

# THOM SERIES OF CONTACT SINGULARITIES

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*Dedicated to Jim Damon*

ABSTRACT. Thom polynomials measure how global topology forces singularities. The power of Thom polynomials predestine them to be a useful tool not only in differential topology, but also in algebraic geometry (enumerative geometry, moduli spaces) and algebraic combinatorics. The main obstacle of their widespread application is that only a few, sporadic Thom polynomials have been known explicitly. In this paper we develop a general method for calculating Thom polynomials of singularities. Along the way, relations with the equivariant geometry of (punctual, local) Hilbert schemes, and with iterated residue identities are revealed.

## 1. INTRODUCTION

For a holomorphic map  $F : N^n \rightarrow P^p$  between compact complex manifolds one can consider the set of points  $\eta(F)$  in the source manifold  $N$  where the map has a certain kind of singularity  $\eta$ . The Thom polynomial  $\text{Tp}(\eta)$  of  $\eta$  is a multivariate polynomial depending only on  $\eta$ , with the property that the cohomology class represented by the closure of  $\eta(F)$  is equal to the specialization of  $\text{Tp}(\eta)$  at the characteristic classes of  $c_i(N)$ ,  $F^*(c_i(P))$ . For this statement to hold, the map  $F$  must satisfy transversality conditions. There is an analogous theory for real smooth maps, where one studies a polynomial of the Stiefel-Whitney classes of  $TN$  and  $F^*TP$  expressing  $[\eta(F)] \in H^*(N; \mathbf{Z}_2)$ . These real Thom polynomials can be calculated from the complex Thom polynomials [BH61], hence we restrict our study to the complex case.

We must specify what the singularity  $\eta$  means. For the definition of  $\eta(F)$  to make sense,  $\eta$  must be a subset of the vector space of holomorphic germs  $\mathcal{E}_0(n, p) = \{(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^p, 0)\}$ , invariant under the action of the holomorphic reparametrization groups of  $(\mathbf{C}^n, 0)$  and  $(\mathbf{C}^p, 0)$ . A natural choice for such a subset is obtained by considering those germs whose local algebras (see definition below) are isomorphic. Subsets obtained by this way are called contact singularities. In the language of equivariant cohomology, the Thom polynomial of the first paragraph is the  $\text{GL}_n(\mathbf{C}) \times \text{GL}_p(\mathbf{C})$ -equivariant cohomology class represented by the closure of the contact class  $\eta$  in  $H_{\text{GL}_n(\mathbf{C}) \times \text{GL}_p(\mathbf{C})}^*(\mathcal{E}_0(n, p))$ .

Thom polynomials have applications in various parts of differential topology, algebraic geometry, and algebraic combinatorics, let us just allude to the simplest case, the celebrated Giambelli-Thom-Porteous formula—where  $\eta$  is the set of germs with corank  $k$  differential. The present paper is devoted to the problem of calculating Thom polynomials in the  $n \leq p$  case, as well as the study of their interior structure.

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The first author was supported by the Alfred Renyi Mathematical Institute, OTKA grant T046365MAT, and is presently supported by the Bolyai Janos Scholarship

Keywords: contact singularities, Thom polynomial, partial resolution, singular-base localization

AMS Subject classification 32S20.

As discussed above, Thom polynomials are parameterized by an algebra  $Q$ , and two integers  $n$ , and  $p$ ; we will call such a Thom polynomial  $\text{Tp}_Q(n, p)$ . It turns out that the corresponding  $\eta \subset \mathcal{E}_0(n, p)$  is finite codimensional if and only if  $Q$  is a finite dimensional, commutative, local algebra. We recently showed in [FR07] that—under technical conditions—the Thom polynomials  $\text{Tp}_Q(n, p)$  for the same  $Q$  but varying  $n$  and  $p$  can be organized into a formal power series in infinitely many variables (cf. Section 7.3).

A powerful technique to calculate Thom polynomials is the method of Restriction Equations [Rim01]. It calculates certain individual Thom polynomials, but not whole Thom series (unless  $Q$  has very small dimension, see Section 3).

In a recent paper [BS06] Bérczi and Szenes introduced a new method of studying Thom polynomials of so-called Morin singularities, ie. singularities corresponding to algebras  $A_i = \mathbf{C}[[x]]/(x^{i+1})$ . One of their key ideas is the usage of (improved versions of) equivariant localization formulas. Their method naturally presents the whole Thom series. As a result, they reduced the calculation of the Thom series of Morin singularities to a finite set of data, as well as determined this data for  $i \leq 6$ . Another important novelty of [BS06] is the encoding of the Thom series of Morin singularities by iterated residues of certain rational functions.

In the present paper we revisit a partial resolution construction of J. Damon for all contact singularities. The Bérczi-Szenes equivariant localization formula applied to this construction leads to our main result, a Localization Formula for  $\text{Tp}_Q(n, p)$ , see Theorem 6.1. The form of this formula implies different stabilization properties of Thom polynomials, including the long-hidden  $d$ -stability property of Thom series. The inputs of the Localization formula for a fixed  $Q$  is a *finite* set of various Euler classes inside of a Grassmannian, or Hilbert scheme; showing how a finite set can encode a whole Thom series. Below we develop different techniques to find these Euler classes, and hence we will calculate several new Thom series. These examples include Thom series corresponding to local algebras of dimension  $< 6$ , as well as a two-parameter list of algebras with corresponding singularities outside the region of “nice” singularities (cf. [Mat71])—for not nice singularities only a handful individual Thom polynomials were known before.

The present work relies on the recent rapid developments of Thom polynomial theory; such as the method of restriction equations of the authors, and the various extensions and applications made by M. Kazarian. It is however particularly triggered by the new ideas and results of Bérczi and Szenes [BS06]. We believe that our general method of calculating Thom polynomials of contact singularities will start a new phase in global singularity theory, where the abundance of examples will help us to understand not only the interior structure of Thom polynomials, but also to discover patterns in enumerative geometry and global singularity theory in general.

**1.1. The plan of the paper.** In Section 2 we recall contact singularities and give a firm foundation of their Thom polynomial theory. In Section 3 we summarize known Thom polynomials. In Section 4, 5 and 6 we review Damon’s partial resolution of contact singularities and explain how it yields an equivariant Localization Formula for Thom polynomials. In Section 7 we explore stabilization properties that our main formula implies. In Section 8 we develop geometric and algebraic techniques to calculate the inputs of the Localization Formula. In Section 9 we explain the connection with the equivariant geometry of the local punctual Hilbert scheme. Section 10 presents the calculation of Thom series of singularities  $\Phi_{n,r}$ . In Section 11 we study generating functions of Thom series, iterated residue formulas, and their relations to geometry (as well as

iterated residue identities they depend on). The interesting phenomena of formally applying Thom polynomial formulas to dimensions where they are not defined is discussed in Section 12.

Throughout the paper we will work in the complex analytic category. Cohomology will be meant with rational coefficients.

**1.2. Acknowledgements.** The authors are indebted to M. Kazarian for several helpful discussions on the topic. He also informed us about his work in progress [Kaz], in which he calculates Thom series using the Gysin-map. We are also grateful to P. Frenkel, B. K om uves, T. Ohmoto, G. Smith, A. Szenes, G. B erczi and E. Szab o for valuable comments. The first author thanks T. Ohmoto for the opportunity to visit Hokkaido University and RIMS which greatly helped this work.

## 2. THOM POLYNOMIALS OF CONTACT SINGULARITIES

**2.1. Contact equivalence of finite germs.** Consider  $\mathcal{E}_0(n, p)$ , the vector space of holomorphic map germs  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^p, 0)$ . Throughout the paper we assume that  $n \leq p$ . The vector space  $\mathcal{E}_0(n) := \mathcal{E}_0(n, 1)$  is an algebra without an identity. The space  $\mathcal{E}_0(n, p)$  is a module over  $\mathcal{E}_0(n)$ . A map germ  $g \in \mathcal{E}_0(n, p)$  induces a pullback  $g^* : \mathcal{E}_0(p) \rightarrow \mathcal{E}_0(n)$  by composition.

**Definition 2.1.** The ideal  $I_g$  of a germ  $g \in \mathcal{E}_0(n, p)$  is the ideal in  $\mathcal{E}_0(n)$  generated by elements in  $g^* \mathcal{E}_0(p)$ . The *quotient algebra*  $Q_g$  of a germ  $g \in \mathcal{E}_0(n, p)$  is defined by  $Q_g = \mathcal{E}_0(n)/I_g$ .

Here, and in the whole paper, an ideal *generated* by some ring elements is the smallest ideal containing the specified ring elements, even if the ring has no identity. In singularity theory one usually considers the one dimension larger *local algebra*—defined as  $\mathcal{Q}_g := \mathcal{E}(n)/I_g$ , where  $\mathcal{E}(n)$  is the ring of function germs  $(\mathbf{C}^n, 0) \rightarrow \mathbf{C}$ —which has an identity. The two versions can easily be obtained from each other, but the quotient algebra comes up more naturally in our geometric setting. We will be concerned with germs  $g$  for which the quotient algebra is finite dimensional. We call these germs *finite*. Finite germs only exist for  $n \leq p$ , this is the reason of our overall assumption of  $n \leq p$ . Finiteness is also equivalent to the property that the ideal  $(g^* \mathcal{E}_0(p))$  contains a power of  $\mathcal{E}_0(n)$ . For a finite germ, in local coordinates  $g = (g_1(x_1, \dots, x_n), \dots, g_p(x_1, \dots, x_n))$ , we have

$$\mathcal{Q}_g = \mathbf{C}[[x_1, \dots, x_n]]/(g_1, \dots, g_p), \quad \text{and} \quad Q_g = \mathcal{M}_n/(g_1, \dots, g_p),$$

where  $\mathcal{M}_n$  is the maximal ideal of  $\mathbf{C}[[x_1, \dots, x_n]]$ , that is, the ideal generated by the variables  $(x_1, \dots, x_n)$ . For finite germs the quotient algebra is nilpotent, so we will also call  $Q_g$  the *nilpotent algebra* of the germ  $g$  to distinguish it from the local algebra.

The invertible holomorphic germs  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$  form the *right group*  $\mathcal{R}(n)$ . This group acts on  $\mathcal{E}_0(n)$  by composition, and hence it acts on the set of ideals of  $\mathcal{E}_0(n)$ .

**Definition 2.2.** Two germs  $f, g \in \mathcal{E}_0(n, p)$  are *contact equivalent* if their ideals are in the same  $\mathcal{R}(n)$  orbit. An equivalence class  $\eta \subset \mathcal{E}_0(m, n)$  is called a (contact) singularity.

In singularity theory one considers the so called contact group [AVGL93, Ch3, 1.6]

$$\mathcal{K} = \mathcal{K}(n, p) = \{(h, M) : h \in \mathcal{R}(n), M \text{ is a germ } (\mathbf{C}^n, 0) \rightarrow (\text{GL}(p), 1)\},$$

acting on the vector space  $\mathcal{E}_0(n, p)$  by  $((h, M)g)(x) = M(x)g(h^{-1}(x))$ , and defines germs to be contact equivalent if they are in the same orbit. It is a theorem of Mather [Mat69, Thm 2.9] that

for finite germs the two definitions are equivalent. Hence we will denote the contact equivalence class of a germ  $g$  by  $\mathcal{K}g$ . It is also true that two finite germs are contact equivalent if and only if their nilpotent algebras are isomorphic (see the discussion after Corollary 2.13).

**2.2. Thom polynomials.** Given a map  $F : N^n \rightarrow P^p$  between complex manifolds, and a point  $x \in N$ , we can choose charts around  $x$  and  $F(x)$ , and consider the germ of  $F$  at  $x$  in these charts. The contact singularity of this germ does not depend on the choice of the charts. Indeed, it is a part of Mather's theorem cited above that after reparametrizing the source  $(\mathbf{C}^n, 0)$ , and target  $(\mathbf{C}^p, 0)$  spaces, the ideal of the germ will be in the same  $\mathcal{R}(n)$  orbit. Equivalently, we can refer to the fact that the group  $\mathcal{K}(n, p)$  contains the group  $\mathcal{R}(n) \times \mathcal{R}(p)$  of holomorphic reparametrizations of the source  $(\mathbf{C}^n, 0)$  and target spaces  $(\mathbf{C}^p, 0)$ . (In this context the group  $\mathcal{R}(p)$  is usually called the left group and denoted by  $\mathcal{L}(p)$ .)

Therefore it makes sense to talk about the contact singularity of  $F$  at a point  $x \in N$ . Hence, for a map  $F : N^n \rightarrow P^p$  and a singularity  $\eta \in \mathcal{E}(n, p)$ , we define the singularity subsets

$$\eta(F) = \{x \in N \mid \text{the germ of } F \text{ at } x \text{ is in } \eta\}.$$

After some preparations (Sections 2.3-2.4), in Theorem 2.15 we explain the following statements.

*If  $F$  satisfies certain transversality conditions, then the subset  $\eta(F)$  defines a cohomology class  $[\eta(F)] \in H^*(N)$ . Moreover, this class can be expressed as a universal polynomial (the Thom polynomial,  $\text{Tp}(\eta)$ ) of the Chern classes of  $TN$  and  $f^*TP$ .*

First, in Section 2.3, we will discuss degeneracy loci, and how universal cohomology classes are associated with them. Then we will interpret  $\eta(F)$  as a degeneracy locus in Section 2.4. These two sections serve as a rigorous definition of the Thom polynomial for a contact singularity class.

**2.3. Poincaré dual, equivariant cohomology and degeneracy loci.** In this section we discuss degeneracy loci, and cohomology class represented by them. First recall that subvarieties  $Y \subset X$  represent cohomology classes in the underlying space (see e.g. [Ful97, p.219]).

**Proposition 2.3 (Definition).** *If  $X$  is a smooth algebraic variety and  $Y$  is an irreducible subvariety of complex codimension  $d$  then there is a unique element  $[Y \subset X] \in H^{2d}(X)$  such that*

- (1)  $[Y \subset X]$  is supported on  $Y$ , i.e.  $[Y \subset X]$  restricted to  $X \setminus Y$  is zero,
- (2)  $[Y \subset X]|_{X \setminus \text{Sing } Y} = [Y^\circ \subset (X \setminus \text{Sing } Y)]$ .

*Here  $\text{Sing } Y$  denotes the singular subvariety of  $Y$  and  $Y^\circ = Y \setminus \text{Sing } Y$ . The cohomology class  $[Y^\circ \subset (X \setminus \text{Sing } Y)]$  is defined by extending the Thom class of a tubular neighborhood of the proper submanifold  $Y^\circ \subset (X \setminus \text{Sing } Y)$  via excision.*

If  $Y$  has several components  $Y_i$  (usually of the same codimension) then  $[Y \subset X]$  is defined to be the sum of the classes  $[Y_i \subset X]$ . When the underlying space  $X$  is clear from the context, we denote  $[Y \subset X]$  by  $[Y]$ .

We need the equivariant version of Proposition 2.3 above. Let  $G$  be a complex algebraic Lie group. If  $X$  is a smooth algebraic variety with a  $G$ -action, and  $Y$  is a  $G$ -invariant subvariety then  $Y$  represents a  $G$ -equivariant cohomology class in the equivariant cohomology of  $X$ , as follows (see e.g. [Kaz03]).

**Theorem 2.4 (Definition).** *Let  $X$  be a smooth algebraic variety with a  $G$ -action, and  $Y \subset X$  be a  $G$ -invariant irreducible subvariety of complex codimension  $d$ . Then there is a unique element*

$[Y \subset X]_G \in H_G^{2d}(X)$  (called the  $G$ -equivariant Poincaré dual of  $Y$  in  $X$ ) such that for all algebraic principal  $G$ -bundles  $\pi : P \rightarrow M$  over a smooth algebraic variety  $M$  with classifying map  $k : M \rightarrow BG$  we have

$$(1) \quad [P \times_G Y \subset P \times_G X] = \tilde{k}^*[Y \subset X]_G,$$

where  $\tilde{k} : P \times_G X \rightarrow EG \times_G X$  is induced by  $k$ .

Intuitively  $[Y \subset X]_G$  is the class represented by  $EG \times_G Y$  in  $EG \times_G X$ . From the next section on, we will be mainly interested in the case when  $X$  is a vector space. Then  $H_G^*(X) \cong H^*(BG)$  canonically. The class  $[Y \subset X]_G$  has various definitions and names in the literature (see [FP07] for an account). We will also use the notation  $[Y]_G$  or simply  $[Y]$  for  $[Y \subset X]_G$ , when the underlying space  $X$  and the group action is clear from the context.

**Remark 2.5.** We can make this construction more explicit for the torus  $T = \mathrm{GL}(1)^r$  (this is the case we need in the Localization Formula below): repeat the construction above for the principal  $T$ -bundle  $P = (\mathbf{C}^{d+1} \setminus \{0\})^r \rightarrow (\mathbf{P}^d)^r$ . Here the classifying map  $k$  is the standard inclusion  $(\mathbf{P}^d)^r \rightarrow (\mathbf{P}^\infty)^r = BT$ . It is not difficult to show that

$$\tilde{k}^* : H^j(ET \times_T X) \rightarrow H^j(P \times_T X)$$

is bijective for  $j \leq 2d$ . So for large enough  $d$ , equation (1) defines  $[Y \subset X]_T$  uniquely:

$$[Y \subset X]_T = (\tilde{k}^*)^{-1}[P \times_T Y \subset P \times_T X].$$

Suppose that  $s : M \rightarrow E$  is an algebraic section of the fiber bundle  $\varphi : E = P \times_G X \rightarrow M$ . If  $Y \subset X$  is a  $G$ -invariant subvariety, then we use the notation  $Y(\varphi) := P \times_G Y$  for the set of ‘ $Y$ -points’ in  $E$  and  $Y(s) := s^{-1}(Y(\varphi))$  for the set of ‘ $Y$ -points of  $s$ ’. We call  $Y(s)$  the *degeneracy locus* corresponding to  $Y$  and  $s$ . To make a statement about the class represented by  $Y(s)$  (Corollary 2.9) we need to discuss transversality.

