reason for this is that the linear model cannot account for the relationships represented in D_{CC} because these relationships arise from an underlying structure where the effects of the factors on the surface attributes are curvelinear.

It is interesting to note the pattern of weights found in the solution shown in Table 3.6. Factor 1 seems to differentiate items of medium difficulty from items of extreme difficulty (i.e., either very difficult or very easy). Factor 2 shows a continuum of weights from low to high according to the difficulty of the items. Such factors are routinely observed when factor analyzing binary data in practice, and have often been called "difficulty factors". For some time they have been thought to be artifacts related to the presence of items of varying difficulty (Gourlay, 1951; Guilford, 1941; Wherry & Gaylord, 1951).. However, they have been recognized more recently as arising from non-linearities in the relationships of the items to the factors (Gibson, 1959, 1960; McDonald, 1965a). The general point to recognize is that when

Table 3.5
Item Difficulties and Intercorrelations for Demonstration of
Curvelinear Relations of Attributes to Factors

Item Difficulties

Item	1	2	3	4	5	6	7	8	9	10
Difficulties	.05	.15	.25	.35	.45	.55	.65	.75	.85	.95

Item Intercorrelations

Item	1	2	3	4	5	6	7	8	9	10
1	1.000									
2	.546	1.000								
3	.397	.728	1.000							
4	.313	.572	.787	1.000						
5	.254	.464	.638	.811	1.000					
6	.208	.380	.522	.664	.818	1.000				
7	.068	.308	.424	.538	.664	.811	1.000			
8	.132	.243	.333	.424	.522	.638	.787	1.000		
9	.096	.176	.243	.308	.380	.464	.572	.728	1.000	
10	.053	.096	.132	.168	.208	.254	.313	.397	.546	1.000

Table 3.6
Two-Factor Solution for Demonstration of
Curvelinear Relations of Attributes to Tactors

	Factor	
Item	1	2
1	.345	-.330
2	.590	-.489
3	.742	-.486
4	.813	-.318
5	.840	-.109
6	.840	.106
7	.814	.315
8	.744	.483
9	.591	.487
10	.346	.329

there exist curvelinear relations of attributes to factors, the linear factor model is not appropriate. If used, it will yield artifactual factors arising from non-linearities in the relations of attributes to factors.

To summarize the issue of linearity in the common factor model, it should be understood that the model assumes that the relations of the attributes to the factors are linear, but allows for non-linear relations of attributes to each other, or of factors to each other. The model can be used effectively when non-linear relations exist among attributes or among factors, as long as relations of the attributes to the factors are linear. Of course, in the real world the relations of the attributes to the factors in a given domain almost surely would not be exactly linear. It is likely that such relations would be at least slightly nonlinear in many cases. However, as long as these relations are approximately linear, the linear common factor model generally would provide an adequate and useful representation of the relations. But when the relations of the attributes to the factors are severely nonlinear, problems arise in the use of the model. As a result of the recognition of this phenomenon, non-linear varieties of factor analysis have been developed (e.g., McDonald, 1967). However, such developments are beyond the scope of the present text, which focuses on the linear common factor model.