EXTENDED JOSEPH POLYNOMIALS, QUANTIZED CONFORMAL BLOCKS, AND A \( q \)-SELBERG TYPE INTEGRAL

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To the memory of Yu. Stroganov

Abstract. We consider the tensor power \( V = (\mathbb{C}^N)^\otimes n \) of the vector representation of \( \mathfrak{gl}_N \) and its weight decomposition \( V = \oplus_{\lambda=(\lambda_1,\ldots,\lambda_N)} V[\lambda] \). For \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \), the trivial bundle \( V[\lambda] \times \mathbb{C}^n \to \mathbb{C}^n \) has a subbundle of \( q \)-conformal blocks at level \( \ell \), where \( \ell = \lambda_1 - \lambda_N \) if \( \lambda_1 - \lambda_N > 0 \) and \( \ell = 1 \) if \( \lambda_1 - \lambda_N = 0 \). We construct a polynomial section \( I_\lambda(z_1,\ldots,z_n,h) \) of the subbundle. The section is the main object of the paper. We identify the section with the generating function \( J_\lambda(z_1,\ldots,z_n,h) \) of the extended Joseph polynomials of orbital varieties, defined in [DFZ05, KZ09].

For \( \ell = 1 \), we show that the subbundle of \( q \)-conformal blocks has rank 1 and \( I_\lambda(z_1,\ldots,z_n,h) \) is flat with respect to the quantum Knizhnik–Zamolodchikov discrete connection.

For \( N = 2 \) and \( \ell = 1 \), we represent our polynomial as a multidimensional \( q \)-hypergeometric integral and obtain a \( q \)-Selberg type identity, which says that the integral is an explicit polynomial.

1. Introduction

The bundle of conformal blocks was introduced in conformal field theory. The bundle has a projectively flat Knizhnik–Zamolodchikov (KZ) connection which is a flat connection for conformal blocks on the sphere, see, for example, [KZ84, KL93]. The equations for flat sections of the bundle of conformal blocks on the sphere are called the KZ differential equations. In [SV91, FSV94a, FSV94b] solutions of the KZ differential equations were constructed as multidimensional hypergeometric integrals.

The \( q \)KZ difference equations were introduced in [FR92]. The \( \mathfrak{gl}_N \) \( q \)-conformal blocks were defined in [MV98, MV99], cf. [EF99]. The bundle of \( q \)-conformal blocks has a discrete flat connection defined by \( q \)KZ operators, see [MV98, MV99]. In [TV97] solutions of the \( \mathfrak{gl}_2 \) \( q \)KZ equations were constructed as multidimensional \( q \)-hypergeometric integrals.

We consider the tensor power \( V = (\mathbb{C}^N)^\otimes n \) of the vector representation of \( \mathfrak{gl}_N \) and its weight decomposition \( V = \oplus_{\lambda=(\lambda_1,\ldots,\lambda_N)} V[\lambda] \). For \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \), the trivial bundle \( V[\lambda] \times \mathbb{C}^n \to \mathbb{C}^n \) has a subbundle of \( q \)-conformal blocks at level \( \ell \), where \( \ell = \lambda_1 - \lambda_N \) if \( \lambda_1 - \lambda_N > 0 \) and \( \ell = 1 \) if \( \lambda_1 - \lambda_N = 0 \). We construct a polynomial section \( I_\lambda(z_1,\ldots,z_n,h) \) of the subbundle. The section is the main object of the paper. We identify the section with

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the generating function $J_\lambda(z_1, \ldots, z_n, h)$ of the extended Joseph polynomials of the orbital varieties, defined in [DFZJ05, KZJ09].

For $\ell = 1$, we show that the subbundle of $q$-conformal blocks has rank 1 and $I_\lambda(z_1, \ldots, z_n, h)$ is flat with respect to the $q$KZ discrete connection.

For $N = 2$ and $\ell = 1$, we represent our polynomial as a multidimensional $q$-hypergeometric integral and obtain a $q$-Selberg type identity, which says that the integral is an explicit polynomial. The simplest of these identities is

\[\int_{-i\infty}^{i\infty} \frac{\Gamma(a + s)\Gamma(b + s)\Gamma(c - s)\Gamma(d - s)\, ds}{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)} = 2\pi i \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)},\]

which is a formula for the Barnes integral in [WW27]. The integral representation for $I_\lambda$ gives an integral representation for the extended Joseph polynomials if $N = 2$ and $\ell = 1$.

For $N = 2$ we also give a presentation for $I_\lambda$ as a multiple residue of a suitable rational function.

The fact that the generating function $J_\lambda(z_1, \ldots, z_n, h)$ with $\lambda_1 - \lambda_N \leq 1$ satisfies the $q$KZ equations at level 1 was conjectured in [DFZJ05] and proved in [KNST09] by a different method, by relating the generating function with non-symmetric Jack polynomials.

The results of this paper may be considered as a “quantization” of the results of [Var10, RV11, RSV10], where the bundle of (non-quantum) conformal blocks at level 1 in $(\mathbb{C}^N)^\otimes n$ was considered. The bundle is of rank 1 and has a flat connection defined by the KZ differential operators. A rational flat section of the bundle was constructed. The section was interpreted as a generating function of the Euler classes of the fixed points of the $GL_n$-action on a suitable space of partial flags in $\mathbb{C}^n$. A Selberg type identity was obtained that equates the rational section and a multidimensional hypergeometric integral.

In Section 2 we introduce $q$-conformal blocks and $q$KZ equations. In Section 3 we define our main object – the polynomial $I_\lambda(z_1, \ldots, z_n, h)$. In Section 4 we identify the polynomial with the generating function of the extended Joseph polynomials of orbital varieties. In Section 5 we prove all properties of the generating function. In Section 6 we prove a $q$-Selberg type identity. In Section 7 we give an alternative integral formula for $I_\lambda(z_1, \ldots, z_n, h)$, if $N = 2$.

2. Quantum conformal blocks and $q$KZ equations

2.1. Operators on representation-valued functions. Let $N \geq 2$ be a positive integer. Let $e_{i,j}$, for $i, j = 1, \ldots, N$, be the standard generators of the complex Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{i,j}, e_{s,k}] = \delta_{j,s}e_{i,k} - \delta_{i,k}e_{s,j}$. Consider the standard vector representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$, and its $n$-th tensor power $V = (\mathbb{C}^N)^\otimes n$. The space $V$ splits into the direct sum of weight subspaces

\[V = \bigoplus_{\lambda = (\lambda_1, \ldots, \lambda_N)} V[\lambda],\]

where $\sum_{i=1}^N \lambda_i = n$, and $V[\lambda] = \{v \in V \mid e_i v = \lambda_i v\}$.

In this paper we will be concerned with $V$-valued functions of $z_1, \ldots, z_n$ also depending on a complex parameter $h$. Now we recall some operators acting on the space of such functions.

- Elements of $\mathfrak{gl}_N$ act naturally on any factor of the tensor power. When $x \in \mathfrak{gl}_N$ acts in the $i$-th factor, we denote its action by $x^{(i)}$. 