**Definition 2.6.** Let  $f : A \rightarrow B$  be an algebraic map between algebraic manifolds and  $Y \subset B$  be a subvariety. The map  $f$  is *transversal* to  $Y$  if it is transversal to all singularity strata of  $Y$ , i.e. to the manifolds  $Y^\circ = Y \setminus \mathrm{Sing} Y$ ,  $\mathrm{Sing} Y \setminus \mathrm{Sing}(\mathrm{Sing} Y)$  and so on.

The following well known fact can be proved by straightforward diagram chasing.

**Proposition 2.7.** *If  $f : A \rightarrow B$  is transversal to  $Y \subset B$ , then  $f^*([Y]) = [f^{-1}(Y)]$ .*

This statement easily generalizes to the equivariant setting.

**Proposition 2.8.** *Let the  $G$ -equivariant map  $f : A \rightarrow B$  be transversal to the  $G$ -invariant subvariety  $Y \subset B$ . Then  $f^*([Y]_G) = [f^{-1}(Y)]_G$ .*

As a consequence, if  $s$  is a transversal section of  $P \times_G X \rightarrow M$ , then the equivariant Poincaré dual  $[Y]_G$  gives the Poincaré dual  $[Y(s)]$  of the degeneracy locus as a  $G$ -characteristic class of the  $G$ -principal bundle  $P \rightarrow M$ . This is formulated in

**Corollary 2.9.** *If the section  $s : M \rightarrow E$  of the vector bundle  $\varphi : E = P \times_G X \rightarrow M$  is transversal to  $Y(\varphi) = P \times_G Y$  then  $[Y(s)] = k^*[Y]_G$  where  $k : M \rightarrow BG$  is the classifying map of  $P$ .*

**Remark 2.10.** In the complex algebraic setting the existence of a transversal section is not guaranteed. Nevertheless  $k^*[Y]_G$  is always an obstruction: if  $k^*[Y]_G$  is non-zero then there is no section such that the degeneracy locus  $Y(s)$  is empty since  $k^*[Y]_G$  is supported on  $Y(s)$ . The theory can be extended to the real smooth category. In that case the existence of the Poincaré dual is not automatic, but a generic section is transversal.

**2.4. Jet approximation: reduction to finite dimension.** In this section we interpret  $\eta(F)$  (from Section 2.2) as a degeneracy locus. For this we need a finite dimensional approximation of  $\mathcal{E}_0(n, p)$ , and related notions.

The vector space of  $k$ -jets is defined to be the vector space of degree  $k$  polynomials  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^p, 0)$ . That is, we have

$$J^k(n, p) = \bigoplus_{i=1}^k \text{Hom}(S^i \mathbf{C}^n, \mathbf{C}^p),$$

where  $S^i \mathbf{C}^n$  is the  $i^{\text{th}}$  symmetric power of the vector space  $\mathbf{C}^n$ . Let  $J^k(n) = J^k(n, 1)$ . The map  $\mathcal{E}_0(n, p) \rightarrow J^k(n, p)$ , defined by taking the degree  $k$  Taylor polynomial at 0, will be denoted by  $j^k$ . The space  $J^k(n)$  is an algebra (without identity) with multiplication  $h_1 \cdot h_2 = j^k(h_1 \cdot h_2)$ . The  $j^k$ -image ( $k$ -jets) of elements in  $\mathcal{R}(n)$  form a group  $\mathcal{R}^k(n)$ . The group  $\mathcal{R}^k(p)$  will also be denoted by  $\mathcal{L}^k(p)$ . The group  $\mathcal{R}^k(n)$  acts on the vector space  $J^k(n)$  by

$$\alpha \cdot h = j^k(h \circ \alpha^{-1}) \quad (\alpha \in \mathcal{R}^k(n), h \in J^k(n)).$$

Hence the group  $\mathcal{R}^k(n)$  also acts on the set of ideals of  $J^k(n)$ . Similarly we can define the group  $\mathcal{K}^k = \mathcal{K}^k(n, p)$  acting on the vector space  $J^k(n, p)$ .

Let  $h \in J^k(n, p)$ . The ideal in  $J^k(n)$ , generated by the coordinate functions of  $h$ , will be denoted by  $I_h$ . We call  $Q_h = J^k(n)/I_h$  the nilpotent algebra of the jet  $h$ . Two  $k$ -jets are defined to be contact equivalent, if their ideals are in the same  $\mathcal{R}^k(n)$ -orbit. Equivalently we have

**Theorem 2.11.** [Mat69, Thm 2.9, 2.1] *Two  $k$ -jets are contact equivalent if and only if they are in the same  $\mathcal{K}^k$ -orbit if and only if their nilpotent algebras are isomorphic.*

Mather used local algebras instead of nilpotent ones, but the two versions clearly give equivalent conditions. The following result (see [Tou66, Tou70], [Mat68, Thm 3.5]) asserts that finite germs are “finitely determined”.

**Theorem 2.12** (Tougeron). *If a germ  $f \in \mathcal{E}_0(n, p)$  is finite, then there is a  $k$  such that for every  $g \in \mathcal{E}_0(n, p)$  the germ  $g$  is contact equivalent to  $f$  if their  $k$ -jets are equal. In this case we say that the germ  $f$  is  $k$ -determined.*

In the literature, instead of the condition of  $f$  being finite, authors use a different condition (“ $d(f, \mathcal{K}) < \infty$ ”); but it is a theorem of Wall [Wal81, Prop 2.4], that for  $n \leq p$  the two conditions are equivalent. Using the observation  $j^k(H(g)) = j^k(H)(j^k(g))$  ( $H \in \mathcal{K}, g \in \mathcal{E}_0(n, p)$ ) we immediately get the following

**Corollary 2.13.** *Suppose that  $f \in \mathcal{E}_0(n, p)$  is  $k$ -determined. Then  $f$  is contact equivalent to  $g$  if and only if  $j^k f$  is contact equivalent to  $j^k g$ .*

Notice that this corollary together with Theorem 2.11 implies the statement mentioned in Section 2.1, that is, finite germs are contact equivalent if and only if their nilpotent algebras are isomorphic. (Choose a  $k$  such that the germ  $g$  is  $k$ -determined and  $I_g \supset \mathcal{E}_0^{k+1}$ . Then  $Q_g \cong Q_{j^k g}$ .)

Now we can give the promised degeneracy locus description of the singularity set  $\eta(F)$ . Given a map  $F : N^n \rightarrow P^p$  between manifolds, and a positive integer  $k$ , we construct a fiber bundle

$$(2) \quad \begin{aligned} \phi_F : \{ (x, h) : x \in N, h \text{ is the } k\text{-jet of a germ } (N, x) \rightarrow (P, F(x)) \} \\ (x, h) \mapsto x, \end{aligned}$$

with fiber  $J^k(n, p)$ . For  $k = 1$  this is a vector bundle  $\text{Hom}(TN, F^*TP)$ . In general the fiber is a vector space, but the structure group (the left-right group  $\mathcal{R}^k(n) \times \mathcal{L}^k(p)$  acting on  $J^k(n, p)$  by composition on the two sides) does not act linearly. We also have the natural section of this bundle induced by  $F$ :

$$(3) \quad j^k F : x \mapsto (x, j^k(\text{germ of } F \text{ at } x)).$$

Now let  $g \in \mathcal{E}_0(n, p)$  be a  $k$ -determined germ, let  $\eta = \mathcal{K}g$  be its contact equivalence class, and let  $\eta^k$  be the contact equivalence class of  $j^k g \in J^k(n, p)$ . Since  $\eta^k$  is  $\mathcal{R}^k(n) \times \mathcal{L}^k(p)$ -invariant, it defines a degeneracy locus and Corollary 2.13 implies that we have the following degeneracy locus description of the  $\eta$  singularity subset of a map  $F$

$$(4) \quad \eta(F) = \eta^k(j^k(F)).$$

**2.5. The Thom polynomial.** Now we are ready to define the Thom polynomial of a contact singularity. Let  $g \in \mathcal{E}_0(n, p)$  be a  $k$ -determined finite germ. Let  $\eta \subset \mathcal{E}_0(n, p)$  be the closure of  $\mathcal{K}g$ , and let  $\eta^k \subset J^k(n, p)$  be the closure of  $\mathcal{K}^k j^k g$ . Notice that  $\mathcal{K}^k$  is a connected algebraic group acting algebraically, so the orbit-closure is the same in the Zariski and the metric topology. Connectedness implies that the closure of the orbit is irreducible.

Recall that  $\mathcal{R}^k(n) \times \mathcal{L}^k(p)$  acts on  $J^k(n, p)$  leaving  $\eta^k$  invariant.

**Definition 2.14.** The Thom polynomial  $\text{Tp}(\eta)$  of  $\eta$  is defined to be the class represented by  $\eta^k$  in the  $\mathcal{R}^k(n) \times \mathcal{L}^k(p)$ -equivariant cohomology of  $J^k(n, p)$ . We will also use the notation  $\text{Tp}(g)$ .

Since  $J^k(n, p)$  is a vector space (hence contractible), and  $\mathcal{R}^k(n) \times \mathcal{L}^k(p)$  is homotopy equivalent to  $\text{GL}(n) \times \text{GL}(p)$ , we have

$$\text{Tp}(\eta) = [\eta^k \subset J^k(n, p)]_{\mathcal{R}^k(n) \times \mathcal{L}^k(p)} = [\eta^k \subset J^k(n, p)]_{\text{GL}(n) \times \text{GL}(p)} \in H^*(B(\text{GL}(n) \times \text{GL}(p))).$$

The degree of the Thom polynomial is the codimension of  $\eta^k$  in  $J^k(n, p)$ . We will also refer to this degree as the *codimension of the germ*  $g$ .

The cohomology ring  $H^*(B(\text{GL}(n) \times \text{GL}(p)))$  is a polynomial ring generated by the universal Chern classes  $a_1, \dots, a_n, b_1, \dots, b_p$  of the groups  $\text{GL}(n)$ ,  $\text{GL}(p)$ , hence the Thom polynomial is indeed a polynomial. Moreover  $\text{Tp}(\eta)$  can be evaluated if two sets of cohomology classes  $a_i$  ( $i = 1, \dots, n$ ), and  $b_j$  ( $j = 1, \dots, p$ ) are given.

The meaning of the Thom polynomial is enlightened by putting together expression (4) with Definition 2.4. We obtain the following

**Proposition 2.15.** *Let  $g \in \mathcal{E}_0(n, p)$  be a  $k$ -determined germ,  $\eta^k$  the closure of  $\mathcal{K}^k(j^k g)$  in  $J^k(n, p)$ , and let  $F : N^n \rightarrow P^p$  be a map between compact complex manifolds. Suppose that the section  $j^k F$  (see (3)) is transversal to  $\eta^k(\phi_F)$ —the  $\eta^k$ -points of the of the fibration (2). Then the cohomology class  $[\eta(F) \subset N]$  represented by the  $\eta$ -points of the map  $F$  is equal to the Thom polynomial of  $\eta$  evaluated at the Chern classes of  $TN$  and  $F^*TP$ .*

We have not included the letter  $k$  in the notation  $\text{Tp}(\eta)$ , since, we will show in Section 7.1 that the Thom polynomial does not depend on the choice of  $k$  (as long as  $g$  is  $k$ -determined). Observe that Proposition 2.15 proves this statement, provided there are sufficiently many maps  $F$  satisfying the conditions of the Proposition. We will use a different approach in Section 7.1.

### 3. EXAMPLES, KNOWN RESULTS

Suppose a complex, commutative, finite dimensional, local algebra  $\mathcal{Q}$  is given. Then there exists a contact singularity in  $\mathcal{E}_0(n, p)$  with local algebra  $\mathcal{Q}$  for each  $n$ , and  $p$  (if  $n$  and  $p - n$  are large enough). A general Thom polynomial is a formula (containing  $n$ , and  $p$  as parameters) expressing the Thom polynomial of all these singularities together. For example, the Thom polynomial of a singularity in  $\mathcal{E}_0(n, p)$  with local algebra  $\mathcal{Q} = \mathbf{C}[[x]]/(x^3)$  is

$$c_{l+1}^2 + \sum_{i=1}^{\infty} 2^{i-1} c_{l+1-i} c_{l+1+i},$$

where  $l = p - n$ , and the classes  $c_i$  are defined by

$$(5) \quad 1 + c_1 t + c_2 t^2 + \dots = \frac{1 + b_1 t + b_2 t^2 + \dots + b_p t^p}{1 + a_1 t + a_2 t^2 + \dots + a_n t^n},$$

and the conventions  $c_0 = 1$ ,  $c_{<0} = 0$ . Using Schur polynomials

$$(6) \quad \Delta_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r} := \det(c_{\lambda_i + j - i})_{i,j=1,\dots,r}$$

we can further write the Thom polynomial in the form

$$\Delta_{l+1,l+1} + 2\Delta_{l+2,l} + 4\Delta_{l+3,l-1} + \dots$$

This example displays four important properties:

- the Thom polynomial can be expressed in the “quotient variables” (5);
- when the Thom polynomial is expressed in quotient variables, then the dependence on  $p$  and  $n$  is only through  $l = p - n$ ;
- if the general Thom polynomial is expressed in quotient variables, and the indexes are shifted by  $l + 1$  (i.e. substituting  $d_i = c_{l+1+i}$ ), the expression does not depend on  $l$  either;
- the coefficients of Thom polynomials in the basis of Schur polynomials are non-negative.

The general Thom polynomial after shifting the indices by substituting  $d_i = c_{l+1+i}$  is called the Thom series of the local algebra  $\mathcal{Q}$ , and is denoted by  $\text{Ts}_{\mathcal{Q}}$ . For example

$$\text{Ts}_{\mathbf{C}[[x]]/(x^3)} = d_0^2 + d_{-1} d_1 + 2d_{-2} d_2 + 4d_{-3} d_3 + \dots$$

Alternatively we can work with nilpotent algebras. We will also use the notation  $\text{Ts}_Q$  for  $Q$  being a nilpotent algebra.

All four properties above hold in general. The first three we will prove in Section 7. The first two are classical facts, we will call them the Thom-Damon-Ronga theorem, the third (in a special case) is a theorem from [FR07]. The fourth property was recently proved in [PW07].

Several individual Thom polynomials are known for small values of  $l$  (see e.g. [Rim01], [Kaz03]), but hardly any general Thom polynomials, i.e. Thom series are known. Here is a complete list of local algebras whose Thom series is known:

- $\Sigma^n = \mathbf{C}[[x_1, \dots, x_n]] / \mathcal{M}_n^2$  (Giambelli-Thom-Porteous formula)
- $A_2$  [Ron72]

- $A_3$  [BFR03] (announced), [Pra05] (sketched), [BS06, LP08] (proved)
- $A_4, A_5, A_6$  [BS06]
- The Thom-Boardman classes  $\Sigma^{n,1}$  [FK06]
- $I_{2,2}$ . [FR07, Pra07, Kaz]

Here and in what follows we use standard notations of singularity theory, as follows.  $A_i = \mathbf{C}[[x]]/(x^{i+1})$ ,  $I_{a,b} = \mathbf{C}[[x,y]]/(xy, x^a + y^b)$ ,  $III_{a,b} = \mathbf{C}[[x,y]]/(xy, x^a, y^b)$ . For Thom series of Thom-Boardman classes see also Section 12.

Below we will develop a method to calculate general Thom polynomials. It leads to formulas where the stability properties are not apparent, as these formulas are given in Chern roots. In Section 10.2 we show how to find formulas in quotient variables. In Section 11 we explore even more compact descriptions in terms of generating functions.

#### 4. A PARTIAL RESOLUTION

In this section we introduce the key geometric idea leading to our cohomological localization formula. We present a partial resolution (i.e. a birational map of varieties) of contact invariant subvarieties of  $J^k(n, p)$ , in particular of closures of contact singularities. The construction is originally due to J. Damon [Dam72]. Similar ideas are present in works of J. Mather on Thom-Boardman singularities [Mat73]. The idea is that a contact invariant subvariety of  $J^k(n, p)$  is the union of large linear subspaces.

Let  $m$  be a nonnegative integer, and let  $\text{Gr}^m = \text{Gr}^m(J^k(n))$  be the Grassmannian of  $m$ -codimensional subspaces of  $J^k(n)$ .

**Definition 4.1.** Let  $Y \subset \text{Gr}^m$  be a subvariety and fix  $p \geq 1$ . The *correspondence variety* of  $Y$  is

$$C(Y) = \{(I, g) \in \text{Gr}^m \times J^k(n, p) \mid I \in Y, I_g \subset I\}.$$

We have now the following diagram

$$(7) \quad \begin{array}{ccccc} C(Y) & \hookrightarrow & \text{Gr}^m \times J^k(n, p) & \xrightarrow{\pi_2} & J^k(n, p) \\ \downarrow & & \downarrow \pi_1 & & \\ Y & \hookrightarrow & \text{Gr}^m & & \end{array}$$

where  $\pi_1, \pi_2, i$  are the obvious projections and imbedding. The projection  $C(Y) \rightarrow Y$  makes  $C(Y)$  a vector bundle with fiber  $C_I = I \otimes \mathbf{C}^p \subset J^k(n) \otimes \mathbf{C}^p = J^k(n, p)$ .