Following \[MV98\] define an operator
\[
e(z) = \sum_{j=1}^{n} \left( z_j - h e_{N,N}^{(j)} + h \sum_{s=j+1}^{n} (e_{1,1}^{(s)} - e_{N,N}^{(s)}) \right) e_{1,N}^{(j)} + h \sum_{j=2}^{N-1} \sum_{1 \leq r < s \leq N} e_{j,N}^{(r)} e_{1,j}^{(s)}.
\]

- Let \( P^{(i,j)} \) be the permutation of the \( i \)-th and \( j \)-th factors of \( (\mathbb{C}^N)^{\otimes n} \).
- The deformed \( S_n \)-action on \( V \)-valued functions of \( z_1, \ldots, z_n \). The \( i \)-th elementary transposition \( s_i \in S_n \) acts by the formula
\[
s_i : I(z_1, \ldots, z_n) \mapsto I(z_1, \ldots, z_{i+1}, z_i, \ldots) - I(z_1, \ldots, z_i, z_{i+1}, \ldots) \frac{h}{z_i - z_{i+1}}.
\]

This defines an action of \( S_n \). Observe that, despite the presence of denominators, polynomials are mapped to polynomials by elements of \( S_n \). In the whole paper an \( S_n \)-action will always mean this deformed action unless otherwise stated.

- Let \( u \) be a new variable. We define the following \( R \)-matrix operator
\[
R^{(i,j)}(u) = \frac{u - h P^{(i,j)}}{u + h}.
\]

Observe that \( R^{(i,j)}(u) R^{(i,j)}(-u) = 1 \) and
\[
R^{(i,j)}(u - v) R^{(i,k)}(u) R^{(j,k)}(v) = R^{(j,k)}(v) R^{(i,k)}(u) R^{(i,j)}(u - v).
\]

### 2.2. Yangian \( Y(\mathfrak{gl}_N) \).

The Yangian \( Y(\mathfrak{gl}_N) \) is a unital associative algebra with generators \( T^{(s)}_{i,j}, i, j = 1, \ldots, N, \ s \in \mathbb{N} \). Organize them into generating series
\[
T_{i,j}(u) = \delta_{i,j} + \sum_{s=1}^{\infty} T^{(s)}_{i,j} u^{-s}, \quad i, j = 1, \ldots, N.
\]

The defining relation in \( Y(\mathfrak{gl}_N) \) have the form
\[
(u - v) [T_{i,j}(u), T_{k,l}(v)] = T_{k,j}(v) T_{i,l}(u) - T_{k,i}(u) T_{i,l}(v),
\]
for all \( i, j, k, l = 1, \ldots, N \).

The Yangian \( Y(\mathfrak{gl}_N) \) contains \( U(\mathfrak{gl}_N) \) as a subalgebra. The embedding is given by \( e_{i,j} \mapsto T^{(1)}_{i,j} \) for any \( i, j = 1, \ldots, N \).

Let
\[
T(u) = \sum_{i,j=1}^{N} E_{i,j} \otimes T_{i,j}(u)
\]

where \( E_{i,j} \) is the image of \( e_{i,j} \in \mathfrak{gl}_N \) in \( \text{End}(\mathbb{C}^N) \). Relations (2.2) can be written as the equality of series with coefficients in \( \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \):
\[
(u - v + P) T^{(1)}(u) T^{(2)}(v) = T^{(2)}(v) T^{(1)}(u) (u - v + P),
\]
where \( P \) is the permutation of the \( \mathbb{C}^N \) factors, \( T^{(1)}(u) = \sum_{i,j=1}^{N} E_{i,j} \otimes 1 \otimes T_{i,j}(u) \) and \( T^{(2)}(u) = 1 \otimes T(u) \).
More information on the Yangian $Y(\mathfrak{gl}_N)$ can be found in [Mol07]. Notice that the series $T_{i,j}(u)$ here corresponds to the series $T_{j,i}(u)$ in [Mol07].

The assignment

$$T(u) \mapsto R^{(0,1)}(z_1 - hu) \cdots R^{(0,n)}(z_n - hu) \prod_{i=1}^{n} \frac{z_i - hu + h}{z_i - hu}$$

defines an action of the Yangian $Y(\mathfrak{gl}_N)$ on $V$-valued functions of $z_1, \ldots, z_n$. We identify here the space $(\mathbb{C}^N)^{(n+1)}$ with $\mathbb{C}^N \otimes V$ and count the tensor factors by $0, 1, \ldots, n$.

The action of $e(z)$ on $V$-valued functions coincide with that of $h(T_{N,1}^{(2)} - T_{N,N}^{(1)}, T_{N,1}^{(1)})$.

**Lemma 2.1.** The Yangian action commutes with the $S_n$ action (2.1).

**Proof.** The commutativity with the first term in (2.1) follows from the Yang-Baxter equation for $R(u)$, the last formula in Section 2.1. The commutativity with the second term in (2.1) is the commutativity with multiplication by functions of $z_1, \ldots, z_n$. □

2.3. **Singular vectors, $q$-conformal blocks, and qKZ equations.** Let $\lambda$ be a partition, i.e. assume that $\lambda_1 \geq \ldots \geq \lambda_N$. Define $d(\lambda) = \lambda_1 - \lambda_N$.

A vector $v \in V[\lambda]$ is a singular vector, if $\sum_{a=1}^{N} e_{i,j}^{(a)} v = 0$ for all $i < j$.

Let $\ell \geq d(\lambda)$ be a positive integer, and $(z_1, \ldots, z_n) \in \mathbb{C}^n$. Following [MV98] we call $v \in V[\lambda]$ a level $\ell$ $q$-conformal block, if it is a singular vector and

$$e(z)^{\ell - d(\lambda) + 1} v = 0.$$ 

Note that if $v \in V[\lambda]$ is a level $\ell$ $q$-conformal block, then $v$ is a level $\ell'$ $q$-conformal block for any $\ell' > \ell$.

For $i = 1, \ldots, n$, define the qKZ operators at level 1 by the formula

$$K_i(z_1, \ldots, z_n) = R^{(i,i-1)}(z_i - z_{i-1} - (N + 1)h) \cdots R^{(i,1)}(z_i - z_1 - (N + 1)h) \times$$
$$\times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}).$$

The qKZ difference equations at level 1 for a $V[\lambda]$-valued function $I$ is the system of equations (2.3)

$$I(z_1, \ldots, z_i - (N + 1)h, \ldots, z_n) = K_i(z_1, \ldots, z_n) I(z_1, \ldots, z_i, \ldots, z_n), \quad i = 1, \ldots, n.$$

**Lemma 2.2.** Let $d(\lambda) \leq 1$. For generic $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ the space of $q$-conformal blocks at level 1 is at most one-dimensional.

**Proof.** The space of conformal blocks for $h = 0$ is defined as

$$CB_\lambda(z) = \{ v \in V[\lambda] \text{ is a singular vector and } \sum_{j=1}^{N} z_a e_{1,N}^{(j)} (\ell - d(\lambda) + 1) v = 0 \}.$$ 

For generic $z \in \mathbb{C}^n$ the dimension of $CB_\lambda(z)$ is calculated by the Verlinde formula. For $\ell = 1$ and $d(\lambda) \leq 1$ the Verlinde formula gives 1, so in this case the space of (non-quantum) conformal blocks for generic $z$ is one-dimensional. The space of $q$-conformal blocks specializes to $CB_\lambda(z)$ at $h = 0$. At this specialization the dimension may only increase, hence the dimension of $q$-conformal blocks is at most 1. □

Below we will show that for generic $z$ the dimension is equal to 1.
THE MINIMAL DEGREE SKEW-SYMMETRIC POLYNOMIAL $I_\lambda$

Recall that $\lambda \in \mathbb{N}^N$ is a partition of $n$. Define $k(\lambda) = \sum_{i=1}^{N} \lambda_i(\lambda_i - 1)/2$.

Let $v_1, \ldots, v_N$ be the standard basis in $\mathbb{C}^N$, $e_{i,j}v_k = \delta_{j,k}v_i$. For a multi-index $L = (l_1, \ldots, l_n)$ define $v_L = v_{l_1} \otimes \cdots \otimes v_{l_n}$. A $V[\lambda]$-valued function $I$ can be expressed as

$$I = \sum_{L} f_L(z_1, \ldots, z_n, h) v_L$$

for multi-indices $L = (l_1, \ldots, l_n)$ with $|\{j : l_j = i\}| = \lambda_i$. Denote the multi-index

$$L_0 = (1, \ldots, 1, 2, \ldots, 2, \ldots, N, \ldots, N)_{\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_N}$$

In what follows we will be concerned with the degree of polynomials and rational functions in $z_i, h$. Our convention is that $\deg z_i = \deg h = 1$. With this convention, the deformed $S_n$-action of Section 2.1 is homogeneous. Hence, if $I$ is a skew-symmetric $V[\lambda]$-valued polynomial, then its homogeneous parts are also such. Now we study what the homogeneous degrees of skew-symmetric polynomials can be.

Define another $S_n$-action on functions of $z_1, \ldots, z_n$, where the $i$-th elementary transposition $s_i \in S_n$ is acting by the formula

$$s_i : f \mapsto \hat{s}_i f,$$

$$\hat{s}_i f(z_1, \ldots, z_n) = \frac{z_i - z_{i+1} + h}{z_i - z_{i+1}} f(\ldots, z_{i+1}, z_i, \ldots) - \frac{h}{z_i - z_{i+1}} f(\ldots, z_i, z_{i+1}, \ldots).$$

For a permutation $\sigma \in S_n$ and a multi-index $L = (l_1, \ldots, l_n)$ set $\sigma(L) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(n)})$. The following lemma is obvious.

**Lemma 3.1.** A $V[\lambda]$-valued function $I$ is skew-symmetric with respect to action (2.1) if and only if $f_{s_i(L)} = -\hat{s}_i f_L$ for every multi-index $L$ and every $i = 1, \ldots, n - 1$.

**Lemma 3.2.** If a polynomial $f(z_1, \ldots, z_n)$ is skew-symmetric with respect to the $S_n$-action (3.2), then it is divisible by

$$\prod_{1 \leq i < j \leq n} (z_i - z_j + h).$$

**Proof.** Skew-symmetry with respect to $\hat{s}_i$ implies

$$(z_i - z_{i+1} + h) f(\ldots, z_{i+1}, z_i, \ldots) = (z_{i+1} - z_i + h) f(\ldots, z_i, z_{i+1}, \ldots).$$

Therefore $z_i - z_{i+1} + h$ divides $f$. This further implies that $z_{i-1} - z_{i+1} + h$ divides $f(\ldots, z_i, z_{i-1}, \ldots)$, which using (3.3) again yields that $z_{i-1} - z_{i+1} + h$ divides $f$. Iterating this idea we obtain the statement of the lemma.

**Lemma 3.3.**

(i) If $I \neq 0$ is a $V[\lambda]$-valued skew-symmetric polynomial, then its degree is at least $k(\lambda)$.

(ii) A $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$ is unique up to multiplication by a number.

(iii) There exists a nonzero $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$. 

Proof. (i) By Lemma 3.1 a $V[\lambda]$-valued skew-symmetric polynomial $I$ is uniquely determined by the coefficient $f_{L_0}$, and $\deg I = \deg f_{L_0}$.

Denote by $S_{\lambda_1} \times \ldots \times S_{\lambda_N} \subset S_n$ the isotropy subgroup of $L_0$. By Lemma 3.1 for $s_i \in S_{\lambda_1} \times \ldots \times S_{\lambda_N}$ we have $s_if_{L_0} = -f_{L_0}$. Using Lemma 3.2 for each $S_{\lambda_i}$ we obtain that $f_{L_0}$ is divisible by

$$D_0 = \prod_{1 \leq a < b \leq \lambda_1} (z_a - z_b + h) \prod_{\lambda_1 < a < b \leq \lambda_1 + \lambda_2} (z_a - z_b + h) \cdots \prod_{n - \lambda_N \leq a < b \leq n} (z_a - z_b + h)$$

and has degree at least $k(\lambda)$.

(ii) If $f_{L_0}$ has degree $k(\lambda)$, then it is proportional to $D_0$.

(iii) Define

$$I_\lambda = \sum_{\sigma \in S_n / S_{\lambda_1} \times \ldots \times S_{\lambda_N}} \text{sgn}(\sigma) \, \hat{\sigma}(D_0) \, v_{\sigma(L_0)},$$

where $\text{sgn}(\sigma)$ is the sign of the shortest permutation in the coset, and $\hat{\sigma}$ denotes action (3.2). Then $I_\lambda$ is a nonzero $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$.

The $V[\lambda]$-valued polynomial $I_\lambda$, defined by (3.3), is the main object of this paper. Now we reformulate its definition.

Definition 3.4. Let $\lambda \in \mathbb{N}^N$ be a partition of $n$. Let $I_\lambda$ be the $V[\lambda]$-valued skew-symmetric polynomial of degree $k(\lambda)$ normalized in such a way that the coefficient of $v_{L_0}$ is $D_0$, see (3.1), (3.4).

Example. We have

$I_{(1,1)} = v_{12} - v_{21}$,

$I_{(2,1)} = (z_1 - z_2 + h)v_{112} + (z_3 - z_1 - 2h)v_{121} + (z_2 - z_3 + h)v_{211},$

$I_{(2,2)} = (z_1 - z_2 + h)(z_3 - z_1 + h)(v_{1122} + v_{2211}) + (z_1 - z_4 + 2h)(z_2 - z_3 + h)(v_{1221} + v_{2112})$

$$+ (-(z_1 - z_2 + h)(z_3 - z_4 + h) - (z_1 - z_4 + 2h)(z_2 - z_3 + h))(v_{1212} + v_{2121}).$$

Note that the last coefficient function in the third formula does not factor.