**Proposition 4.2.** Let  $g \in J^k(n, p)$  with  $\mu(g) := \text{codim } I_g = m$ . Let  $Y$  be  $\overline{\mathcal{R}I_g} \subset \text{Gr}^m$  for  $\mathcal{R} = \mathcal{R}^k(n)$ . Then

$$\phi = \pi_2 \circ i : C(\overline{\mathcal{R}I_g}) \rightarrow \overline{\mathcal{K}g}$$

is a birational map.

*Proof.* For  $\tilde{\mathcal{K}}g := \{(I_h, h) : h \in \mathcal{K}g\}$  we see that  $\phi|_{\tilde{\mathcal{K}}g} : \tilde{\mathcal{K}}g \rightarrow \mathcal{K}g$  is a bijection, so it is enough to show that  $\tilde{\mathcal{K}}g \subset C(\overline{\mathcal{R}I_g})$  is open (and therefore dense since  $C(\overline{\mathcal{R}I_g})$  is irreducible). For that it is enough to show that  $\tilde{\mathcal{K}}g$  intersected with a fiber is open in the fiber; hence we need the following lemma.

**Lemma 4.3.** For any jet  $g \in J^k(n, p)$  the set  $A_g := \{h \in I_g \otimes \mathbf{C}^p : I_h = I_g\}$  is Zariski open in  $I_g \otimes \mathbf{C}^p$ .

*Proof.* Let  $h = (h_1, \dots, h_p) \in I_g \otimes \mathbf{C}^p$ , and let  $a_i^j$  be the coefficients of  $h_i$  in some linear basis of  $I_g$ . The property that the  $h_i$ 's generate  $I_g$  as an ideal is equivalent to the property that an appropriate matrix, whose entries are linear functions of the  $a_i^j$ 's, has full rank. Therefore, the property that the  $h_i$ 's generate  $I_g$  cuts out a Zariski open subset.  $\square$

This finishes the proof of Proposition 4.2.  $\square$

Using the Gysin map  $\phi_*$  we have that  $\phi_*(1) = [\tilde{\mathcal{K}}g] = \text{Tp}(g)$ . We calculate the Gysin map using localization in the next section.

## 5. SINGULAR-BASE EQUIVARIANT LOCALIZATION

In this section we recall a version of the Berline-Vergne-Atiyah-Bott equivariant localization formula, due to Bérczi and Szenes. For completeness we give a proof. This version presents the ‘localization’ of an equivariant cohomology class on the total space of a vector bundle over a compact singular base space.

Let  $V$  be a vector space. Suppose that  $M$  is a compact algebraic manifold, and  $Y \subset M$  a subvariety. Let  $E \rightarrow Y$  be a sub-vector bundle of  $M \times V \rightarrow M$  restricted to  $Y$ . Let  $\pi_2 : M \times V \rightarrow V$  be the projection,  $i : E \subset M \times V$  the embedding, and  $\phi = \pi_2 \circ i$ , as in the diagram

$$\begin{array}{ccccc}
 & & \phi & & \\
 & & \curvearrowright & & \\
 E & \xrightarrow{i} & M \times V & \xrightarrow{\pi_2} & V \\
 \downarrow & & \downarrow & & \\
 Y & \xrightarrow{\quad} & M & & 
 \end{array}$$

Recall that for  $A \subset B$ , by  $[A]$  or  $[A \subset B]$  we mean the cohomology class represented by  $A$  in the cohomology of  $B$ .

**Proposition 5.1.** [BS06, (3.8)] *Suppose that the torus  $T$  acts on all spaces in the diagram above, and that all maps are  $T$ -equivariant. Assume that the fixed point set  $F(M)$  of the  $T$ -action on  $M$  is finite. Then for the push-forward map  $\phi_* : H_T^*(E) \rightarrow H_T^*(V)$  we have*

$$(8) \quad \phi_*(1) = \sum_{f \in F(M)} \frac{[Y \subset M]|_f \cdot [E_f \subset V]}{e(T_f M)} = \sum_{f \in F(Y)} \frac{[Y \subset M]|_f \cdot [E_f \subset V]}{e(T_f M)}.$$

Consequently, if  $\phi$  is birational to its image, then the right hand side of (8) is equal to  $[\phi(E)] \in H_T^*(V)$ .

*Proof.* We have to calculate the integral  $\pi_{2*}(i_*(1)) = \int_M i_*(1) = \int_M [E \subset M \times V]$  (we identify the cohomology of  $M \times V$  with the cohomology of  $M$ ) for which we apply the Berline-Vergne-Atiyah-Bott localization formula, that we recall now.

**Proposition 5.2.** [AB84] *Suppose that  $M$  is a compact manifold and  $T$  is a torus acting smoothly on  $M$ , and the fixed point set  $F(M)$  of the  $T$ -action on  $M$  is finite. Then for any cohomology class  $\alpha \in H_T^*(M)$*

$$(9) \quad \int_M \alpha = \sum_{f \in F(M)} \frac{\alpha|_f}{e(T_f M)}.$$

Here  $e(T_f M)$  is the  $T$ -equivariant Euler class of the tangent space  $T_f M$ . The right side is considered in the fraction field of the polynomial ring of  $H_T^*(\text{point}) = H^*(BT)$  (see more on details in [AB84]): part of the statement is that the denominators cancel when the sum is simplified.

We complete the proof of Proposition 5.1 by noticing that

$$(10) \quad [E \subset M \times V]|_f = [E_f \subset V] \cdot [Y \subset M]|_f.$$

The second equality in (8) follows from the fact that the cohomology class  $[Y \subset M]$  is supported on  $Y$ , so other fixed points give zero contribution.  $\square$

**Remark 5.3.** If  $\phi$  decreases the dimension, then  $\phi_*(1)$  is zero—since supported on a too small subset—so the right hand side of (8) is zero.

**Remark 5.4.** If  $f$  is a smooth point of  $Y$ , then

$$\frac{e(T_f M)}{[Y \subset M]|_f} = e(T_f Y),$$

hence if  $Y$  is smooth then formula (8) further simplifies to

$$\phi_*(1) = \sum_{f \in F(M)} \frac{[E_f \subset V]}{e(T_f Y)}.$$

This formula holds in the general case too, if we define the *virtual (tangential) Euler class*  $e(T_f Y)$  to be  $\frac{e(T_f M)}{[Y \subset M]|_f}$  even if  $f$  is not a smooth point of  $Y$ .

**Remark 5.5.** The moral of Proposition 5.1 is that if we want to calculate the equivariant class of a variety with localization, we should look for high-dimensional linear spaces in it. Precisely saying, we need another variety, birational to the original, which is the total space of a vector bundle over a compact base space. The higher the rank of the bundle, the simpler the formula is. Usually the variety we start with is a cone, hence there is a canonical line bundle whose total space is birational to it. Therefore formula (8) can be applied to find the Thom polynomial. This case is used in [FNR06]. Certain quiver varieties are birational to total spaces of vector bundles with higher rank, over a smooth compact space [Rei03]. Hence formula (8) can be effectively applied, yielding similar formulas for quiver polynomials as in [KS06].

## 6. LOCALIZATION FOR CONTACT CLASSES

Now we apply the equivariant localization formula above to the construction of Section 4. This construction is different from the resolution used in [BS06] for Morin singularities; it is more general (it covers all contact singularities), but numerically less effective.

Let  $\text{GL}(n, p) = \text{GL}(n) \times \text{GL}(p)$ . Recall that the spaces in diagram (7) have  $\text{GL}(n, p)$ -actions, and the maps in the diagram are  $\text{GL}(n, p)$ -equivariant. Let  $T(n, p) = T(n) \times T(p) \cong U(1)^n \times U(1)^p$  be the maximal torus of  $\text{GL}(n, p)$ , and restrict the action on the spaces and maps of diagram (7) to  $T(n, p)$ . Recall also that the map  $H_{G(n,p)}^*(\text{pt}) \rightarrow H_{T(n,p)}^*(\text{pt})$  is injective (splitting lemma), hence by this restriction we do not lose any cohomological information. Now we can apply Proposition 5.1 and Proposition 4.2 to the diagram (7), and we obtain our main result. Let  $F$

denote the set of monomial ideals in  $\text{Gr}^m$  being the fixed points of the  $T(n, p)$ -action on  $\text{Gr}^m$  (where  $T(p)$  acts trivially).

**Theorem 6.1** (Localization Formula). *Let  $g \in J^k(n, p)$  be a finite germ. Then*

$$\text{Tp}(g) = \sum_{I \in \mathcal{F}} \frac{[C_I \subset J^k(n, p)] \cdot [\overline{\mathcal{R}I_g} \subset \text{Gr}^m]|_I}{e(T_I \text{Gr}^m)}.$$

Using the virtual tangent Euler classes we get

$$(11) \quad \text{Tp}(g) = \sum_{I \in \mathcal{F}} \frac{[C_I \subset J^k(n, p)]}{e(T_I \overline{\mathcal{R}I_g})}.$$

In the rest of this section we present two lemmas in which we study the factors  $[C_I \subset J^k(n, p)]$  and  $e(T_I \text{Gr}^m)$  in the Localization Formula. For this we choose the following notations. Let

$$H_{T(n,p)}^*(J^k(n, p)) = H_{T(n,p)}^*(\text{pt}) = \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p],$$

where  $\alpha_i$  (resp  $\beta_i$ ) denotes the universal first Chern class in the  $i$ 'th factor of  $H_{T(n)}^*(\text{pt}) = \otimes_{i=1}^n H^*(BU(1))$  (resp.  $H_{T(p)}^*(\text{pt}) = \otimes_{i=1}^p H^*(BU(1))$ ). We call the  $\alpha_i$ 's and the  $\beta_i$ 's the Chern roots of the group  $T(n, p)$ . As usual, we identify weights of a  $T(n, p)$ -representation with linear combinations of the Chern roots. For a  $T(n, p)$ -representation  $A$ , let  $W_A$  denote the multiset of its weights. The Euler class of a representation  $A$  is  $e(A) = \prod_{w \in W_A} w$ .

We define resultants by  $\text{Res}(S|T) = \prod_{s \in S, t \in T} (s - t)$  for the finite multisets  $S$  and  $T$ . For example, the representation of  $T(n) \times T(p)$  on the vector space  $\text{Hom}(\mathbf{C}^n, \mathbf{C}^p)$  (by  $(A, B) \cdot F = B \circ F \circ A^{-1}$ ) has weights  $W_{\text{Hom}(\mathbf{C}^n, \mathbf{C}^p)} = \{\beta_i - \alpha_j : i = 1, \dots, p; j = 1, \dots, n\}$  and Euler class  $\text{Res}(\{\beta_1, \dots, \beta_p\} | \{\alpha_1, \dots, \alpha_n\})$ . (In the sequel, we will drop the brackets  $\{ \}$  from the notation.) Similarly,

$$W_{J^k(n,p)} = \left\{ \beta_i - \sum_{j=1}^n a_{ij} \alpha_j : i = 1, \dots, p; a_{ij} \geq 0, 1 \leq \sum_{j=1}^n a_{ij} \leq k \right\}$$

and for the  $T(n)$ -representation  $J^k(n)$  we have

$$W_{J^k(n)} = \left\{ - \sum_{j=1}^n a_j \alpha_j : a_j \geq 0, 1 \leq \sum_{j=1}^n a_j \leq k \right\}.$$

The equivariant cohomology class represented by a subrepresentation in a representation space is the Euler class of the factor representation. Hence we have the following lemma.

**Lemma 6.2.** *Let  $I$  be a monomial ideal. Then*

$$[C_I \subset J^k(n, p)] = e(Q_I \otimes \mathbf{C}^p) = \prod_{i=1}^p \prod_{w \in W_{Q_I}} (\beta_i + w) = \text{Res}(\beta_1, \dots, \beta_p | -W_{Q_I}),$$

where  $Q_I$  is the quotient space  $J^k(n)/I$ . If  $I$  is monomial then  $Q_I$  is equipped with the induced representation of  $T(n) \leq \text{GL}(n)$ .

□

Notice that all coefficients in  $W_{Q_I}$  are negative, so in applications the form  $\text{Res}(\beta_1, \dots, \beta_p | -W_{Q_I})$  seems more natural.

The tangent bundle of a Grassmannian is  $\text{Hom}(A, B)$  where  $A$  and  $B$  are the tautological sub- and quotient bundles. Therefore the following lemma calculates the denominator of the Localization Formula explicitly.

**Lemma 6.3.** *We have*

$$e(T_I \text{Gr}^m) = \text{Res}(W_{Q_I} | W_I).$$

□

Again, if we want positive coefficients, we can write  $e(T_I \text{Gr}^m) = \text{Res}(-W_I | -W_{Q_I})$ .

The factor  $[\overline{\mathcal{R}I_g} \subset \text{Gr}^m]_I$  in the Localization Formula is a subtle invariant of the set  $\overline{\mathcal{R}I_g}$  at  $I$ . Its calculation is difficult in general. In Section 8 we calculate special cases.

**6.1. First application of the Localization Formula.** Let  $g \in J^k(n, p)$  be the jet with the degree  $d$  monomials as coordinate functions for  $k \geq d$  and  $p = \binom{n+d-1}{d}$ . That is,

$$I_g = (J)^d = \langle x^K : |K| = d, K \subset \{1, \dots, n\} \rangle,$$

where  $J = J^k(n)$ . The ideal  $I_g$  is a fixed point of the right group  $\mathcal{R}$ , so the localization formula immediately gives

$$(12) \quad [(J)^d(n, p)] = \text{Res}(\beta_1, \dots, \beta_p | -W_{Q((J)^d)}),$$

where  $-W_{Q((J)^d)} = \{\sum a_i \alpha_i : a_i \geq 0, 0 < \sum a_i < d\}$ .

The singularity whose  $k$ -jet is  $g$  is called the Thom-Boardman singularity  $\Sigma^{n, \dots, n}(n, p)$  (the number of  $n$ 's in the superscript is  $d$ ). Hence (12) is the Thom polynomial of this singularity. This is not a new result though it might be the first appearance in the literature in this generality. The  $d = 1$  case recovers a special case of the Giambelli-Thom-Porteous formula in the Chern root format:  $\text{Tp}(\Sigma^n(n, p)) = \text{Res}(\beta_1, \dots, \beta_p | \alpha_1, \dots, \alpha_n)$  (cf. Theorem 7.13).

## 7. STABILITY PROPERTIES OF THE THOM POLYNOMIAL

**7.1. Dependence of Thom polynomials on  $k$ .** Our definition of the Thom polynomial of a finite germ  $f$ , used its  $k$ -jet. In this section, temporarily, we will use the notation  $\text{Tp}(j^k(f))$  to emphasize this dependence on  $k$ . Our goal is to show that the Thom polynomial does not depend on  $k$ , that is, we prove the following

**Theorem 7.1.** *Suppose that  $g \in J^k(n, p)$  and  $\tilde{g} \in J^{k+1}(n, p)$  such that  $I_{\tilde{g}} = \iota I_g$ , where  $\iota : \text{Gr}^m(J^k(n)) \rightarrow \text{Gr}^m(J^{k+1}(n))$  is induced by the obvious embedding  $J^k(n) \rightarrow J^{k+1}(n)$ . Then*

$$\text{Tp}(\tilde{g}) = \text{Tp}(g).$$

If  $f$  is a germ with  $j^k(f) = g$ , and  $I_f \supset \mathcal{E}_0(n)^{k+1}$ , then the Theorem above implies that the definition of  $\text{Tp}(f)$  does not depend on  $k$ . The condition  $I_f \supset \mathcal{E}_0(n)^{k+1}$  can be achieved by increasing  $p$  and adding monomials of degree  $k + 1$  in the new coordinates. This is the best possible result we can hope for. It says that we can take the smallest possible  $k$  in the localization formula, we just have to make sure that  $I_f \supset \mathcal{E}_0(n)^{k+1}$ .

*Proof of Theorem 7.1.* It is based on the following geometric statement.

**Proposition 7.2.** *Let  $I \in \text{Gr}^m(J^k(n))$  be an ideal of  $J^k(n)$ . Then*

$$\iota \mathcal{R}^k I = \mathcal{R}^{k+1} \iota I.$$

Since the image of  $\iota$  is closed, we also have

$$(13) \quad \overline{\iota \mathcal{R}^k I} = \overline{\mathcal{R}^{k+1} \iota I}$$

so the localization formula for the two Thom polynomials are identical.  $\square$

*Proof of Proposition 7.2.* We first calculate infinitesimally. The tangent space  $T_e \mathcal{R}^k(n)$  can be identified with polynomial vector fields on  $\mathbf{C}^n$ :

$$(14) \quad T_e \mathcal{R}^k(n) = \left\{ \sum_{i=1}^n p_i \partial_i : p_i \in J^k(n) \right\}.$$

The action of  $\mathcal{R}^k = \mathcal{R}^k(n)$  on  $Gr^m(J^k(n))$  induces a surjective map

$$T_e \mathcal{R}^k \rightarrow T_I \mathcal{R}^k I \subset T_I Gr^m \cong \text{Hom}(I, Q_I)$$

with image

$$(15) \quad T_I \mathcal{R}^k I = \left\{ \sum_{i=1}^n p_i \omega_i : p_i \in J^k(n) \right\},$$

where  $\omega_i \in \text{Hom}(I, Q)$ ,  $\omega_i(f) = [\partial_i(f)] \in Q_I$ . To compare with  $T_{\iota I} \mathcal{R}^{k+1} \iota I$  notice that  $\iota I \supset (J)^{k+1}$  for  $J = J^{k+1}(n)$ , which implies that  $(J)^{k+1} Q_{\iota I} = 0$ . Also we have that  $Q_{\iota I} \cong Q_I$ , therefore

$$(16) \quad \iota_* T_I \mathcal{R} I = T_{\iota I} \mathcal{R} \iota I.$$

The last equation implies that if  $\mathcal{R}^{k+1}$  glues together some  $\mathcal{R}^k$ -orbits then they must have the same dimension as their union—which is impossible, since  $\mathcal{R}^{k+1}$  is connected.  $\square$

**7.2. Dependence of Thom polynomials on  $p - n$ .** In this section we prove the classical stability result on Thom polynomials of contact singularities. Let  $\sigma : J^k(n, p) \rightarrow J^k(n+1, p+1)$  denote the stabilization map

$$\sigma g(x_1, \dots, x_{n+1}) := (g_1(x_1, \dots, x_n), \dots, g_p(x_1, \dots, x_n), x_{n+1}).$$
 Then

**Theorem 7.3** (Stability).

$$\text{Tp}(g) = \sigma^* \text{Tp}(\sigma g),$$

where  $\sigma^* : H_{G(n,p)}^* \rightarrow H_{G(n+1,p+1)}^*$  is the homomorphism induced by the map

$$G(n, p) \rightarrow G(n+1, p+1), \quad (M, N) \mapsto \left( \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \right).$$

The reason we include the proof here is twofold. First, we would like to strengthen the stability theorem and show that these Thom polynomials are *supersymmetric*. Second, it gives us a chance to study the geometry related to the Localization Formula.

The Localization Formula gives the Thom polynomial in Chern roots i.e. in generators of  $H_{T(n,p)}^* \cong \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]$ . Since  $\text{Tp}(g)$  is in the image of  $H_{G(n,p)}^* \rightarrow H_{T(n,p)}^*$ , it is symmetric in both the  $\alpha$  and the  $\beta$  variables. But it has more symmetry.

**Definition 7.4.** The polynomial  $q \in \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]$  is *supersymmetric* (see [Las81]) if

- (1) symmetric in both the  $\alpha$  and the  $\beta$  variables,
- (2)  $q(\alpha_1, \dots, \alpha_{n-1}, t, \beta_1, \dots, \beta_{p-1}, t)$  does not depend on  $t$ .

**Theorem 7.5.** *The Thom polynomial of a contact orbit is supersymmetric.*

An important property of supersymmetric polynomials is that they can be expressed in *quotient variables*: We define a map

$$\rho_{n,p} : \mathbf{Z}[c_1, \dots, c_i, \dots] \rightarrow \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]$$

by the formal power series

$$(17) \quad 1 + c_1 t + c_2 t^2 + \dots = \frac{\prod_{j=1}^p (1 + t\beta_j)}{\prod_{i=1}^n (1 + t\alpha_i)},$$

i.e.  $\rho_{n,p}(c_1) = \beta_1 + \dots + \beta_p - \alpha_1 - \dots - \alpha_n$  and so on. We say that  $q \in \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]$  can be expressed in quotient variables if it is in the image of  $\rho_{n,p}$ .

**Theorem 7.6** (Lascoux [Las81]). *The polynomial  $q \in \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]$  is supersymmetric if and only if it can be expressed in quotient variables.*

The expression of supersymmetric polynomials in terms of the quotient variables is unique if the degree of  $q$  is not too high compared to  $n$  and  $p$ .

**Proposition 7.7.** *If  $\rho_{n,p}(h) = 0$  for a non-zero polynomial  $h \in \mathbf{Z}[c_1, \dots, c_i, \dots]$  then  $\deg(h) \geq (n+1)(p+1)$  with the convention  $\deg c_i = i$ .*

In fact, the kernel of  $\rho_{n,p}$  is known explicitly (see e.g. [FP98, §3.2]):  $\ker(\rho_{n,p}) = \langle \Delta_\lambda : (n+1)^{(p+1)} \subset \lambda \rangle$ , where  $\langle \rangle$  means the generated  $\mathbf{Z}$ -module (for the definition of  $\Delta$  see (6)).

Now we translate supersymmetry to geometry. Notice that the stabilization map  $\sigma : J^k(n, p) \rightarrow J^k(n+1, p+1)$  is  $G'$ -equivariant for  $G' = G(n, p) \times \mathrm{GL}(1)$ , where  $\mathrm{GL}(1)$  acts trivially on  $J^k(n, p)$  and *diagonally on the last variables* of elements of  $J^k(n+1, p+1)$ . Supersymmetry and stability is equivalent to the following strengthening of the stability Theorem 7.3:

**Theorem 7.8.** *(Strong stability.) For any  $g \in J^k(n, p)$*

$$\mathrm{Tp}_{G'}(g) = \mathrm{Tp}_{G'}(\sigma g).$$

This theorem immediately follows from the two lemmas below and Proposition 2.8 on the transversal pull back of Thom polynomials.

**Lemma 7.9.** *The stabilization map  $\sigma$  is transversal to every contact class in  $J^k(n+1, p+1)$ .*

**Lemma 7.10.**  $\sigma^{-1}(\mathcal{K}\sigma g) = \mathcal{K}g$  for any  $g \in J^k(n, p)$ .

*Proof of Lemma 7.9.* The results of Section 4 imply that for  $g \in J^k(n, p)$  the tangent space of its contact class is

$$(18) \quad T_g \mathcal{K}g = I_g \otimes \mathbf{C}^p + T_g \mathcal{R}g,$$

where

$$(19) \quad T_g \mathcal{R}g = \left\{ \sum_{i=1}^n p_i \partial_i g : p_i \in J^k(n) \right\}.$$

Applying this to the germ  $\sigma g \in J^k(n+1, p+1)$  we see that transversality is equivalent to the property that the three subspaces  $I_{\sigma g} \otimes \mathbf{C}^{p+1}$ ,  $T_{\sigma g} \mathcal{R}\sigma g$  and  $\sigma J^k(n, p)$  span  $J^k(n+1, p+1)$ .

Let  $h = \sum_{i=1}^{p+1} h_i \otimes e_i$  any element of  $J^k(n+1, p+1)$ , where  $\{e_i : i = 1, \dots, p+1\}$  is the standard basis of  $\mathbf{C}^{p+1}$  and  $h_i \in J^k(n+1)$ . We can write  $h_i$  in the form  $h_i = a_i + b_i x_{n+1}$  where

$a_i \in J^k(n)$ . Since  $x_{n+1} \in I_{\sigma g}$ , it is enough to show that  $a_i \otimes e_i$  is in the span. For  $i \leq p$  we have  $a_i \otimes e_i \in \sigma J^k(n, p)$  and for the last coordinate notice that  $a_{p+1} \partial_{n+1} \sigma g = a_{p+1} \otimes e_{p+1}$ .  $\square$

*Proof of Lemma 7.10.* The statement follows from Theorem 2.12 and that  $Q_{\sigma g} \cong Q_g$  for any  $g \in J^k(n, p)$ .  $\square$

The proof of Theorem 7.8—and hence Theorems 7.3 and 7.5—is complete. These essentially imply the *Thom-Damon-Ronga theorem*. Recall that  $\mu(g) := \text{codim}(I_g \subset J^k(n))$ .

**Proposition 7.11.** *Let  $g \in J^k(n, p)$  with  $n \geq \mu(g) - 1$ . Then there is a unique polynomial  $\text{tp}(g) \in \mathbf{Z}[c_1, c_2, \dots]$  such that  $\rho_{n,p}(\text{tp}(g)) = \text{Tp}(g)$ .*

*Proof.* Theorem 7.6 implies existence. Since  $\deg(\text{tp}(g)) = \mu(g)p - \dim(\mathcal{R}g)$ , we have  $\deg(\text{tp}(g)) < (n+1)(p+1)$ , and Proposition 7.7 implies uniqueness.  $\square$

**Definition 7.12.** If the condition  $n \geq \mu(g) - 1$  is not satisfied, then we can take an iterated stabilization of  $g$  to get a unique polynomial, what we will also denote by  $\text{tp}(g)$ . Also, we will use the notation  $\text{tp}_Q(l) := \text{tp}(g)$ , where  $g \in J^k(n, p)$ , such that its nilpotent algebra  $Q_g$  is isomorphic to  $Q$ , and  $p - n = l$ . Stability justifies this definition.

As we already remarked, formula (12) implies that for  $p \geq \binom{n+1}{2}$

$$(20) \quad \text{Tp}_{\Sigma^n}(n, p) = \text{Res}(\beta_1, \dots, \beta_p | \alpha_1, \dots, \alpha_n).$$

Since  $\mu(\Sigma^n(n, p)) = n$ , Proposition 7.11 can be applied. We obtain that the polynomial in quotient variables which is equal to the right hand side of (20) will express the Thom polynomial of any  $\Sigma^n(*, * + l)$  (at least for  $l \geq \binom{n}{2}$ ). This argument reproves the following classical theorem.

**Theorem 7.13** (Giambelli-Thom-Porteous). *The Thom polynomial of  $\Sigma^n$  is*

$$(21) \quad \text{tp}_{\Sigma^n}(l) = \underbrace{\Delta_{n+l, \dots, n+l}}_n = \det(c_{n+l+j-i})_{1 \leq i, j \leq n}.$$

**7.3. Dependence of Thom polynomials on  $p$ .** In this section we study the relation between the Thom polynomial of the germs  $g$  and  $\delta g$  where

$$\delta : J^k(n, p) \rightarrow J^k(n, p+1), \quad \delta g : (x_1, \dots, x_n) \mapsto (g_1, \dots, g_p, 0).$$

In other words we are interested in the dependence of the Thom polynomial  $\text{tp}_Q(l)$  on  $l$ . In [FR07] we showed that under a technical condition one can calculate  $\text{tp}(g)$  from  $\text{tp}(\delta g)$  by ‘lowering the indices’. Notice that  $Q_{\delta g} \cong Q_g$ . Consequently the Thom polynomials of all germs with a given quotient algebra  $Q$  (or local algebra  $\mathcal{Q}$ ) can be arranged into a series what we called the *Thom series* of  $Q$ . The variables of this series are normalized Chern classes what we denoted by  $d_i$ , and hence this stabilization property will be called *d-stability*.

Using the Localization Formula we prove d-stability without the technical condition of [FR07] and we show that Thom polynomials of Thom-Boardman classes are also d-stable—extending a weaker statement of [FR07].

**Definition 7.14.** Fix  $m \in \mathbf{N}$  and assume that the polynomial  $q \in \mathbf{Z}[c_0, c_1, \dots]$  has *width*  $m$ , i.e.

$$q = \sum_{|K|=m} a_K c^K, \quad \text{where } K \in \mathbf{N}^m \text{ and } c^K = \prod_{i=1}^m c_{K(i)},$$

using the  $c_0 = 1$  convention. We define the *lowering operator*  $\flat = \flat(m)$  by

$$q^\flat := \sum_{|K|=m} a_K c^{K^\flat}, \quad \text{where } K^\flat(i) = K(i) - 1,$$

using the  $c_{-1} = 0$  convention.

E.g. for  $m = 2$  and  $q = c_2^2 + c_1c_3 + 2c_4$  we have  $q^\flat = c_1^2 + c_2$ , where we did not write out the  $c_0 = 1$  factors.

**Theorem 7.15.** *Let  $g$  be a jet with  $\mu(g) = m$ . Then  $\text{tp}(g)$  has width  $m$  and*

$$\text{tp}(\delta g)^\flat = \text{tp}(g).$$

A simple calculation shows (see [FR07, 2.3]) that to prove Theorem 7.15 it is enough to prove the following.

**Proposition 7.16.** *Let  $g \in J^k(n, p)$  with  $\mu(g) = m$  and write*

$$\text{Tp}(\delta g)(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p, \beta_{p+1}) = \sum p_i \beta_{p+1}^{m-i} \quad \text{for } p_i \in \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p].$$

*Then  $p_0 = \text{Tp}(g)$ .*

*Proof.* Notice that changing  $g$  to  $\delta g$  in the Localization Formula only changes the factors  $[C_I]$  by multiplying them with  $\text{Res}(\beta_{p+1}|W_{Q_I})$ , i.e. if

$$\text{Tp}(g) = \sum_{I \in F} a_I \quad \text{for } a_i \in \mathbf{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p] \quad \text{then} \quad \text{Tp}(\delta g) = \sum_{I \in F} a_I \text{Res}(\beta_{p+1}|W_{Q_I}),$$

which implies the Proposition, therefore the Theorem.  $\square$

We can rephrase Theorem 7.15 in terms of Thom series. Let  $Q$  be an  $m$ -dimensional nilpotent algebra over  $\mathbf{C}$  and let  $g(n, p) \in J^k(n, p)$  be a jet with  $Q_g \cong Q$ . If  $k > m$ ,  $n \geq m$  and  $p \geq b(Q) - a(Q) + n$ —where  $a(Q)$  is the minimal number of generators for  $Q$  and  $b(Q)$  is the minimal number of relations for  $Q$ —then such a jet exists. Let  $\text{tp}_Q(n, p) := \text{tp}(g(n, p))$  denote the Thom polynomial in quotient variables.

**Theorem 7.17.** *Let  $Q$  be an  $m$ -dimensional nilpotent algebra over  $\mathbf{C}$ . Then there is a unique homogeneous ( $\deg d_i = i$ ) formal power series*

$$\text{Ts}_Q = \sum_{|K|=m} a_K d^K \in \mathbf{Z}[[\dots, d_{-i}, \dots, d_0, \dots, d_i, \dots]],$$

*such that the Thom polynomials  $\text{tp}_Q(n, p)$  can be obtained by substituting  $d_i = c_{i+(p-n)+1}$  with the usual  $c_0 = 1$ ,  $c_i = 0$  for  $i < 0$  convention.  $\square$*

This is an improvement of [FR07, Th.4.1], where the assumption of non-zero normal Euler class was assumed. The degree of  $\text{Ts}_Q$  can be calculated by finding the degree of  $\text{tp}_Q(n, p)$  for some  $n$  and  $p$  which requires the calculation of the dimension of an  $\mathcal{R}$ -orbit (or, equivalently, of the corresponding unfolding space).

From this proof we can see that the d-stability property is not as deep as stability. It is a curious fact of the history of Thom polynomials that it remained hidden for so long.

**Remark 7.18.** The Localization Formula and Proposition 7.11 shows that if we know the tangent Euler classes for  $g \in J^k(m-1, p)$  with  $\mu(g) = m$  at the monomial ideals (the number of these depend only on  $m$  and not on  $p$ , for a precise formulation, see Proposition 8.4), then we have a simple algorithm to calculate the Thom series of  $Q_g$ . But to give a closed formula in the  $d$ -variables is a different problem in algebraic combinatorics. The difficulty is to move from formulas in Chern roots to formulas in Chern classes. There are many unsolved problems in this context, like expressing Chern classes of various vector bundle constructions in terms of the Chern classes of the input bundles (see e.g. [Pra96, §.2]). It is sometimes a question of taste which form one prefers. In some cases we succeeded to find formulas in the  $d$ -variables, see Section 10.2.

## 8. FURTHER CALCULATIONS

**8.1. Extrapolation.** The tangent Euler classes  $e(T_I \overline{\mathcal{R}I}_g)$  are difficult to calculate directly. At this point we do not have a general method to do it. One of our strategies is to use the Localization Formula backwards: knowing the Thom polynomial  $\text{Tp}_Q(n, p)$  for some  $n$  and  $p$  we can calculate the tangent Euler classes and then we can calculate the whole Thom series. This method is based on relating the tangent Euler classes to incidences (in the sense of [Rim01]).

We will use the following shorthand notations for the tangent Euler classes:

$$e(g, f) = e(T_{I_f} \overline{\mathcal{R}I}_g), \quad e(g, I) = e(T_I \overline{\mathcal{R}I}_g), \quad e(Q, I) = e(T_I \eta_Q),$$

where  $\eta_Q \subset \text{Gr}^\mu$  is the set of ideals  $I$  with quotient algebra  $J^k(n)/I$  isomorphic to  $Q$ .

**Theorem 8.1** (Interpolation Formula). *Let  $g \in J^k(n, p)$  and let  $f \in J^k(n, p)$  be a monomial germ with  $\mu(f) = \mu(g)$ . Then*

$$e(g, f) = \frac{\text{Res}(W_f|W_{Q_f})}{\text{Tp}(g)|_f},$$

where  $W_f = \{w_1, \dots, w_p\}$ ,  $w_i = \sum w_{i,j} \alpha_j$  with  $f_i = \prod x_j^{w_{i,j}}$  and  $|_f$  denotes the restriction to the  $n$ -dimensional subtorus  $T(f)$  of  $\mathcal{K}$  fixing  $f$ , identifying the generators of  $H_{T(f)}^*$  with  $\alpha_1, \dots, \alpha_n$ . In other words  $\alpha_i|_f = \alpha_i$  and  $\beta_i|_f = \sum w_{i,j} \alpha_j$ .

*Proof.* Restricting the Localization Formula we obtain

$$\text{Tp}(g)|_f = \sum_{I \in \mathcal{F}} \frac{\text{Res}(W_f|W_{Q_I})}{e(g, I)}.$$

If  $I$  is a monomial ideal different from  $I_f$  with  $\mu(I) = \mu(f)$  then there is a  $w_i \in W_f \cap W_{Q_I}$  therefore  $\text{Res}(W_f|W_{Q_I}) = 0$ .  $\square$

The next lemma will further simplify our calculations by allowing us to use as small  $n$  as possible. Recall that the stabilization map  $\sigma : J^k(n, p) \rightarrow J^k(n+1, p+1)$  is defined by

$$\sigma g(x_1, \dots, x_{n+1}) = (g_1(x_1, \dots, x_n), \dots, g_p(x_1, \dots, x_n), x_{n+1}).$$

**Lemma 8.2** (Tangent Lemma). *Let  $f, g \in J^k(n, p)$  and let  $f$  be a monomial germ. Then*

$$e(\sigma g, \sigma f) = e(g, f) \text{Res}(\alpha_{n+1}|W_{Q_f}),$$

where  $Q_f$  is the shorthand notation for  $Q_{I_f}$ .