Remark. In the quasiclassical limit $h = 0$, the vector $I_\lambda$ is the minimal degree skew-symmetric polynomial under the $S_n$-action

$s_i^{h=0} : I \mapsto P^{(i,i+1)}I(\ldots, z_{i+1}, z_i, \ldots).$

Its explicit form is

$$I_{\lambda}^{h=0} = \sum \left( \text{sgn}(L) \prod_{a < b, l_a = l_b} (z_a - z_b) \right) v_L,$$

with an appropriately defined $\text{sgn}(L)$. Rescaled by the the discriminant $\prod_{a < b} (z_a - z_b)$, this function was studied in [RV11], see also [RSV10]. It is shown there that this function satisfies (non-quantum) conformal block properties and (non-quantum) KZ differential equations.
4. Geometric description of $I_\lambda$

In this section we provide the connection of the $I_\lambda$ to geometry which was advertised in the introduction.

4.1. Orbital varieties and Joseph representation.

4.1.1. Orbital varieties. Consider some conjugacy class of nilpotent elements inside $\mathfrak{g} = \mathfrak{gl}_n$. Such a conjugacy class is characterized by the unordered set of sizes of the Jordan blocks, which form a partition $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_K)$ of $n$. It is more convenient to use instead of $\lambda'$ its conjugate partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$. If we depict partitions as Young diagrams, then the diagram of $\lambda$ is the transpose of that of $\lambda'$: the lengths of its columns are the sizes of Jordan blocks. For example, for one block of size 3 and one block of size 1, we use $\lambda = (2, 1, 1)$, that is $\lambda = (2, 1, 1)$.

Let $\bar{D}_\lambda \subset \mathfrak{g}$ be the closure of the conjugacy class $D_\lambda$ associated to the partition $\lambda$. $\bar{D}_\lambda$ is known to be an irreducible algebraic variety, but if we denote by $n$ the space of strict upper triangular matrices, then the intersection $O_\lambda := \bar{D}_\lambda \cap n$ is in general reducible: its geometric components (i.e., reduced irreducible components) are called orbital varieties.

Given an element of $x \in O_\lambda$, note that $x$ leaves stable the natural flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ associated to the standard basis. So the restriction of $x$ to $\mathbb{C}^i$, $i = 0, \ldots, n$, is a nilpotent element to which can be attached a partition of $i$ as described above, say $\varphi_i(x)$. Note that generically, $\varphi_n(x) = \lambda$. The following results were found:

**Theorem** (Spaltenstein [Spa82]). Let $\lambda$ be a partition of $n$, and $x \in O_\lambda$.

- The sequence $\varphi_i(x)$ forms an increasing chain of Young diagrams, so there is a map $\varphi$ from $O_\lambda$ to the set of standard Young tableaux with $n$ boxes (i.e., fillings of Young diagrams with numbers $\{1, \ldots, n\}$ which are increasing along rows and columns) such that the subdiagram of $\varphi(x)$ made of the boxes labelled from 1 to $i$ is $\varphi_i(x)$.
- The irreducible components $O_{\lambda, \alpha}$ of $O_\lambda$ are the closures of $\varphi^{-1}(\alpha)$, where $\alpha$ runs over SYT($\lambda$), the set of standard Young tableaux of shape $\lambda$.
- The $O_{\lambda, \alpha}$ all have the same dimension which is one half of that of $D_\lambda$.

The dimension of $D_\lambda$ is easily calculated by computing the stabilizer of any element of the orbit and its dimension $\sum_{i,j} \min(\lambda'_i, \lambda'_j) = \sum_i \lambda_i^2$ (cf [Hum95, p. 11]), so that we find:

$$\dim O_{\lambda, \alpha} = \frac{n(n-1)}{2} - \frac{1}{2} \sum_{i=1}^{n} \lambda_i(\lambda_i - 1).$$

When there is no risk of confusion, we shall drop the index $\lambda$: $O_{\lambda, \alpha} = O_{\alpha}$.

4.1.2. The case $\lambda_1 - \lambda_N \leq 1$. Define the dominance order on partitions by $\lambda \prec \mu$ iff $\sum_{i \leq k} \lambda_i \leq \sum_{i \leq k} \mu_i$ for all $k$. Then one has [Hum95, p. 139] $D_\mu \subset D_\lambda$ iff $\lambda \prec \mu$. The next proposition gives a more explicit description of $D_\lambda$ and $O_\lambda$ in a special case which is important for our purposes:
Proposition 4.1. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ be a partition such that $\lambda_1 - \lambda_N \leq 1$. Then
\[
D_\lambda = \{x \in g : x^N = 0\},
\]
\[
\mathcal{O}_\lambda = \{x \in n : x^N = 0\}.
\]

Proof. According to the discussion above, the first equality amounts to saying that among all the partitions $\mu$ of $n$ with at most $N$ parts, $\lambda$ is the smallest for the dominance order. By direct computation, if $n = Nq + r$, $\lambda = (q + 1, \ldots, q + 1, q, \ldots, q)$; and assuming a $\mu = (\mu_1, \ldots, \mu_N)$, $\sum_i \mu_i = n$ breaks one of the inequalities $\sum_{i \leq k} \lambda_i \leq \sum_{i \leq k}$ leads to a contradiction with $\mu$ being decreasing.

The second equality follows immediately from the first. \qed

Example. If $\lambda = (1, 1)$ there is only one tableau \[\begin{array}{cc}
\ast & \ast \\
\ast & \ast \\
\end{array}\] and \[\begin{array}{cc}
\ast & \ast \\
\ast & \ast \\
\end{array}\] is a free entry and \(\ast\) a zero in the lower triangle.

Next,
\[
\mathcal{O}_{(2,1)} = \begin{pmatrix}
\ast & x_{12} & x_{13} \\
\ast & \ast & x_{23} \\
\ast & \ast & \ast
\end{pmatrix} : x_{12}x_{23} = 0 = \mathcal{O}_{(2,1)} \cup \mathcal{O}_{(2,1)}
\]
\[
\mathcal{O}_{(2,1)} = \begin{pmatrix}
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{pmatrix}, \quad \mathcal{O}_{(2,1)} = \begin{pmatrix}
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{pmatrix}.
\]

Similarly, one computes
\[
\mathcal{O}_{(2,2)} = \{x \in 4 \times 4 : x^2 = 0\} = \mathcal{O}_{(2,2)} \cup \mathcal{O}_{(2,2)}
\]
\[
\mathcal{O}_{(2,2)} = \begin{pmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{pmatrix}, \quad \mathcal{O}_{(2,2)} = \begin{pmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{pmatrix} : x_{12}x_{24} + x_{13}x_{34} = 0
\]

4.1.3. Hotta’s construction of the Joseph representation. In the rest of Section 4 we fix a partition $\lambda$. Define $W_\lambda$ to be the finite-dimensional space of maps from SYT($\lambda$) to $\mathbb{C}$. Its dimension is that of the irreducible representation of $S_n$ associated to $\lambda$. Orbital varieties provide us with a natural action of $S_n$ on $W_\lambda$, which we describe now following [Hot84].

Given $i = 1, \ldots, n - 1$, define $n_i$ to be the subspace of $x \in n$ whose entry $x_{i,i+1}$ vanishes; and $P_i$ to be the parabolic subgroup of $GL_n$ made of invertible matrices $x$ which are upper triangular except possibly at $x_{i,i+1}$. Note that the map $f : P_i \times g \to g$, $f(p, x) = pxp^{-1}$ sends $P_i \times n_i$ to $n_i$. We shall now describe the action of the elementary transposition $(i, i+1) \in S_n$ by giving its matrix elements.

Given a $\alpha \in SYT(\lambda)$, two situations can occur:

(1) Either $\mathcal{O}_\alpha \subset n_i$, in which case set $m_{i;\alpha,\beta} = -\delta_{\alpha,\beta}$ for all $\beta$.

(2) Or $\mathcal{O}_\alpha \not\subset n_i$, in which case consider the scheme-theoretic intersection (i.e., with multiplicities) $\mathcal{O}_\alpha \cap n_i$, and then its image by $f$, i.e., $f(P_i \times (\mathcal{O}_\alpha \cap n_i))$ (again keeping
track of the degree of the map on each irreducible component of \(O_\alpha \cap n_i\); Clearly \(f(P_i \times (O_\alpha \cap n_i)) \subset O_\lambda \cap n_i\), so its top-dimensional components are again orbital varieties (necessarily distinct from \(\alpha\)). Then set
\[
m_{i;\alpha,\beta} = \begin{cases} 
1 & \beta = \alpha \\
\text{multiplicity of } O_\beta \text{ in } f(P_i \times (O_\alpha \cap n_i)) & \beta \neq \alpha
\end{cases}
\]
Finally, if \((e_\alpha)_{\alpha \in SYT(\lambda)}\) is the standard basis of \(W_\lambda\): \(e_\beta(\alpha) = \delta_{\alpha,\beta}\), then define
\[
(4.2) \quad \rho_{(i,i+1)} e_\beta = - \sum_{\alpha \in SYT(\lambda)} m_{i;\alpha,\beta} e_\alpha.
\]

**Theorem (Hotta).** The \((m_{i;\alpha,\beta})_{\alpha,\beta \in SYT(\lambda)}\), \(i = 1, \ldots, n - 1\), satisfy the symmetric group relations; and equipped with the action \(\rho_{(i,i+1)}\) above, \(W_\lambda\) is the standard \(S_n\)-module associated to the partition \(\lambda\).

**Example.** For the three cases \((1,1), (2,1), (2,2)\), we find:
\[
\lambda = (1,1): \quad \rho_{(1,2)} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},
\]
\[
\lambda = (2,1): \quad \rho_{(1,2)} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \rho_{(2,3)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},
\]
\[
\lambda = (2,2): \quad \rho_{(1,2)} = \rho_{(3,4)} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \rho_{(2,3)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

### 4.2. Extended Joseph polynomials.
Now consider the (complex) torus \(T = (\mathbb{C}^\times)^{n+1}\) acting on \(n\) as follows: the first \(n\) variables correspond to conjugation by diagonal matrices, whereas the last variable corresponds to scaling. Explicitly, if \(x \in n\) has entries \(x_{ij}\) and \(t = (t_1, \ldots, t_n, q) \in T\), then \((t \cdot x)_{ij} = q t_j t_j^{-1} x_{ij}\).

Observe that \(O_\lambda\), and therefore its irreducible components \(O_{\lambda,\alpha}\), are invariant by \(T\)-action. Thus, they have natural Poincaré-dual classes in equivariant cohomology. It is convenient to describe them in the language of multidegrees (see [MS05]).

#### 4.2.1. Multidegrees.
Given a torus \(T\) acting linearly on a complex vector space \(W\), we assign to a closed \(T\)-invariant sub-scheme \(X \subseteq W\) its **multidegree**\(\text{mdeg}_W X \in \text{Sym}(T^*)\) (here \(T^*\) is viewed as a lattice inside the dual of the Lie algebra of \(T\)), which can be computed inductively using the following properties (as in [Jos97]):

1. If \(X = W = \{0\}\), then \(\text{mdeg}_W X = 1\).
2. If the scheme \(X\) has top-dimensional components \(X_i\), where \(m_i > 0\) denotes the multiplicity of \(X_i\) in \(X\), then \(\text{mdeg}_W X = \sum_i m_i \text{mdeg}_W X_i\).
3. Assume \(X\) is a variety, and \(H\) is a \(T\)-invariant hyperplane in \(W\).
   a. If \(X \not\subseteq H\), then \(\text{mdeg}_W X = \text{mdeg}_H (X \cap H)\).
(b) If $X \subset H$, then $\text{mdeg}_W X = [W/H]_T \text{mdeg}_H X$, where $[\cdot]_T \in T^*$ denotes the weight of the $T$-action.

One can readily see from these properties that $\text{mdeg}_W X$ is homogeneous of degree $\text{codim}_W X$, and is a positive sum of products of the weights of $T$ on $W$.

In our case, $\text{Sym}(T^*) \cong \mathbb{Z}[z_1, \ldots, z_n, h]$, and the weights on $\mathbb{C}[x_{ij}]_{1 \leq i < j \leq n}$ of the $T$-action are defined by

$$[x_{ij}]_T = h + z_i - z_j$$

The multidegree of $O_\alpha$ with respect to this $T$-action, $J_\alpha := \text{mdeg}_n O_\alpha$, is called the extended Joseph polynomial of $O_\alpha$. $J_\alpha$ is by definition a homogeneous polynomial in $\mathbb{Z}[z_1, \ldots, z_n, h]$, of degree the codimension of $O_\alpha$, which is nothing but $k(\lambda) = \frac{1}{2} \sum_i \lambda_i(\lambda_i - 1)$ defined in Section 3, according to Eq. (4.1). The reason for the name, first given in [DFZJ05], is that if we remove the scaling equivariance, i.e., set the variable $h = 0$, these polynomials reduce to the ones Joseph introduced in [Jos84].

**Example.** All the examples of orbital varieties given above are complete intersections (the number of equations is equal to the codimension); their multidegree is therefore the product of weights of the equations:

$$J_{12} = 1,$$
$$J_{13} = h + z_1 - z_2,$$
$$J_{14} = h + z_2 - z_3,$$
$$J_{23} = (h + z_1 - z_2)(h + z_3 - z_4),$$
$$J_{24} = (h + z_2 - z_3)(2h + z_1 - z_4).$$

### 4.2.2. Divided Differences.

The geometric construction given in Section 4.1.3 has a direct counterpart for multidegrees. Here our reference is [KZJ09, Sect. 5.1.1].