*Proof.* Theorem 7.8 on strong stability implies that  $\mathrm{Tp}(\sigma g)|_{\sigma f} = \mathrm{Tp}(g)|_f$ . Using the Interpolation Theorem 8.1 we get

$$e(\sigma g, \sigma f) = \frac{\mathrm{Res}(W_{\sigma f}|W_{Q_{\sigma f}})}{\mathrm{Tp}(\sigma g)|_{\sigma f}} = \frac{\mathrm{Res}(W_f|W_{Q_f}) \mathrm{Res}(\alpha_{n+1}|W_{Q_f})}{\mathrm{Tp}(g)|_f},$$

by noticing that  $W_{\sigma f} = W_f \cup \{\alpha_{n+1}\}$  and  $Q_{\sigma f} = Q_f$ .  $\square$

Using the Localization Formula it is easy to see that the Tangent Lemma is equivalent to Theorem 7.8, so it would be interesting to find a direct proof of it.

**Example 8.3. The Thom polynomial of  $A_3$ :** The first case not covered in Section 6.1 is the Thom series of the Morin singularity  $A_3$ , the contact class corresponding to the algebra  $\mathbb{C}[[x]]/(x^4)$ . Since  $\mu(g) = 3$  it is enough to write down the Localization Formula for  $n = 2$  (see Remark 7.18). The monomial ideals for  $n = 2$  can be identified with partitions of  $\mu + 1 = 4$ : (4), (31), (211), (22). These monomial ideals will be denoted by  $I_4, I_{31}, I_{211}$  and  $I_{22}$ . Germs with these monomial ideals will be denoted by  $f_4, f_{31}, f_{211}$  and  $f_{22}$ .

Since  $f_4$  is the suspension (c.f. Section 7.2) of the jet  $x_1 \mapsto x_1^4 \in J^4(1, 1)$ , we can apply the Tangent Lemma 8.2. The ideal  $(x_1^4)$  of  $J^k(1)$  is a fixed point of the  $\mathcal{R}$ -action, so  $e(x_1^4, x_1^4) = 1$ , therefore

$$e(f_4, f_4) = \mathrm{Res}(\alpha_1, 2\alpha_1, 3\alpha_1|\alpha_2).$$

We could alternatively use the calculation (15) of the tangent space of an  $\mathcal{R}$ -orbit.

For  $I_{22} = (x_1^2, x_2^2)$  we use the Interpolation Formula. The ideal  $I_{22}$  has 2 generators so we need that  $\mathrm{tp}_{A_3}(0) = c_1^3 + 3c_1c_2 + 2c_3$ . We write this polynomial in the Chern roots  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and restrict to  $f_{22}$  ( $\beta_1 \mapsto 2\alpha_1, \beta_2 \mapsto 2\alpha_2$ ) i.e. make the substitutions

$$c_1 = 2\alpha_1 + 2\alpha_2 - \alpha_1 - \alpha_2 = \alpha_1 + \alpha_2, \quad c_2 = \alpha_1\alpha_2 - \alpha_1^2 - \alpha_2^2, \quad c_3 = (\alpha_1 + \alpha_2)(\alpha_2 - \alpha_1)^2,$$

and we get that

$$\mathrm{Tp}_{A_3}(2, 2)|_{f_{22}} = (\alpha_1 + \alpha_2)\alpha_1\alpha_2.$$

The interpolation theorem implies that

$$e(f_4, f_{22}) = \frac{\mathrm{Res}(\alpha_1, \alpha_2, \alpha_1 + \alpha_2|2\alpha_1, 2\alpha_2)}{(\alpha_1 + \alpha_2)\alpha_1\alpha_2} = \frac{(\alpha_2 - \alpha_1)^2(2\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2)}{\alpha_1 + \alpha_2}.$$

We also need to calculate the Euler class at  $f_{31} = (x_1^3, x_1x_2, x_2)$  (the Euler class at  $f_{211}$  can be obtained by permuting  $\alpha_1$  and  $\alpha_2$ ). The ideal of  $I_{31}$  has three generators so we need  $\mathrm{tp}_{A_3}(1)$  to apply the interpolation formula. This calculation is better to do with computer, the result can be found in Section 8.2 at  $\mu = 3$ .

This method of calculating all the ingredients of the Localization Formula from a concrete Thom polynomial  $\mathrm{tp}_Q(l)$  (for an appropriate  $l$ ), will be called the Extrapolation method. Now we estimate the value of  $l$  for which this method works.

**Proposition 8.4.** *For a monomial algebra  $Q$  (i.e.  $Q = J^k(n)/I$  for a monomial ideal  $I$ ) with generators  $a(Q) \leq \mu(Q) - 1$ , the number of relations  $b(Q)$  is at most  $\binom{\mu(Q)}{2}$ .*

*Proof.* If  $a(Q) = \mu(Q) - 1$ , then  $I$  contains all but one quadratic monomials. They are all generators and, if the missing monomial is of the form  $x_i^2$ , there can be one extra generator, all together maximum  $\binom{\mu(Q)}{2}$  generators. If  $a(Q) < \mu(Q) - 1$ , then ‘‘cut off’’ a maximal degree monomial from  $Q$ : let us call the resulting algebra  $Q'$ . Then  $\mu(Q') = \mu(Q) - 1$  and  $a(Q') \leq$

$\mu(Q) - 2$  so by an induction argument we can assume that  $b(Q') \leq \binom{\mu(Q)-1}{2}$ . But  $b(Q) \leq b(Q') + (\mu(Q) - 2) < \binom{\mu(Q)}{2}$ .  $\square$

**Corollary 8.5.** *To calculate the Thom series of the nilpotent algebra  $Q$  with the Extrapolation method, it is enough to know  $\text{tp}_Q(\binom{\mu(Q)-1}{2})$ .*  $\square$

We saw already that the Thom series depends only on the finite data of tangent Euler classes at monomial ideals for  $n = \mu(Q) - 1$ . Now we see that the same information is stored in the polynomial  $\text{tp}_Q(\binom{\mu(Q)-1}{2})$  in a compact way.

Let us emphasize that the Localization Formula shows that it is theoretically possible to calculate a closed formula for *any* Thom series, as we have an algorithm based on Groebner degeneration to calculate any Thom polynomial. However this algorithm is extremely ineffective for explicit calculations (works only for trivial cases).

There is a remarkable relation among the tangent Euler classes  $e(Q, I)$  for different monomial ideals  $I$ . Consider the Localization formula for  $n = \mu(Q)$ , and  $p = 0$ :

$$(22) \quad \text{Tp}_Q(\mu(Q), 0) = \sum_I \frac{1}{e(Q, I)}.$$

Since the left hand side is 1 for the Morin singularity  $A_{\mu(Q)}$ , and is 0 for all other algebras, this formula is an identity among the Euler classes. As a consequence, the value of  $e(Q, \mathcal{M}_{\mu(Q)}^2)$  can be calculated algebraically from the other Euler classes.

**Remark 8.6.** In this section we showed that if the method of Restriction Equations [Rim01] calculates  $\text{tp}_Q(\binom{\mu(Q)-1}{2})$ , then—using the interpolation formula—we have the Thom polynomial in any relative codimension  $l$ . However, the Extrapolation method gives more, it clarifies the expected regularity of the coefficients of the Thom series. For  $\mu(Q) \geq 6$  the Restriction Equations method is not sufficient to determine  $\text{tp}_Q(\binom{\mu(Q)-1}{2})$  (except for  $A_6$ ). The reason is the appearance of a continuous family of nilpotent algebras  $C_\lambda$  with  $\mu(C_\lambda) = 6$ . We study this family in Section 9.3. For  $\mu(Q) \leq 4$  and  $n = 2$  we list the tangent Euler classes in the next section. It might be a coincidence that G. Bérczi and A. Szenes in [BS06] calculates the Thom series of  $A_\mu$  explicitly for  $\mu \leq 6$ , exactly in the range where the extrapolation method works.

**8.2. Thom polynomials corresponding to algebras of small dimension.** In what follows the maximal ideal  $\mathcal{M}_i$  of  $\mathbf{C}[[x_1, \dots, x_i]]$  will be considered to be a subset of  $\mathcal{M}_{i+1}$ . If  $I \subset \mathcal{M}_i$  is an ideal we define its descendant in  $\mathcal{M}_{i+1}$  as  $I + (x_{i+1})$ . Descendants of descendants are also called descendants. Observe that the factor ring of  $\mathcal{M}_i$  by  $I$  is isomorphic to the factor ring of  $\mathcal{M}_{i+1}$  by the descendant of  $I$ , in particular  $\text{codim}(I \subset \mathcal{M}_i) = \text{codim}(I' \subset \mathcal{M}_{i+1})$  for the descendant  $I'$  of  $I$ .

Recall also that we consider the right group  $\mathcal{R}(i)$  acting on  $\mathcal{M}_i$ , in particular the symmetric group  $S_i \subset \text{GL}(i) \subset \mathcal{R}(i)$  also acts on  $\mathcal{M}_i$ .

Let us fix  $\mu \geq 1$ . Consider a list of monomial ideals  $I_i \subset \mathcal{M}_{n(i)}$  ( $n(i) \leq \mu$ ) such that

- $\text{codim}(I_i \subset \mathcal{M}_{n(i)}) = \mu$ ;
- no ideal in the  $\text{GL}_{n(i)}$ -orbit of  $I_i$  is the descendant of an ideal in  $\mathcal{M}_{n(i)-1}$ ;
- the  $S_\mu$ -orbits of the descendants of  $I_i$ 's in  $\mathcal{M}_\mu$  form a no-repetition, complete list of the codimension  $\mu$  monomial ideals of  $\mathcal{M}_\mu$ .

**Example 8.7.** Here are examples for small  $\mu$ , with the notation  $x, y, z, \dots = x_1, x_2, x_3, \dots$

$$\mu = 1: I_1 = (x^2) \subset \mathcal{M}_1.$$

$$\mu = 2: I_1 = (x^3) \subset \mathcal{M}_1, I_2 = (x^2, xy, y^2) = \mathcal{M}_2^2 \subset \mathcal{M}_2.$$

$$\mu = 3: I_1 = (x^4) \subset \mathcal{M}_1, I_2 = (x^2, y^2) \subset \mathcal{M}_2, I_3 = (x^2, xy, y^3) \subset \mathcal{M}_2,$$

$$I_4 = \mathcal{M}_3^2 = (x^2, y^2, z^2, xy, yz, zx) \subset \mathcal{M}_3.$$

$$\mu = 4: I_1 = (x^5) \subset \mathcal{M}_1, I_2 = (x^2, xy, y^4) \subset \mathcal{M}_2, I_3 = (x^3, xy, y^3) \subset \mathcal{M}_2, I_4 = (x^2, xy^2, y^3) \subset \mathcal{M}_2,$$

$$I_5 = (x^2, y^2, z^3, xy, yz, zx) \subset \mathcal{M}_3, I_6 = (x^2, y^2, z^2, xy, xz) \subset \mathcal{M}_3,$$

$$I_7 = \mathcal{M}_4^2 \subset \mathcal{M}_4.$$

**Remark 8.8.** Monomial ideals  $I$  of  $\mathcal{M}_n$  can be visualized by the set  $\{(i_1, i_2, \dots, i_n) \in \mathbf{Z}^n : \prod_{j=1}^n x_j^{i_j} \notin I\}$ . This set can be viewed as the  $n$ -dimensional generalization of (two dimensional) Young diagrams of partitions. In this language, the list  $I_i$  for a given  $\mu$  is the list of all “shapes” of cardinality  $\mu + 1$  Young diagrams of dimension at most  $\mu$ .

The Localization Formula (11) can now be rephrased as follows.

**Theorem 8.9.** *Let  $\mu$  be a positive integer, and  $I_i$  be a list of monomial ideals described above. Then for a nilpotent algebra of dimension  $\mu$  we have*

$$(23) \quad \text{Tp}_Q(n, p) = \sum_i \text{Sym}_{I_i} \frac{\text{Res}(\beta_1, \dots, \beta_p | W_{Q_{I_i}})}{e(Q, I_i) \cdot \text{Res}(\alpha_{n(i)+1}, \dots, \alpha_n | W_{Q_{I_i}})},$$

where  $e(Q, I_i)$  is the virtual tangent Euler class of the closure of the set

$$\{I \triangleleft \text{Gr}^\mu(\mathcal{M}_{n(i)}) : \mathcal{M}_{n(i)} / I \cong Q\}$$

at the point  $I_i$ . The symmetrizer operator acts on a polynomial  $p$  by

$$\text{Sym}_{I_i}(p(\alpha_1, \dots, \alpha_n)) = \frac{1}{|\{\sigma \in S_n : \sigma(I_i) = I_i\}|} \sum_{\sigma \in S_n} p(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}).$$

If  $n(i) > n$ , or  $e(Q, I_i) = \infty$  for some  $i$ , then the  $i$ 'th term in the sum in (23) is defined to be 0. □

**Corollary 8.10.** *The finitely many rational functions  $e(Q, I_i)$  determine the Thom polynomials  $\text{Tp}_Q(n, p)$  of the nilpotent algebra  $Q$  for all  $n$  and  $p$ .*

Using the convention  $x, y, z, \dots = x_1, x_2, x_3, \dots$ , and the following names of nilpotent algebras:

$$A_i = \mathcal{M}_1 / (x^{i+1}), \quad I_{a,b} = \mathcal{M}_2 / (xy, x^a + y^b),$$

$$III_{a,b} = \mathcal{M}_2 / (x^a, xy, y^b), \quad \Sigma^{2,1} = \mathcal{M}_2 / (x^2, xy^2, y^3),$$

here is a list of some  $E$ -classes:

$$\mu = 1:$$

$$e(A_1, (x^2)) = 1$$

$$\mu = 2:$$

$I =$		$(x^3)$		$(x^2, xy, y^2)$
$e(A_2, I) =$		1		$\frac{1}{3}(\alpha_1 - 2\alpha_2)(\alpha_2 - 2\alpha_1)$

$\mu = 3$ :

$I =$	$(x^4)$	$(x^2, y^2)$	$(x^2, xy, y^3)$
$e(A_3, I) =$	1	$\frac{(\alpha_1 - \alpha_2)^2(2\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2)}{(\alpha_1 + \alpha_2)}$	$\frac{1}{2}(3\alpha_2 - \alpha_1)(\alpha_1 - \alpha_2)^2$
$e(I_{2,2}, I) =$	$\infty$	$-(\alpha_1 - \alpha_2)^2$	$2(\alpha_1 - \alpha_2)^2$
$e(III_{2,3}, I) =$	$\infty$	$\infty$	$\alpha_1 - \alpha_2$

 $\mu = 4$ :

$$\begin{aligned}
e(A_4, (x^5)) &= 1, \\
e(A_4, (x^2, xy, y^4)) &= \frac{1}{5}(\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2)(\alpha_1 - 4\alpha_2)(3\alpha_2 - 2\alpha_1), \\
e(A_4, (x^3, xy, y^3)) &= \frac{1}{5}(\alpha_1 - \alpha_2)^2(2\alpha_1 - 3\alpha_2)(3\alpha_1 - 2\alpha_2), \\
e(A_4, (x^2, xy^2, y^3)) &= \frac{2(\alpha_1 - 2\alpha_2)(2\alpha_1 - \alpha_2)(\alpha_1 - 3\alpha_2)(\alpha_1 - \alpha_2)^2}{5(\alpha_1 + \alpha_2)}, \\
e(A_4, (x^2, y^2, z^2, xy, xz)) &= \clubsuit \cdot \frac{(\alpha_2 + \alpha_3 - 2\alpha_1)(\alpha_2 - 2\alpha_3)(\alpha_3 - 2\alpha_2)}{5(\alpha_2 + \alpha_3)}, \\
e(A_4, (x^2, y^2, z^3, xy, yz, zx)) &= \spadesuit \cdot \frac{-2(\alpha_1 + \alpha_2 - 2\alpha_3)(3\alpha_3 - \alpha_2)(3\alpha_3 - \alpha_1)(\alpha_1 - 2\alpha_2)(\alpha_2 - 2\alpha_1)}{5(4\alpha_1^3 - 9\alpha_1^2\alpha_3 - 5\alpha_1^2\alpha_2 - 4\alpha_1\alpha_3^2 + 15\alpha_1\alpha_2\alpha_3 - 5\alpha_1\alpha_2^2 - 4\alpha_2\alpha_3^2 - 9\alpha_2^2\alpha_3 + 4\alpha_2^3 + 9\alpha_3^3)}.
\end{aligned}$$

$$\begin{aligned}
e(I_{2,3}, (x^5)) &= \infty, \\
e(I_{2,3}, (x^2, xy, y^4)) &= (\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2)(2\alpha_1 - 3\alpha_2), \\
e(I_{2,3}, (x^3, xy, y^3)) &= \frac{(\alpha_1 - \alpha_2)^2(2\alpha_1 - 3\alpha_2)(3\alpha_1 - 2\alpha_2)}{\alpha_1 + \alpha_2}, \\
e(I_{2,3}, (x^2, xy^2, y^3)) &= (\alpha_1 - \alpha_2)^2(2\alpha_2 - \alpha_1), \\
e(I_{2,3}, (x^2, y^2, z^2, xy, xz)) &= \clubsuit \cdot \frac{(2\alpha_3 - \alpha_2)(\alpha_3 - 2\alpha_2)(2\alpha_1 - \alpha_2 - \alpha_3)}{\alpha_1\alpha_2 + \alpha_1\alpha_3 + 4\alpha_2^2 + 4\alpha_3^2 - 16\alpha_2\alpha_3}, \\
e(I_{2,3}, (x^2, y^2, z^3, xy, yz, zx)) &= \spadesuit \cdot \frac{2(\alpha_1 + \alpha_2 - 2\alpha_3)(3\alpha_3 - \alpha_1)(3\alpha_3 - \alpha_2)(\alpha_1 - 2\alpha_2)(\alpha_2 - 2\alpha_1)}{4(\alpha_1^4 + \alpha_2^4) - 6\alpha_1^2\alpha_2^2 - 5\alpha_1\alpha_2(\alpha_1^2 + \alpha_2^2) + \alpha_3(-25(\alpha_1^3 + \alpha_2^3) + 39\alpha_1\alpha_2(\alpha_1 + \alpha_2)) + \alpha_3^2(29(\alpha_1^2 + \alpha_2^2) - 59\alpha_1\alpha_2) + \alpha_3^3(\alpha_1 + \alpha_2)}.
\end{aligned}$$

$$\begin{aligned}
e(III_{2,4}, (x^5)) &= \infty, \\
e(III_{2,4}, (x^3, xy, y^3)) &= \infty, \\
e(III_{2,4}, (x^2, xy, y^4)) &= (\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2), \\
e(III_{2,4}, (x^2, xy^2, y^3)) &= (\alpha_1 - \alpha_2)(2\alpha_2 - \alpha_1), \\
e(III_{2,4}, (x^2, y^2, z^2, xy, xz)) &= \clubsuit \cdot \frac{(\alpha_2 - 2\alpha_3)(\alpha_3 - 2\alpha_2)}{\alpha_1\alpha_2 + \alpha_1\alpha_3 - 4\alpha_2^2 - 4\alpha_3^2 + 4\alpha_2\alpha_3}, \\
e(III_{2,4}, (x^2, y^2, z^3, xy, yz, zx)) &= \spadesuit \cdot \frac{-(\alpha_1 - 3\alpha_3)(\alpha_2 - 3\alpha_3)}{2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3 - \alpha_1\alpha_2)}.
\end{aligned}$$

$$\begin{aligned}
e(III_{3,3}, (x^5)) &= \infty, \\
e(III_{3,3}, (x^2, xy, y^4)) &= \infty, \\
e(III_{3,3}, (x^3, xy, y^3)) &= -(\alpha_1 - \alpha_2)^2, \\
e(III_{2,4}, (x^2, xy^2, y^3)) &= 2(\alpha_1 - \alpha_2)^2, \\
e(III_{3,3}, (x^2, y^2, z^2, xy, xz)) &= \clubsuit \cdot (-1), \\
e(III_{3,3}, (x^2, y^2, z^3, xy, yz, zx)) &= \spadesuit \cdot \frac{2(\alpha_1 - 2\alpha_2)(\alpha_2 - 2\alpha_1)}{2\alpha_1^2 + 2\alpha_2^2 - 3\alpha_3^2 - 2\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}.
\end{aligned}$$

$$\begin{aligned}
e(\Sigma^{2,1}, (x^5)) &= \infty, \\
e(\Sigma^{2,1}, (x^2, xy, y^4)) &= \infty, \\
e(\Sigma^{2,1}, (x^3, xy, y^3)) &= \infty, \\
e(\Sigma^{2,1}, (x^2, xy^2, y^3)) &= \alpha_1 - \alpha_2, \\
e(\Sigma^{2,1}, (x^2, y^2, z^2, xy, xz)) &= \clubsuit \cdot \frac{1}{2(\alpha_1 - \alpha_2 - \alpha_3)}, \\
e(\Sigma^{2,1}, (x^2, y^2, z^3, xy, yz, zx)) &= \spadesuit \cdot \frac{1}{\alpha_1 + \alpha_2 - \alpha_3},
\end{aligned}$$

where

$$\clubsuit = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - 2\alpha_2)(\alpha_1 - 2\alpha_3)(\alpha_2 - \alpha_3)^2,$$

$$\spadesuit = (\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2(\alpha_1 - \alpha_2 - \alpha_3)(\alpha_2 - \alpha_1 - \alpha_3).$$

Theorem 8.9, and the list of  $e(Q, I)$ -classes above give the Thom polynomial of all singularities whose associated algebra has dimension at most 4, with the following exceptions:

- We did not include  $e(Q, I)$  classes for  $Q = \mathcal{M}_i / \mathcal{M}_i^2 = \Sigma^i$  for  $i = 2, 3, 4$ , since those Thom polynomials (Giambelli-Thom-Porteous formulae) are known, see (21).
- For  $\mu = 3$  and  $\mu = 4$  we did not include the classes  $e(Q, \mathcal{M}_\mu^2)$ , since they can be calculated using (22).
- There are three other algebras with  $\mu = 4$ , namely

$$\mathcal{M}_3 / (x^2, y^2, z^3, xy, yz, zx), \mathcal{M}_3 / (x^2, y^2, z^2, xz, yz), \mathcal{M}_3 / (xy, xz, yz, x^2 - y^2, x^2 - z^2).$$

Their Thom polynomials will be studied in Section 10 (under the names  $\Phi_{3,2}, \Phi_{3,1}, \Phi_{3,0}$ ).

## 9. RETURNING TO GEOMETRY

In this section we use some simple geometric observation to calculate the Localization Formula for some singularities. We believe that this is just the beginning. In a similar fashion one can transform the deep geometric knowledge of singularity theorists into further formulas.

**9.1. The punctual Hilbert scheme.** The localization Formula reduces the Thom polynomial calculations to the study of the space of ideals  $\mathcal{H}^m(n) = \mathcal{H}^m(n, k)$  in  $\text{Gr}^m(J^k(n))$ . We suppress  $k$  from the notation since these spaces are isomorphic for all  $k \geq m$ , what we will assume. The variety  $\mathcal{H}^m(n)$  was studied under the name *local punctual Hilbert scheme* (with the reduced scheme structure) by A. Iarrobino, J. Damon, A. Galligo, T. Gaffney and others (see [Iar72a, DG76, Gaf85]). The connection of this Hilbert scheme with singularity theory is well known, however the  $\mathcal{R}(n)$ -equivariant theory needed for Thom polynomial calculations is not developed yet. We do not pursue this approach here but we believe that it would lead to strong results.

The structure of  $\mathcal{H}^m(n)$  is complicated. In general even its dimension is not known. It has many components, one of which is the closure of the orbit  $\mathcal{R}(n)(x_1^{m+1}, x_2, \dots, x_n)$ . For  $n = 2$  this is the only component, for  $n \geq 3$ , there can be others. Nevertheless we can see that the calculation of the Thom polynomials of the Morin singularities  $A_m$  is related to the study of the singularities of this component at monomial ideals.

The Hilbert scheme  $\mathcal{H}^m(n)$  has many smooth  $\mathcal{R}(n)$ -invariant subvarieties. For the corresponding contact classes we can easily calculate the Localization Formula.

**9.2. Subgrassmannians.** Let  $V$  be a  $d$ -dimensional subspace of  $P := \text{Hom}(\text{Sym}^{k+1} \mathbf{C}^n, \mathbf{C})$  for  $d < \binom{n+1}{2}$ . Then

$$I_V := \langle V, \text{Hom}(\text{Sym}^{k+2} \mathbf{C}^n, \mathbf{C}) \rangle$$

is an ideal of  $J^{k+N}(n)$  for any  $N \geq 1$ . We have the map  $j : V \mapsto I_V$  mapping the Grassmannian  $\text{Gr}_d(P)$  into  $\mathcal{H}^m(n)$  where  $m = \sum_{i=2}^{k+1} \binom{n+i-1}{i} - d$ . It is easy to see that  $j$  is an embedding. These subgrassmannians were used by Iarrobino in [Iar72b] to give a lower bound on the dimension of  $\mathcal{H}^m(n)$ . The corresponding contact classes

$$\Sigma^{n^{k(d)}}(n, p) = \Sigma^{\overbrace{n, \dots, n}^k}^{(d)}(n, p) := \{g \in J^{k+N}(n, p) : I_g \in j(\text{Gr}_d(P))\}$$

were studied by J. Damon in [Dam72]. He also calculated the Thom polynomial of some of these classes. The Localization Formula gives the answer for all of these cases:

Let  $W$  be the set of integer  $k+1$ -tuples  $w = (w_1, \dots, w_{k+1})$  with  $1 \leq w_1 \leq w_2 \leq \dots \leq w_{k+1} \leq n$  and let  $\alpha_w$  denote the weight  $\sum w_i \alpha_i$ . Then  $\{\alpha_w : w \in W\}$  is the set of weights of  $P = \text{Hom}(\text{Sym}^{k+1} \mathbf{C}^n, \mathbf{C})$ . The  $T(n)$ -fixed points  $F$  of the Grassmannian  $\text{Gr}_d(P)$  are corresponding to the  $d$ -element subsets of  $W$ . For  $S \in F$  let

$$[E_S] = \prod_{i=1}^p \prod_{\sigma \in S} (\beta_i - \alpha_\sigma) \quad \text{and} \quad e_S = e(T_S \text{Gr}_d(P)) = \prod_{\sigma \in S} \prod_{\sigma' \in W \setminus S} (\alpha_\sigma - \alpha_{\sigma'}).$$

Then by the Localization Formula we get

$$(24) \quad [\Sigma^{n^{k(d)}}(n, p)] = [\Sigma^{n^k}(n, p)] \cdot \sum_{S \in F} \frac{[E_S]}{e_S}.$$

Recall that we calculated  $[\Sigma^{n^k}(n, p)]$  in Section 6.1. If  $k = 1$  and  $d = \binom{n+1}{2} - 1$  then by Proposition 7.11 we can calculate the stable Thom polynomial from (24). In fact these Thom polynomials can be calculated by a direct geometric argument (see the proof of Theorem 10.4). (Knowing the result from the Localization Formula certainly helped to find the geometric argument.)

Even if the  $\mathcal{R}(n)$ -invariant subvariety  $Y$  is not smooth, we can simplify the Localization Formula if we know that  $Y$  is contained in one of these subgrassmannians:

**9.3. Nets of conics.** Consider the jet

$$C_\lambda = (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy) \quad \text{for } \lambda(\lambda^3 - 1)(8\lambda^3 + 1) \neq 0.$$

This one-parameter family was studied by Mather in [Mat70] and Wall in [Wal77]. This is the smallest codimensional example of a family in the equidimensional case. The contact class of  $C_\lambda$  has codimension 10, their union is open in the Thom-Boardmann class  $\Sigma^3$ . This implies that  $\text{tp}(C_\lambda) = A\Delta_{3331} + B\Delta_{433}$  for some  $A, B \in \mathbf{N}$  (see (6)). The restriction equation  $\text{tp}(C_\lambda)|_{C_\mu} = 0$  implies that  $A = 2B$ .

From the definition we can see that  $I_\lambda$ , the ideal of  $C_\lambda$ , is in the subgrassmannian  $X := j(\text{Gr}_3(\text{Hom}(\text{Sym}^2 \mathbf{C}^3, \mathbf{C})))$ . The action of the right group  $\mathcal{R}(3)$  depends only on the linear part—only the  $\text{GL}(3)$ -action is interesting. As it is explained in [Mat70] only the 8-dimensional Lie group  $\text{PGL}(3)$  acts effectively on the 9-dimensional manifold  $X$  and the orbit  $o_\lambda$  of  $C_\lambda$  is 8-dimensional, i.e. its closure is a hypersurface.

For the Localization Formula we need to calculate  $[\overline{o_\lambda} \subset X]|_f$ , where  $f$  is a fixed point of the  $T(3)$ -action, i.e. spanned by three of the six basis vectors  $\{x_i x_j : 1 \leq i \leq j \leq 3\}$  of

$\text{Hom}(\text{Sym}^2 \mathbf{C}^3, \mathbf{C})$ . Calculating these incidences boils down to calculating the degree of the hypersurface  $\overline{o_\lambda}$ , i.e.  $[\overline{o_\lambda}] \in H^2(X) \cong \mathbf{Z}$ . In fact localization can be avoided here, since it can be seen directly using *projective* Thom polynomials ([FNR05, §6.]), that this degree is equal to  $B$ . Using ideas of [Wal77] one can show that this degree is 4 for generic  $\lambda$ , therefore we have

**Theorem 9.1.** *The Thom polynomial of  $C_\lambda$  for generic  $\lambda$  in relative codimension 0 is*

$$(25) \quad \text{tp}_{C_\lambda}(0) = 8\Delta_{3331} + 4\Delta_{433}.$$

## 10. THOM SERIES OF $\Phi_{n,r}$ SINGULARITIES

The subgrassmannian  $j\text{Gr}^1(\text{Hom}(\text{Sym}^2 \mathbf{C}^n, \mathbf{C})) \cong \mathbf{P}(\text{Sym}^2 \mathbf{C}^n)$  splits into orbits  $X(n, r)$  according to the corank of the symmetric matrices in  $\text{Sym}^2 \mathbf{C}^n$ . The orbits correspond to the following algebras.

**Definition 10.1.** Let  $m > r$  be nonnegative integers. The quotient of  $\mathcal{M}_m$  by the ideal

$$J_{m,r} = \left\langle \sum_{1 \leq i < j \leq m} a_{ij} x_i x_j : \sum_{r+1}^m a_{ii} = 0 \right\rangle + \mathcal{M}_m^3$$

will be denoted by  $\Phi_{m,r}$ .

A finite generator set of  $J_{m,r}$  is given by:

$$J_{m,r} = \langle x_i x_j, x_k^2, x_{r+1}^2 - x_l^2 : 1 \leq i < j \leq m, 1 \leq k \leq r, r+2 \leq l \leq m \rangle$$

for  $r < m - 1$ , and

$$J_{m,m-1} = \langle x_i x_j, x_k^2, x_n^3 : 1 \leq i < j \leq m, 1 \leq k \leq m - 1 \rangle.$$

Observe that for small values of the parameters  $m, r$  we recover familiar algebras:

$$\Phi_{1,0} = A_2, \quad \Phi_{2,0} = I_{2,2}, \quad \Phi_{2,1} = III_{2,3}.$$

Following our previous convention, the singularities corresponding to the algebras  $\Phi_{m,r}$  in  $\mathcal{E}_0(n, n+l)$  will be denoted by  $\Phi_{m,r}(n, n+l)$ . Calculation shows that

$$\text{codim}(\Phi_{m,r}(n, n+l) \subset \mathcal{E}_0(n, n+l)) = (m+1)l + \left( \binom{m+1}{2} + \binom{r+1}{2} + 1 \right).$$

**10.1. Thom polynomials of  $\Phi_{m,r}$  in terms of Chern roots.** To calculate the Thom polynomials of these classes—using the Localization Formula—we need the  $\text{GL}(n)$ -equivariant cohomology classes  $[X(n, r) \subset \mathbf{P}(\text{Sym}^2 \mathbf{C}^n)]$  restricted to the  $T(n)$ -fixed points of  $\mathbf{P}(\text{Sym}^2 \mathbf{C}^n)$ . The cohomology class of the cone of  $X(n, r)$  was calculated in [HT84] and [JLP82]:

$$[\text{Cone}X(n, r) \subset \text{Sym}^2 \mathbf{C}^n] = 2^r \Delta_{r,r-1,\dots,2,1},$$

where  $\Delta_{r,r-1,\dots,2,1}$  denotes the Schur polynomial in the Chern classes  $c_1, \dots, c_n$ , corresponding to the partition  $(r, r-1, \dots, 2, 1)$ . Using [FNR05, §6] we can calculate the  $T(n)$ -equivariant *projective Thom polynomial*

$$[X(n, r) \subset \mathbf{P}(\text{Sym}^2 \mathbf{C}^n)] = 2^r \Delta_{r,r-1,\dots,2,1}(\alpha_1 - \frac{1}{2}\xi, \dots, \alpha_n - \frac{1}{2}\xi) \text{ in}$$

$$H_{T(n)}^*(\mathbf{P}(\text{Sym}^2 \mathbf{C}^n)) \cong \mathbf{Z}[\alpha_1, \dots, \alpha_n, \xi] / \prod (\alpha_i + \xi).$$

We need the restriction of this class to the fixed points  $\{f_{ij} : i \leq j \leq n\}$  of  $\mathbf{P}(\text{Sym}^2 \mathbf{C}^n)$ :

$$(26) \quad [X(n, r)]|_{f_{ij}} = 2^r \Delta_{r, r-1, \dots, 2, 1}(\alpha_1 - \frac{1}{2}(\alpha_i + \alpha_j), \dots, \alpha_n - \frac{1}{2}(\alpha_i + \alpha_j)).$$

The other components of the Localization Formula are

$$[E_{ij}] = \text{Res}(\beta_1, \dots, \beta_p | \alpha_i + \alpha_j), \quad e_{ij}^{(n)} = \text{Res}(K_{ij}^{(n)} | \alpha_i + \alpha_j), \quad \text{for } K_{ij}^{(n)} = \{\alpha_k + \alpha_l : k \leq l, (k, l) \neq (i, j)\}.$$

Hence the Interpolation formula yields to the following

**Theorem 10.2.** *The Thom polynomial of  $\Phi_{n,r}$  is*

$$(27) \quad \text{Tp}_{\Phi_{n,r}}(n, p) = \text{Res}(\beta_1, \dots, \beta_p | \alpha_1, \dots, \alpha_n) \sum_{1 \leq i \leq j \leq n} \frac{[E_{ij}]}{e_{ij}^{(n)}} [X(n, r)]|_{f_{ij}}.$$

□

**10.2. Thom polynomials of  $\Phi_{m,r}$  in terms of quotient Chern classes.** Since  $\mu(\Phi_{n,r}) = n + 1$ , the polynomial  $\text{Tp}_{\Phi_{n,r}}(n, p)$  determines the Thom series of  $\Phi_{n,r}$  by Proposition 7.11. We devote this section to the calculation of these Thom series.