Define the divided difference operator $\partial_i = \frac{1}{z_i - z_{i+1}}(\tau_i - 1)$ where $\tau_i$ is permutation of variables $z_i$ and $z_{i+1}$. Note that both $\partial_i$ and $\tau_i$ are operators leaving $\mathbb{Z}[z_1, \ldots, z_n]$ stable.

Let $B$ be the group of invertible upper triangular matrices of size $n$. We use the following special case of [Jos84] (see also [KZJ09, Lemma 8]):

**Lemma 4.2.** Let $X \subset n$ be a $B$-invariant variety such that $f(P_i \times X) \subset n$. Let $k$ be the degree of the map $f|_X : (P_i \times X)/B \to n$, or zero if the generic fiber is infinite (i.e., $X$ is $P_i$-invariant). Then

$$- \frac{1}{h + z_{i+1} - z_i} \partial_i((h + z_{i+1} - z_i) \text{mdeg}_n X) = k \text{mdeg}_n f(P_i \times X).$$

The proof is a standard equivariant localization argument which we shall not repeat here.

We now discuss separately the two cases of the construction of Section 4.1.3. Given a $\alpha \in \text{SYT}(\lambda)$,
(1) If \( \mathcal{O}_\alpha \subset \mathfrak{n}_s \), then \( f(P_t \times \mathcal{O}_\alpha) \subset \mathfrak{n}_s \cap \mathcal{O}_\lambda \) is irreducible and contains \( \mathcal{O}_\alpha \); therefore it is equal (set-theoretically) to \( \mathcal{O}_\alpha \), i.e., \( \mathcal{O}_\alpha \) is \( P_t \)-invariant. Lemma 4.2 implies
\[
\partial_i((h + z_{i+1} - z_i)J_\alpha) = 0.
\]

(2) If \( \mathcal{O}_\alpha \not\subset \mathfrak{n}_s \), we have \( \text{mdeg}_i(\mathcal{O}_\alpha \cap \mathfrak{n}_i) = (h + z_i - z_{i+1})J_\alpha \) by property (3b) of multidegrees, and then by applying Lemma 4.2 to each irreducible component of \( \mathcal{O}_\alpha \cap \mathfrak{n}_i \) we find:
\[
-(h + z_i - z_{i+1})\partial_iJ_\alpha = \sum_{\beta \neq \alpha} m_{i;\alpha,\beta}J_\beta.
\]

Adding the diagonal term to the sum, Eq. (4.4) can be rewritten under the equivalent form
\[
\hat{s}_iJ_\alpha = \sum_{\beta} m_{i;\alpha,\beta}J_\beta
\]
where we used the \( S_n \)-action \( \hat{s}_i = \tau_i - h\partial_i \) of Eq. (3.2). Note now that Eq. (4.3) is a special case of Eq. (4.5) where \( m_{i;\alpha,\beta} = -\delta_{\alpha,\beta} \). So Eq. (4.5) is valid in all cases. At \( h = 0 \), which is the case that Joseph considered in [Jos84], \( \hat{s}_i \) reduces to the action of \( S_n \) on \( \mathbb{C}[z_1, \ldots, z_n] \) by permutation of variables.

4.3. Identification with \( I_\lambda \). There is a natural object in \( W_\lambda \otimes \mathbb{C}[z_1, \ldots, z_n, h] \), namely the map \( J_\lambda : \alpha \in \text{SYT}(\lambda) \mapsto J_\alpha \). Combining Eqs. (4.2) and (4.3), we find:
\[
\rho^{(i,i+1)}J_\lambda = -\hat{s}_iJ_\lambda.
\]

According to Theorem 4.1.3 \( W_\lambda \) carries the structure of \( S_n \)-module which is the same, by Schur–Weyl duality, as that of the space of singular vectors in \( V[\lambda] \) (where \( S_n \) acts by permutation \( P^{(i,i+1)} \) of tensors). Tensoring with \( \mathbb{C}[z_1, \ldots, z_n, h] \) (on which we do not make \( S_n \) act), we obtain an \( S_n \)-intertwiner \( \phi : W_\lambda \otimes \mathbb{C}[z_1, \ldots, z_n, h] \to V[\lambda]_{\text{sing}} \otimes \mathbb{C}[z_1, \ldots, z_n, h] \). The equation above becomes
\[
P^{(i,i+1)}\phi(J_\lambda) = -\hat{s}_i\phi(J_\lambda)
\]
which means \( \phi(J_\lambda) \) satisfies the hypothesis of Lemma 3.1. Note that \( \phi \) is only defined up to a non-zero multiplicative constant.

We now want to identify \( \phi(J_\lambda) \) with \( I_\lambda \) by using Lemma 3.3. By definition the entries of \( \phi(J_\lambda) \) are linear combinations of those of \( J_\lambda \) and therefore are homogeneous polynomials of degree \( k(\lambda) \) in the variables \( z_1, \ldots, z_n, h \). We have just derived the skew-symmetry of \( \phi(J_\lambda) \) from Lemma 3.1. Therefore, we have proved:

**Theorem 4.3.** Let \( \lambda \) be a partition of \( n \). Then there exists a unique intertwiner \( \phi \) such that
\[
\phi(J_\lambda) = I_\lambda.
\]

In particular, all properties that we shall prove for \( I_\lambda \) are true for \( J_\lambda \) as well.

**Example.** By comparing the formulæ for \( I_\lambda \) and \( J_\lambda \), we find
\[
\begin{align*}
\lambda = (1,1) : & \quad \phi([1]) = v_{12} - v_{21}, \\
\lambda = (2,1) : & \quad \phi([2]) = v_{112} - v_{121}, \quad \phi([3]) = v_{211} - v_{121}, \\
\lambda = (2,2) : & \quad \phi([3]) = v_{1122} + v_{2211} - v_{1212} - v_{2121}, \quad \phi([3]) = v_{1221} + v_{2121} - v_{1212} - v_{2211}.
\end{align*}
\]
4.4. **Case of two rows.** In this section, we assume that the partition $\lambda$ has only two rows: $\lambda = (n-p, p)$. This case was investigated in detail in the paper [KZJ09], so that we shall omit proofs of the results that were already contained in it.

4.4.1. **Link patterns.** Call **link pattern** (or non-crossing matching) an unordered collection of disjoint pairs of $\{1, \ldots, n\}$ such that if $\{1, \ldots, n\}$ are represented as ordered vertices on a line, then the two elements of each pair can simultaneously be connected in the upper half plane in a non-crossing fashion (we sometimes call these connecting lines **arches**) and unpaired elements can be connected to upwards infinity (i.e., they must be outside all arches); cf Fig. 1 left.

There is a simple bijection from link patterns with $p$ arches to standard Young tableaux of shape $\lambda = (n-p, p)$, obtained by recording in the first row the locations of openings of arches and of empty spots, and on the second row the locations of closings of arches. see Fig. 1.

Given $\alpha \in \text{SYT}(\lambda)$, we can therefore consider its associated link pattern, and in particular, we shall use the following notation: if $i$ and $j$ are paired in the link pattern, write $\alpha(i) = j$, $\alpha(j) = i$; if $i$ is unpaired, write $\alpha(i) = \emptyset$.