10.2.1. *Notations from algebraic combinatorics.* Let

$$A_n = \{\alpha_1, \dots, \alpha_n\}, \quad B_p = \{\beta_1, \dots, \beta_p\}.$$

For a partition  $\lambda = (\lambda_1, \dots, \lambda_s)$  and variables  $x_1, \dots, x_t$  we define

$$\Delta_\lambda(x_1, \dots, x_t) = \det(\sigma_{\lambda_i + j - i})_{1 \leq i, j \leq s},$$

where  $\sigma_i = \sigma_i(x_1, \dots, x_t)$  is the  $i$ th elementary symmetric polynomial of  $x_1, \dots, x_t$  (i.e.  $\sigma_1 = \sum x_i$ ,  $\sigma_2 = \sum_{i \neq j} x_i x_j$  etc). The symbol  $\Delta_\lambda$  without arguments (as before, in (6)) will denote the determinant

$$\Delta_\lambda = \det(c_{\lambda_i + j - i})_{1 \leq i, j \leq s}$$

with entries the quotient variables (5), (17). We will use partitions, and their notations as are in [Ful97]. For example,  $\bar{\lambda}$  will denote the *conjugate* partition of  $\lambda$ . Addition of partitions is defined coordinatewise,  $a^b$  means  $b$  copies of  $a$ , and concatenation is indicated by a comma. For example  $(3^4 + (2, 1), (1, 1)) = (5, 4, 3, 3, 1, 1)$ . We will need the staircase partition

$$\rho_s = (s, s - 1, \dots, 2, 1).$$

**Definition 10.3.** The Schur coefficients of the equivariant Segre classes of  $\text{Sym}^2 \mathbf{C}^n$  will be denoted by double brackets; namely:

$$\frac{1}{\prod_{1 \leq i \leq j \leq n} (1 - \alpha_i - \alpha_j)} = \sum_I ((I)) \Delta_{I - \rho_{n-1}}(\alpha_1, \dots, \alpha_n).$$

Here  $I$  runs through all length  $n$  sequences  $I = (i_1, i_2, \dots, i_n)$  with  $i_1 > i_2 > \dots > i_n \geq 0$ .

The numbers  $((I))$  are positive; their combinatorics, as well as recursion and Pfaffian formulas are studied in [Pra88, LLT89, Pra96]. For practical purposes, the following recursion is most useful

$$r((i_1, \dots, i_r)) - 2 \sum_{k=1}^r ((i_1, \dots, i_{k-1}, i_k - 1, i_{k+1}, \dots, i_r)) = \begin{cases} 0 & i_r > 0 \\ ((i_1, \dots, i_{r-1})) & i_r = 0, \end{cases}$$

together with the conventions  $((0)) = 1$ , and that  $((I)) = 0$  if  $I$  does not satisfy  $i_1 > i_2 > \dots > i_n \geq 0$ . For example  $((i)) = 2^i$ ,  $((i, 0)) = 2^i - 1$ ,  $((2, 1)) = 3$ ,  $((3, 1)) = 10$ .

It was P. Pragacz in [Pra07] who first identified the coefficients of the Schur expansion of the Thom series of  $I_{2,2}$  with the coefficients  $((I))$ .

Now we are ready to present the Thom series corresponding to the algebras  $\Phi_{m,m-s}$ .

**Theorem 10.4.** *We have*

$$(28) \quad \text{tp}_{\Phi_{m,m-s}}(l) = \sum_I ((I)) \Delta_{I'},$$

where

$$\begin{aligned} I' &= ((l+s)^s + I - \rho_{s-1}, (l+m)^{m-s}, l+m+1-s-|I|) \\ &= (l+1+i_1, l+2+i_2, \dots, l+s+i_s, \underbrace{l+m, \dots, l+m}_{m-s}, l+m+1-s-|I|), \end{aligned}$$

and the summation is for sequences  $I = (i_1, \dots, i_s)$  with  $i_1 > i_2 > \dots > i_s \geq m-s$  and  $|I| = i_1 + \dots + i_s \leq l+m-s+1$ .

**Remark 10.5.** We can formally change the summation for all sequences  $I$  with  $i_1 > i_2 > \dots > i_s \geq 0$  without changing the sum. Indeed, if  $|I|$  is larger than  $l+m-s+1$  then the  $\Delta$  polynomial is zero, since the last part in the partition is negative. If  $i_s < m-s$  then in the determinant expansion of  $\Delta$  the  $s$ th row coincides with one of the next  $m-s$  rows, hence, again  $\Delta = 0$ .

**Remark 10.6.** The sum of the parts of all the partitions  $I'$  above is  $l(m+1) + \binom{m+1}{2} + \binom{m-s+1}{2} + 1$ , consistent with the fact that this is the codimension of the singularity  $\Phi_{m,m-s}(*, *+l)$  in  $\mathcal{E}_0(*, *+l)$ .

**Remark 10.7.** The formula gets particularly simple if  $r = m-1$ :

$$(29) \quad \text{tp}_{\Phi_{n,n-1}}(l) = 2^{n-1} \sum_{i=0}^{l+1} 2^i \Delta_{n+l+i, \underbrace{n+l, \dots, n+l}_{n-1}, l+1-i},$$

which recovers the Ronga-formula for  $A_2 = \Phi_{1,0}$  and gives the Thom series of  $III_{2,3} = \Phi_{2,1}$  for  $n=3$ . This is already a new result <sup>1</sup>.

*Proof.* First we prove Theorem 10.4 for the case  $s = m$ . This is a direct geometric argument, not using localization. We want to calculate the equivariant Poincaré dual of the  $\Phi_{m,0}$ -jets  $X_{m,0} \subset \text{Hom}(\mathbf{C}^m, \mathbf{C}^p) \oplus \text{Hom}(\text{Sym}^2 \mathbf{C}^m, \mathbf{C}^p)$ . By definition:

$$[X_{m,0}] = e(\text{Hom}(\mathbf{C}^m, \mathbf{C}^p)) \cdot [\Sigma^1(\text{Sym}^2 \mathbf{C}^m, \mathbf{C}^p)],$$

where  $\Sigma^1(V, W)$  denotes the corank 1 linear maps from  $V$  to  $W$ . We have that

$$e(\text{Hom}(\mathbf{C}^m, \mathbf{C}^p)) = \text{Res}(B_p|A_m), \text{ and } [\Sigma^1(\text{Sym}^2 \mathbf{C}^m, \mathbf{C}^p)] = c_q((\text{Sym}^2 \mathbf{C}^m, \mathbf{C}^p)),$$

where  $q = p - \binom{m+1}{2} + 1$  and  $c_q((\text{Sym}^2 \mathbf{C}^m, \mathbf{C}^p))$  denotes the  $q^{\text{th}}$  (equivariant) Chern class of the formal difference  $\mathbf{C}^p \ominus \text{Sym}^2 \mathbf{C}^m$ . The second statement is the Giambelli-Thom-Porteous formula.

<sup>1</sup>We recently received a copy of [Ozt09], in which the Thom series of  $III_{2,3}$  is also calculated.

Now

$$c_q((\mathrm{Sym}^2 \mathbf{C}^m, \mathbf{C}^p)) = \sum_{i=0}^q c_{q-i}(\mathbf{C}^p) s_i(\mathrm{Sym}^2 \mathbf{C}^m),$$

where  $s_i$  denote the Segre classes, which are defined by the identity

$$(1 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots) \cdot (1 - s_1 t + s_2 t^2 - s_3 t^3 + \cdots) = 1.$$

As we mentioned in Definition 10.3 the Segre classes of  $\mathrm{Sym}^2 \mathbf{C}^m$  can be expressed from the Chern roots of  $\mathbf{C}^m$ :

$$s_i(\mathrm{Sym}^2 \mathbf{C}^m) = \sum_I ((I)) \Delta_{\overline{I-\rho_{m-1}}}(\alpha_1, \dots, \alpha_m),$$

where  $I$  runs through all length  $m$  sequences  $I = (i_1, i_2, \dots, i_m)$  with  $i_1 > i_2 > \dots > i_m \geq 0$  with  $|I| - \binom{m}{2} = i$ . We finish the proof of the formula (28) for  $s = m$  by recalling the Factorization Property of Schur polynomials.

**Lemma 10.8.** [Mac98, I.3] (*Factorization Formula*) *Let  $n, p$  be nonnegative integers, and let the quotient Chern classes be defined as in (5). Suppose that  $(p^n + \lambda, \mu)$  is a partition. Then*

$$\Delta_{p^n + \lambda, \mu} = \mathrm{Res}(A_n, B_p) \Delta_\mu(B_p) \Delta_{\overline{\lambda}}(A_n).$$

For  $m = 2$  we have  $\Phi_{2,0} = I_{2,2}$ . The Thom series of  $I_{2,2}$  was calculated by several authors [FR07], [Pra07], [Kaz].

Now we go on with the proof of Theorem 10.4. An interpretation of what we proved so far is that the expression in (27) is equal to the expression in (28) for  $s = m$ . In the remaining of the proof we will use this statement to prove that expression (27) agrees with expression (28) for any  $m > s$ . As before,  $b_j$  will denote the  $j$ th elementary symmetric polynomial of the  $\beta_i$ 's.

The equality of formula (27) with (28) for  $s = m$  can be written—using the Factorization Formula, Lemma 10.8—as

$$(30) \quad \sum_{1 \leq i < j \leq s} \frac{\mathrm{Res}(B_{s+l} | \alpha_i + \alpha_j)}{\mathrm{Res}(K_{i,j}^{(s)} | \alpha_i + \alpha_j)} = \sum_I ((I)) b_{l+1-|I|} \Delta_{\overline{I-\rho_{s-1}}}(\alpha_1, \dots, \alpha_s).$$

What we want to prove is the equality of these two formulas for any  $m$  and  $s$ , that is (using the Factorization Formula again)

$$(31) \quad \sum_{1 \leq i < j \leq m} \frac{\mathrm{Res}(B_{m+l} | \alpha_i + \alpha_j)}{\mathrm{Res}(K_{i,j}^{(m)} | \alpha_i + \alpha_j)} 2^{m-s} \Delta_{\rho_{m-s}} \left( \alpha_1 - \frac{\alpha_i + \alpha_j}{2}, \dots, \alpha_m - \frac{\alpha_i + \alpha_j}{2} \right) = \sum_I ((I)) b_{l+1+m-s-|I|} \Delta_{\overline{I-(m-s)-\rho_{s-1}}}(\alpha_1, \dots, \alpha_m).$$

Checking the coefficients of  $b_{s+l-k}$  and  $b_{m+l-k}$  respectively, in these equations we can reduce the theorem to the following problem: Knowing

$$(32) \quad \sum_{1 \leq i < j \leq s} \frac{(\alpha_i + \alpha_j)^k}{\mathrm{Res}(K_{i,j}^{(s)} | \alpha_i + \alpha_j)} = \sum_{|I|=k+1-s} ((I)) \Delta_{\overline{I-\rho_{s-1}}}(\alpha_1, \dots, \alpha_s),$$

we want to prove

$$(33) \quad \sum_{1 \leq i < j \leq m} \frac{(\alpha_i + \alpha_j)^k}{\mathrm{Res}(K_{i,j}^{(m)} | \alpha_i + \alpha_j)} 2^{m-s} \Delta_{\rho_{m-s}} \left( \alpha_1 - \frac{\alpha_i + \alpha_j}{2}, \dots, \alpha_m - \frac{\alpha_i + \alpha_j}{2} \right) =$$

$$\sum_{|I|=k+1-s} ((I)) \Delta_{I-(m-s)s-\rho_{s-1}}(\alpha_1, \dots, \alpha_m).$$

We recall the Gustafson-Milne identity:

**Lemma 10.9.** [GM83], [CL96] *Let  $m \geq s$  be nonnegative integers. If  $H \subset \{1, \dots, m\}$  then the set  $\{\alpha_h\}_{h \in H}$  will be denoted by  $\alpha_H$ , and the set  $\{\alpha_1, \dots, \alpha_m\} \setminus \alpha_H$  will be denoted by  $\alpha_{\bar{H}}$ . Let the partition  $\mu = (\mu_1, \mu_2, \dots)$  satisfy  $\mu_1 \leq s$ . Then we have*

$$\Delta_{\mu}(\alpha_1, \dots, \alpha_m) = \sum_{H \subset \{1, \dots, m\}, |H|=s} \frac{\Delta_{s^{m-s}, \mu}(\alpha_H)}{\text{Res}(\alpha_H | \alpha_{\bar{H}})}.$$

The Gustafson-Milne identity implies that the right hand side of (33) is obtained from the right hand side of (32) by the following operation:

$$p(\alpha_1, \dots, \alpha_s) \mapsto \sum_{H \subset \{1, \dots, m\}, |H|=s} \frac{p(\alpha_H)}{\text{Res}(\alpha_H | \alpha_{\bar{H}})}.$$

Hence it is enough to show that the same operation maps the left hand sides into each other, too. That is we need to prove

$$(34) \quad \sum_{H \subset \{1, \dots, m\}, |H|=s} \frac{\sum_{i \leq j \in H} \frac{(\alpha_i + \alpha_j)^k}{\text{Res}(K_{i,j}^H | \alpha_i + \alpha_j)}}{\text{Res}(\alpha_H | \alpha_{\bar{H}})} = \sum_{1 \leq i \leq j \leq m} \frac{(\alpha_i + \alpha_j)^k}{\text{Res}(K_{i,j}^{(m)} | \alpha_i + \alpha_j)} 2^{m-s} \Delta_{\rho_{m-s}} \left( \alpha_1 - \frac{\alpha_i + \alpha_j}{2}, \dots, \alpha_m - \frac{\alpha_i + \alpha_j}{2} \right).$$

For this, the following Lemma will be useful.

**Lemma 10.10.** *Let  $m > s$  be non-negative integers, and consider the variables  $\gamma_1, \dots, \gamma_m, y$ . For a subset  $H \subset \{1, \dots, m\}$  we set  $\bar{H} = \{1, \dots, m\} - H$ ,  $\gamma_H = \{\gamma_i\}_{i \in H}$ ,  $\gamma_{\bar{H}} = \{\gamma_i\}_{i \in \bar{H}}$ . We have*

$$\sum_{H \subset \{1, \dots, m\}, |H|=s} \frac{\Delta_{\rho_s}(\gamma_H)}{\text{Res}(\gamma_H, \gamma_{\bar{H}})} \prod_{i \in H} \prod_{j \in \bar{H}} (\gamma_i + \gamma_j) = \Delta_{\rho_s}(\gamma_1, \dots, \gamma_m, y, -y).$$

*Proof.* In this proof we use the Thom polynomials of the representation of  $\text{GL}_m$  on  $\Lambda^2 \mathbf{C}^m$ , see e.g. [FNR05, §3]. Consider the canonical exact sequence of vector bundles  $0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$  over the Grassmannian  $Gr_s \mathbf{C}^m$ , and the diagram of maps

$$\begin{array}{ccccc} & & \phi & & \\ & & \curvearrowright & & \\ \Lambda^2 Q & \xrightarrow{i} & \Lambda^2 E & \xrightarrow{\pi_2} & \Lambda^2 \mathbf{C}^m \\ \downarrow & & \downarrow & & \\ Gr_s \mathbf{C}^m & \xrightarrow{id} & Gr_s \mathbf{C}^m, & & \end{array}$$

with the action of the  $n$ -torus. Using the fact that  $\phi(\Lambda^2 Q)$  is the set of two forms of corank at least  $s$ , and the identification  $\Lambda^2 E = \Lambda^2 S \oplus \Lambda^2 Q \oplus (S \otimes Q)$ , Proposition 5.1 gives the statement of the lemma for  $y = 0$ . We leave to the reader to prove the fact that the right hand side is

independent of  $y$ . (One way of proving this is the identification of the right hand side with an *incidence class* of two orbits of the representation of  $\mathrm{GL}_{m+2}$  on  $\Lambda^2 \mathbf{C}^{m+2}$ , see [FP07].)  $\square$

Lemma 10.10 (with the substitution  $\gamma_u = \alpha_u - (\alpha_i + \alpha_j)/2$ ) can be used to show that the coefficient of  $(\alpha_i + \alpha_j)^k$  are the same on the two sides, for all  $i \leq j$ . This completes the proof of Theorem 10.4.  $\square$

## 11. ITERATED RESIDUE FORMULAE AND GENERATING FUNCTIONS

In [BS06, §6.2] (see also [Sze98]) Bérczi and Szenes used one rational function—we will call it *generating function*—and the iterated residue operation, to encode the Thom polynomial of all singularities corresponding to the same nilpotent algebra of type  $A_d$ . We show that generating functions can be assigned to other singularities. We give some examples and indicate how the generating function can be a useful tool in future studies of Thom polynomials.

Certain rational functions in the variables  $z_1, \dots, z_d$  generate polynomials in the quotient variables through the *iterated residue operation*, which we describe now, following [BS06]. Consider  $\mathbf{C}^d$  with coordinates  $z_1, \dots, z_d$ . Let  $\omega_1, \dots, \omega_N$  be affine linear forms on  $\mathbf{C}^d$ , and let  $h(z_1, \dots, z_d)$  be a polynomial. We define the iterated residue operator by

$$(35) \quad \mathbf{RES}_d \frac{h(z_1, \dots, z_d)}{\prod_{i=1}^N \omega_i} = \int_{|z_1|=R_1} \dots \int_{|z_d|=R_d} \frac{h(z_1, \dots, z_d) dz_1 \dots dz_d}{\prod_{i=1}^N \omega_i},$$

where  $0 \ll R_1 \ll R_2 \ll \dots \ll R_d$ . This definition makes sense up to a choice of a sign, but this will be enough for our purposes: in the formulas below we always mean the iterated residue with the appropriate choice of  $\mathbf{RES}_d(dz_1 \dots dz_d) = \pm 1$ . That is, we will describe certain expressions up to sign.