There is an action of the **Temperley–Lieb algebra** on the space of linear combinations of link patterns, which we identify with $W_\lambda$ by identifying $e_\alpha$ with the corresponding link pattern. It is defined graphically by the action of the generators $E^{(i,i+1)}$, $i = 1, \ldots, n - 1$, of the Temperley–Lieb algebra which corresponds to reconnecting vertices $i$ and $i + 1$, e.g.,
with the additional rules that reconnecting two unpaired points produces zero and that closed loops must be erased at a cost of multiplication by 2.

The Temperley–Lieb algebra (at loop weight 2) is a quotient of the symmetric group algebra, and in fact it is shown in [KZJ09, Sect. 5.1.2] that if one sets \( \rho(i,i+1) = 1 - E(i,i+1) \), then the action defined above coincides with the Joseph representation defined in Section 4.1.3.

4.4.2. Description of orbital varieties. When \( \lambda \) has only two rows, \( O_\lambda \) is a “spherical variety”, i.e., the group of invertible upper triangular matrices \( B \) acts on it by conjugation with a finite number of orbits. This gives us a first description of its irreducible components as \( B \)-orbit closures. Define \( \alpha_< \) to be the upper triangular matrix with values in \( \{0, 1\} \) such that \( (\alpha_<)_{ij} = 1 \) iff \( j = \alpha(i) > i \). Then it is easy to show that \( O_\alpha = B \cdot \alpha_< \), where \( \cdot \) denotes conjugation action.

An alternative description is in terms of equations:

**Proposition 4.4.** \( O_\alpha \) is defined by the following equations:

1. \( x^2 = 0 \).
2. The rank of any lower-left submatrix of \( x \) is lower or equal to the rank of the same submatrix of \( \alpha_< \).

This statement can be extracted with some effort from [Mcl06] or can be deduced directly from the results of [KZJ09].

4.4.3. Exchange relation. We rewrite explicitly the dichotomy of the Hotta construction (Sections 4.1.3 and 4.2.2) since it will be needed in the next section.

Consider \( \alpha \in \text{SYT}(\lambda) \) and its associated link pattern. The two cases are:

1. Either \( \alpha(i) \neq i + 1 \) (there is no arch connecting \( i \) and \( i + 1 \) in the link pattern), which means \( (\alpha_<)_{i,i+1} = 0 \) and according to Proposition 4.2.4.2, among the equations of \( O_\alpha \) there is \( x_{i,i+1} = 0 \), i.e., \( O_\alpha \subseteq \mathfrak{nil} \), and Eq. (4.3) holds, or equivalently, \( J_\alpha \) is \( h + z_i - z_{i+1} \) times a symmetric polynomial in \( z_i \), \( z_{i+1} \).

2. Or \( \alpha(i) = i + 1 \) (there is an arch connecting \( i \) and \( i + 1 \) in the link pattern), in which case \( (\alpha_<)_{i,i+1} = 1 \), which implies \( O_\alpha \not\subseteq \mathfrak{nil} \). Then we can rewrite Eq. (4.3) as:

\[
-(h + z_i - z_{i+1}) \partial_i J_\alpha = \sum_{\beta: E(i,i+1) \beta = \alpha} J_\beta
\]

where, to keep notations simple, we have identified standard Young tableaux and link patterns when we write “\( E(i,i+1) \beta = \alpha \)”.

4.4.4. Change of basis. Finally, we investigate the intertwiner \( \phi \). In fact, in the case of two-row diagrams, there is a well-known explicit formula for \( \phi \), which was rediscovered many times and dates back (at least in the special case \( p = n/2 \)) to [RTW32] (see [SZJ10] for more references and background). Roughly speaking, a pairing between \( i \) and \( j \) corresponds to a \( \mathfrak{sl}(2) \) singlet \( v_1^{(i)} \otimes v_2^{(j)} - v_2^{(i)} \otimes v_1^{(j)} \), whereas an unpaired \( i \) is a \( v_1^{(i)} \) (where superscripts are as
usual locations in the tensor product \((\mathbb{C}^2)^\otimes n\). With the present sign conventions, the exact statement is:

**Proposition 4.5.** The intertwiner \(\phi\) is given by

\[
\phi(e_\alpha) = \sum_{L=(l_1, \ldots, l_n) \in \{1,2\}^n} (-1)^{\lceil \frac{n-p}{2} \rceil + \# \{ i \text{ even} : l_i = 1 \}} v_L.
\]

**Proof.** Checking that the \(\phi\) thus defined intertwines the actions of the symmetric group is a routine exercise. Because of the conditions on the conditions on the multi-index \(L\), there are \(k\) 2’s and \(n-k\) 1’s, so \(\phi(e_\alpha) \in V_\lambda\). So the only issue is normalization of \(\phi\), which is fixed by Theorem 4.3. Consider the multi-index \(L_0 = (\underbrace{1, \ldots, 1}_p, \underbrace{2, \ldots, 2}_{n-p})\). We have

\[
I_{L_0} = D_0 = \prod_{1 \leq i < j \leq n-p} (h + z_i - z_j) \prod_{n-p+1 \leq i < j \leq n} (h + z_i - z_j).
\]

On the other hand, by inspection the entry \(\phi(e_\alpha)_{L_0}\) is zero unless \(\alpha\) is the tableau \(\alpha_0 = \begin{array}{c}
\hline
1 \\
\vdots \\
1 \quad \cdots \\
\hline
1 \quad \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\hline
n-p \\
\vdots \\
n-1 \\
\hline
1 \\
\end{array}
\)

corresponding to the link pattern \(J_{\alpha_0}\) is the multidegree of the orbital variety indexed by \(\alpha_0\), which according to Proposition 4.4 is a linear subspace of the form

\[
\mathcal{O}_{\alpha_0} = \left\{ \begin{pmatrix}
\vdots & 0 & \cdots & 0 & \cdots & \cdots & \cdots & p \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \cdots & 0 & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\end{pmatrix} \right\}
\]

so that \(J_{\alpha_0} = I_{L_0}\). We conclude that the normalization of \(\phi\) is fixed by \(\phi(e_{\alpha_0})_{L_0} = 1\). This fits with the formula of the proposition.

\[\square\]

**4.4.5. Cyclicity.** Consider the special case \(n = 2p\), i.e., the Young diagram is rectangular, and the corresponding link patterns have no unpaired vertices. One can then define a rotation of link patterns in the natural way, i.e., move the vertices cyclically \(1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1\) keeping the pairings intact. Via the one-to-one correspondence from link patterns to SYT(\(\lambda\)), this defines a bijection \(\rho\) from SYT(\(\lambda\)) to SYT(\(\lambda\)). One observes empirically the following relation:

\[
J_{\rho\alpha}(z_1, \ldots, z_n) = (-1)^{p-1} J_\alpha(z_2, \ldots, z_n, z_1 - 3h)
\]
It is well-known that if Eq. (4.5) is satisfied, then Eq. (4.7) is equivalent to the qKZ equation (2.3) (in which \( N = 2 \)). Indeed we shall prove in Section 5 that the case \( \lambda = (p, p) \) is among the cases where \( I_\lambda \) and therefore \( J_\lambda \) satisfy the qKZ equation.