We will use the notations  $D_j = \sum_{i=0}^{\infty} \frac{c_i}{z_j^i}$  and  $\mathrm{disc}_\mu = \prod_{i=1}^\mu \prod_{j=i+1}^\mu (z_i - z_j)$

The following conjecture is an extension of the Formula (7.2) in [BS06], where it is proved for Morin singularities. We arrived at this conjecture while discussing the problem with M. Kazarian. Not only he shares this conjecture with us, but he also has other supporting arguments [Kaz], but no proof at the moment.

**Conjecture 11.1.** *Let  $Q$  be a  $\mu$ -dimensional, commutative, nilpotent algebra with  $\deg(\mathrm{tp}_Q(l)) = \mu \cdot l + \gamma$ .*

(a) *Then there exists a rational function  $k_Q$ —called the generating function of  $Q$ —in the variables  $z_1, \dots, z_\mu$ , of degree  $\gamma - \binom{\mu+1}{2}$  such that*

$$(36) \quad \mathrm{tp}_Q(l) = \mathbf{RES}(k_Q \cdot \mathrm{disc}_\mu \cdot \prod_{i=1}^\mu z_i^l D_i).$$

(b) *The generating function  $k_Q$  has the form*

$$(37) \quad k_Q(z_1, \dots, z_\mu) = \frac{h(z_1, \dots, z_\mu)}{\prod_{a \in A} (z_{i_a} + z_{j_a} - z_{s_a})},$$

where  $h$  is a polynomial, and  $\{i_a, j_a, s_a\}_{a \in A}$  is a repetition-free list of indexes with  $i_a \leq j_a < s_a$  for all  $a \in A$ .

The function  $k_Q$  is not unique in general. The Giambelli-Thom-Porteous formula (21) can be encoded by setting

$$k_{\Sigma^r} = \prod_{i=1}^{r-1} z_i^i \quad (\text{here } \mu = r, \gamma = r^2).$$

Formula (7.2) of [BS06] can be interpreted as the the existence of  $k_Q$  for  $Q = A_i$ , ( $i = 1, 2, \dots$ ), as well as a concrete form of its denominator (all indices with  $i_a + j_a \leq s_a$ ). For the  $A_i$  singularity  $\mu = \gamma = i$ , hence the degree of  $k_{A_i}$  is  $-\binom{i}{2}$ . Bérczi and Szenes also calculated  $k_{A_i}$  for  $i = 1, \dots, 6$ . Here is the first three of their results:

$$k_{A_1} = 1, \quad k_{A_2} = \frac{1}{2z_1 - z_2}, \quad k_{A_3} = \frac{1}{(2z_1 - z_2)(2z_1 - z_3)(z_1 + z_2 - z_3)}.$$

In Section 10 the singularities  $\Phi_{m+r,r}$  were considered ( $r = 0, 1, \dots, m = 1, 2, \dots$ ). For these singularities we have  $\mu = m + r + 1$ , and  $\gamma = \binom{m+r+1}{2} + \binom{r}{2} + 1$ , and hence  $\deg k_{\Phi_{m+r,r}} = \binom{r+1}{2} - m$ . The results of Section 10.2 can be summarized by the following generating functions

$$k_{\Phi_{m+r,r}}(z_1, \dots, z_{m+r+1}) = \frac{\prod_{i=1}^{r-1} z_{m+1+i}^i}{2^{m-1}(2z_1 - z_{m+1}) \prod_{i=1}^{m-1} (z_i + z_{i+1} - z_{m+1})}.$$

For example we have  $k_{\Phi_{2,0}} = k_{I_{2,2}} = \frac{1}{2(2z_1 - z_3)(z_1 + z_2 - z_3)}$  and  $k_{\Phi_{2,1}} = k_{III_{2,3}} = \frac{1}{(2z_1 - z_2)}$ . With some experimenting with the computer one can find generating functions for the remaining nilpotent algebras with  $\mu \leq 4$ :

$$\begin{aligned} k_{III_{2,4}} &= \frac{1}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_4)(z_1 + z_2 - z_4)}, \\ k_{III_{3,3}} &= \frac{1}{4(2z_1 - z_3)(z_1 + z_2 - z_3)(2z_1 - z_4)(z_1 + z_2 - z_4)}, \\ k_{I_{2,3}} &= \frac{1}{(2z_1 - z_4)(2z_2 - z_3)(2z_2 - z_4)(z_1 + z_2 - z_4)(z_2 + z_3 - z_4)}, \\ k_{III_{2,4}} &= \frac{1}{(2z_1 - z_3)(z_1 + z_2 - z_3)(2z_1 - z_4)}. \end{aligned}$$

Now we want to explore the connection of the conjectured residue-form and the localization form of Thom polynomials. Recall that for a nilpotent algebra  $Q$ , and an ideal  $I$ , the virtual Euler class  $e(Q, I)$  was defined in Remark 5.4. For a function  $f$  in variables  $z_1, \dots, z_\mu$  define the asymetrization operator

$$\text{ASym}_\mu(f) = \sum_{\sigma \in S_\mu} \varepsilon(\sigma) f(z_{\sigma(1)}, \dots, z_{\sigma(\mu)}),$$

where  $\varepsilon(\sigma)$  is the sign of the permutation  $\sigma$ . For a function  $f$  with variables  $\alpha_i$  let  $f|_{\alpha_i := z_i}$  be the same function with the variables changed to  $z_i$ .

**Conjecture 11.2.** *Suppose Conjecture 11.1(a) holds. Then*

$$\text{ASym}_\mu(k_Q) = \frac{\text{disc}_\mu}{e(Q, \mathcal{M}_\mu^2)|_{\alpha_i := z_i}}.$$

If we assume part (b) of Conjecture 11.1 as well, then Conjecture 11.2 reduces to a remarkable (conjectured) identity for iterated residues (or the Orlik-Salamon algebra) associated with the hyperplane arrangement  $\cup_A \{z_{i_a} + z_{j_a} - z_{s_a} = 0\} \cup \cup_{i,j} \{z_i = \alpha_j\}$ .

**Remark 11.3.** Conjecture 11.2 can be used to guess the function  $k_Q$ , as soon as the class  $e(Q, \mathcal{M}_\mu^2)$  is known. In practice  $e(Q, \mathcal{M}_\mu^2)$  can be calculated using Theorem 8.1, or equation (22). Its denominator is a symmetric function, which is a product of factors  $\alpha_i + \alpha_j - \alpha_s$ . Then one has finitely many choices to guess for the denominator of  $k_Q$ . (Eg. for  $2\alpha_1 - \alpha_2$  we can choose  $2z_1 - z_2$  or  $2z_3 - z_5$ , etc.) Knowing the degree of  $k_Q$  we also find the degree of the numerator. Putting all this together one arrives an oftentimes effective procedure to find the  $k_Q$  function.

Conjectures 11.1, 11.2, their geometric meaning, their connection to residue identities and interpolation theory, as well as more examples will be considered in a future study.

## 12. THE LOCALIZATION FORMULA FOR THE “SMALL $p$ ” CASE

After we wrote down the Localization Formula  $\text{Tp}_Q(n, p)$  for a nilpotent algebra  $Q$ , we can evaluate it for any  $p$ , even if  $Q$  cannot be defined by  $p$  relations. Sometimes it is not zero and we would like to interpret these “small  $p$ ” cases.

Fix  $n$  and  $p$  and assume that  $\dim(Q) = m$ . Then the correspondence variety—introduced in Definition 4.1—of  $Q$  is

$$C(Y_Q) = \{(I, g) \in \text{Gr}^m \times J^k(n, p) \mid I \in Y_Q, I_g \subset I\},$$

where  $Y_Q = \overline{\{I \in \text{Gr}^m : Q_I \cong Q\}}$ . We have the restriction of the second projection

$$\phi : C(Y_Q) \rightarrow \eta_Q(n, p) := \phi(C(Y_Q)).$$

In other words

$$\eta_Q(n, p) = \{g \in J^k(n, p) : \exists I \in Y_Q \text{ such that } I \supset I_g\}.$$

We showed in Proposition 4.2 that if  $n \geq a(Q)$  (minimal number of generators of  $Q$ ) and  $p \geq b(Q)$  (minimal number of relations of  $Q$ ), then  $\phi$  is birational. In some cases  $\phi$  is birational for smaller  $p$  as well. In these cases the Localization Formula still calculates  $[\eta_Q(n, p)]$ .

**Remark 12.1.** Recall that by Remark 5.3 if  $\phi$  drops dimension then  $\phi_*(1) = 0$ . On the other hand  $[\eta_Q(n, p)]$  is never zero if  $\eta_Q(n, p)$  is not empty. The reason is that the representation of  $\text{GL}(n) \times \text{GL}(p)$  on  $J^k(n, p)$  ‘contains the scalars’, so by an appropriate substitution we can recover the degree of an invariant subvariety, which is not zero (see [FNR05, §6] for a detailed argument). So  $\text{Tp}_Q(n, p)$  is not zero exactly when  $\phi$  is generically finite to one. If we know the Thom polynomial in the quotient variables, then decreasing  $p$  correspond to applying the lowering operation  $\flat$  defined in 7.14. Now we show some examples.

**12.1. Lowering the Thom polynomial of  $\Phi_{n, n-1}$ .** Recall formula (29), the Thom polynomial of  $\Phi_{n, n-1}$ :

$$\text{tp}_{\Phi_{n, n-1}}(l) = 2^{n-1} \sum_{i=0}^{l+1} 2^i \Delta_{n+l+i, \underbrace{n+l, \dots, n+l}_{n-1}, l+1-i}.$$

Though  $\Phi_{n,n-1}$ -singularities exist only for  $l \geq \binom{n+1}{2} - n = \binom{n}{2}$ , the formula gives nonzero even for  $l = -1$ . This not just a formal argument. Elementary calculation shows that

$$\Delta_{(i_1, \dots, i_m)}^{b(m)} = \Delta_{(i_1-1, \dots, i_m-1)},$$

or geometrically: we erase the first column of the Young diagram of the partition  $\lambda = (i_1, \dots, i_m)$ . If the partition is shorter than  $m$ , then we get zero. So we can see that decreasing the value of  $l$  by one in this formula agrees with applying the lowering operation  $b = b(m)$ , where  $m = n + 1$  is the dimension of the nilpotent algebra  $\Phi_{n,n-1}$ .

For  $l = -1$  we have  $\text{tp}_{\Phi_{n,n-1}}(-1) = 2^{n-1} \underbrace{\Delta_{n-1, \dots, n-1}}_n$ . Notice, that by the Giambelli-Thom-Porteous formula (21) we have  $\underbrace{\Delta_{n-1, \dots, n-1}}_n = \text{tp}_{\Sigma^n}(-1)$ , so we expect that  $\phi$  will be a generically

$2^{n-1}$  to 1 map with image the set of  $\Sigma^n$ -jets in  $J^k(n, n-1)$ . We can fix  $k = 2$ , and notice that  $Y_{\Phi_{n,n-1}} = X(n, n-1) \subset \mathbf{P}(\text{Sym}^2 \mathbf{C}^n) \subset \text{Gr}^{n+1}(J^2(n))$ , where  $X(n, n-1)$  corresponds to the corank  $n-1$  symmetric matrices. This is how we defined  $\Phi_{n,n-1}$  in Section 10.  $X(n, n-1)$  is closed and can be identified with the *Veronese variety*, i.e. the image of the Veronese map  $\mathbf{P}(\mathbf{C}^n) \rightarrow \mathbf{P}(\text{Sym}^2 \mathbf{C}^n)$ . It is easy to see that the image of  $\phi$  is  $\eta_{\Sigma^n}(n, n-1) = \text{Hom}(\text{Sym}^2 \mathbf{C}^n, \mathbf{C}^{n-1}) \subset J^2(n, n-1)$ , so  $[\eta_{\Sigma^n}(n, n-1)] = e(\text{Hom}(\mathbf{C}^n, \mathbf{C}^{n-1}))$ , as we expected. We need the number of  $\phi$ -preimages of a given jet  $g$  in the image. But for generic  $g$  the set  $\pi_1(\pi_2^{-1})$  intersects  $\mathbf{P}(\text{Sym}^2 \mathbf{C}^n) \subset \text{Gr}^{n+1}(J^2(n))$  in an  $n-1$ -codimensional linear subspace, so the number of  $\phi$ -preimages is the degree of the Veronese variety, which is known to be  $2^{n-1}$  in this case (see e.g. [Har92, p.231]), agreeing with our calculation.

We can study the cases  $l > -1$ , but the geometry gets more complicated, and we do not know for which singularity we calculate the Thom polynomial. For example  $\text{tp}_{\Phi_{3,2}}(0) = 8\Delta_{3331} + 4\Delta_{433}$ —agreeing with (25)—so we calculated up to constant multiple the Thom polynomial of one of the singularities corresponding to nets of conics. One needs to decide which one, and calculate the multiplicity, what we skip now.

**12.2. The singularities  $III_{a,b}$  and  $I_{a,b}$ .** Let  $Q$  be the nilpotent algebra corresponding to the germ  $III_{a,b} = (x^a, y^b, xy)$ . Then  $\eta_Q(n, n)$  contains the  $I_{a,b}$ -germs (corresponding to  $(x^a + y^b, xy)$ ). By calculating the dimensions we see that in fact they form an open set in  $\eta_Q(n, n)$ . It is a routine to calculate that the only ideal in the  $\mathcal{R}$ -orbit of  $(x^a, y^b, xy, x_3, \dots, x_n)$  containing the ideal  $(x^a + y^b, xy, x_3, \dots, x_n)$  is  $(x^a, y^b, xy, x_3, \dots, x_n)$  itself, therefore  $\phi$  is generically one to one. It implies that (see Section 7.3)

**Theorem 12.2.** *For  $m = a + b - 2$  and  $a, b \geq 2$*

$$\text{tp}(I_{a,b}(0)) = \text{tp}^{b(m)}(III_{a,b}(1)).$$

For  $a = b = 2$  this was well known since both  $I_{2,2}(0)$  and  $III_{2,2}(1)$  are open in the corresponding  $\Sigma^2$  Thom-Boardmann class. The Thom polynomial of  $I_{2,3}(0)$  and  $III_{2,3}(1)$  were calculated in [Rim01] but this relation went unnoticed.

**12.3. Thom series for Thom-Boardmann classes.** Let  $\Sigma^K$  denote the Thom-Boardmann class corresponding to  $K = (i_1, \dots, i_s)$  for  $i_1 \geq \dots \geq i_s \geq 1$  (see e.g. [Mat73]). For  $n \geq i_1$  and  $p \geq p_0$  ( $p_0$  depending on  $K$ ) there is a jet  $g_K(n, p) \in J^k(n, p)$ , such that  $\mathcal{K}g_K$  is open in  $\Sigma^K$ , so  $\text{Tp}(g_K(n, p)) = \text{Tp}(\Sigma^K(n, p))$ . For  $n - i_1 < p < p_0$  the Thom-Boardmann class  $\Sigma^K(n, p)$

is still not empty, but it may split into lower dimensional contact classes. The question was raised in [FR07] whether the Thom series of  $g_K$  calculates  $[\Sigma^K(n, p)]$  for  $n - i_1 < p < p_0$ , too. Similarly to the previous exercise the dimensions are correct, so it is enough to check that for a jet  $g \in \Sigma^K(n, p)$  there is a unique ideal  $I \supset I_g$  such that  $Q_I \cong Q_{I_{g_K}}$ . But this unique ideal is exactly the *Boardmannization* of  $g$  (the  $\beta(I_g)$  of [Mat73, §.2]). (Boardmannization gives a section of the possible covering, but the correspondence variety  $C$  is connected.) Consequently we have

**Theorem 12.3.** *The Thom series  $\text{Ts}_{g_K}$  calculates the Thom polynomial of  $\Sigma^K(n, p)$  for all  $n, p$ .*

We can call  $\text{Ts}_{g_K}$  the Thom series  $\text{Ts}(\Sigma^K)$  of  $\Sigma^K$ . Notice that this way we can get Thom polynomials for  $p < n$  as well which cannot be done directly with the Localization Formula. We should mention that at this point this is only a theoretical possibility as  $\text{Ts}(\Sigma^K)$  is known only in the few cases listed in Section 3.

### 13. FINAL REMARKS

The results and examples of this paper may give the wrong impression that now the Thom polynomials of all singularities are calculated. Although we indeed reached far beyond the known Thom polynomials (and series), let us demonstrate the boundaries of our knowledge by some open problems.

We do not know how to calculate the Thom series of  $A_n$  for  $n > 6$ . For  $n > 8$  we do not even know the first Thom polynomial  $\text{tp}_{A_n}(0)$ . We do not know the Thom series of the Thom-Boardmann class  $\Sigma^{211}$ . Are there closed formulas for classes of singularities, for example  $\{A_n : n \geq 0\}$  or  $\{I_{a,b} : a, b \geq 2\}$ ? We repeat a conjecture of the second author [Rim01] in a slightly strengthened form:

**Conjecture:** Every coefficient of the Thom polynomials  $\text{tp}_{A_n}(l)$ —written as a linear combination of Chern monomials—is non-negative, and all coefficients of width at most  $n$  are strictly positive. (By  $d$ -stability, the other coefficients are 0.)

In [BS06, Sect. 8.7] this conjecture is verified for  $n = 3$  and 4.

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