The case \( \lambda = (p, p) \) will be considered again in Section 6.

4.5. **Case of two columns.** Let \( \lambda = (\underbrace{2, \ldots, 2}_{p}, \underbrace{1, \ldots, 1}_{N-p}) \), \( n = N + p \). In this section we show that all orbital varieties \( O_\alpha \) for such \( \lambda \) are complete intersections and we describe explicitly their equations as well as the extended Joseph polynomials \( J_\alpha \).

Note that the codimension of \( O_\lambda \) is simply \( k(\lambda) = p \). Furthermore, \( \lambda \) satisfies the hypothesis of Proposition 4.1, so that

\[
O_\lambda = \{ x \in \mathbb{R}^n : x^N = 0 \} \quad n = N + p, \quad p \leq N.
\]

There is a general duality of orbital varieties which corresponds to conjugation of partitions and standard Young tableaux (related to the duality of [Spa82, Chapter 3]). It means that the cases of two rows and two columns are dual to each other.

4.5.1. **The dual symmetric group action.** As mentioned above, there is a bijection \( \iota \) from \( \text{SYT}(\lambda) \) to \( \text{SYT}(\lambda') \) which is just conjugation (reflection through the diagonal) of Young tableaux. Using this bijection we shall define a new “dual” action on \( W_\lambda \) starting from that on \( W_{\lambda'} \) as defined in Section 4.4.1. Recall that the action is defined by the \( m_{i;\alpha,\beta} \), cf Eq. (4.2).

Given \( \alpha \in \text{SYT}(\lambda) \), define the sign of \( \alpha \) to be

\[
\varepsilon_\alpha = (-1)^{\# \{ i < j : j \text{ strictly south-west of } i \text{ in } \alpha \}}.
\]

Now given \( \alpha, \beta \in \text{SYT}(\lambda) \) and their conjugate \( \alpha', \beta' \), define

\[
m_{i;\alpha,\beta} = -\varepsilon_\alpha \varepsilon_\beta m_{i;\beta',\alpha'}.
\]

Due to the easy lemma the \( m_{i;\alpha,\beta} \neq 0 \), \( \alpha \neq \beta \), implies \( \varepsilon_\alpha \neq \varepsilon_\beta \), we can write equivalently

\[
m_{i;\alpha,\beta} = (-1)^{\delta_{\alpha,\beta}} m_{i;\beta',\alpha'}
\]

i.e., negate the diagonal entries and transpose.

4.5.2. **Defining equations of the orbital varieties.** There is again a bijection between \( \text{SYT}(\lambda) \) and the set of link patterns in size \( n \) with \( p \) arches, obtained by composing the bijection of Section 4.4.1 with conjugation of Young tableaux. So we shall use the same notations \( \alpha(i) = j, \alpha(i) = \varnothing \), for paired \( i, j \) and unpaired \( i \), respectively.

For \( \alpha \in \text{SYT}(\lambda) \) and \( 1 \leq i < j \leq 2n \), denote

\[
p_\alpha(i, j) := j - i + 1 - \# \{ k : i \leq k < \alpha(k) \leq j \}.
\]

We have \( p_\alpha(i, j) \geq \frac{i+j+1}{2} \), with equality if and only if all elements of \( [i, j] \) are paired between themselves; in particular,

\[
p_\alpha(i, j) = \frac{j - i + 1}{2} \quad i < j = \alpha(i).
\]

An important property is that if \( i \leq i' < j' \leq j \), then \( p_\alpha(i', j') \leq p_\alpha(i, j) \) (enlarging an interval by one can either leave \( p_\alpha \) unchanged if a new pairing has been absorbed in the interval, or increase \( p_\alpha \) by 1 otherwise). This implies that \( p_\alpha(i, j) \leq p_\alpha(1, n) = n - p = N \) for all \( i < j \).
Define
\begin{equation}
\hat{\mathcal{O}}_\alpha := \left\{ x \in \mathbb{n} : (x^{p\alpha(i,j)})_{i,j} = 0, \ i < j = \alpha(i) \right\}.
\end{equation}

We want to show that \( \hat{\mathcal{O}}_\alpha = \mathcal{O}_\alpha \). First we prove the following lemma:

**Lemma 4.6.** If \( x \in \hat{\mathcal{O}}_\alpha \) then \((x^a)_{i,j} = 0\) for all \( i < j \) and \( a \geq p_\alpha(i,j) \).

In fact, these are all the \((x^a)_{i,j}\) in the ideal of equations of \( \hat{\mathcal{O}}_\alpha \).

**Proof.** By induction on \( j - i \).

If \( j = i + 1 \) then either \( a > 1 \) in which case \((x^a)_{i,i+1} = 0\) because \( x \) is strict upper triangular; or \( a = 1 = p_\alpha(i,i+1) \) which implies \( i + 1 = \alpha(i) \), in which case \( x_{i,i+1} = 0 \) is part of the defining equations of \( \hat{\mathcal{O}}_\alpha \).

Next, assume \( j > i + 1 \). We are going to divide into cases depending on the position of \( \alpha(i) \) (and similarly for \( \alpha(j) \)). If \( \alpha(i) = j \) and \( a = p_\alpha(i,j) \), once again \((x^a)_{i,j} = 0\) is part of the defining equations of \( \hat{\mathcal{O}}_\alpha \). If \( \alpha(i) \notin [i,j] \) or \( \alpha(i) = j \) and \( a > p_\alpha(i,j) \), consider
\[(x^a)_{i,j} = \sum_{i < k < j} x_{i,k} (x^{a-1})_{k,j}.\]

We claim that every term in the sum is zero. Indeed if \( \alpha(i) \notin [i,j] \), \( p_\alpha(i+1,j) = p_\alpha(i,j) - 1 \) so that \( a - 1 \geq p_\alpha(i,j) - 1 = p_\alpha(i+1,j) \geq p_\alpha(k,j) \) and we apply the induction to \((x^{a-1})_{k,j}\).

Similarly if \( \alpha(i) = j \) and \( a > p_\alpha(i,j) \), \( a - 1 \geq p_\alpha(i,j) = p_\alpha(i+1,j) \geq p_\alpha(k,j) \).

So we can assume in what follows that \( \alpha(i) \in [i,j] \). The exact same reasoning applied to \( j \) allows to conclude that \( \alpha(j) \in [i,j] \), so that \( i \leq \alpha(i) < \alpha(j) < j \).

We now come to the crucial remark that \( p_\alpha(i,j) = j - i + 1 - \#\{\text{pairings of } \alpha \text{ inside } [i,j]\} \)
\[= j - \alpha(j) + 1 - \#\{\text{pairings of } \alpha \text{ inside } [\alpha(j),j]\} \]
\[+ \alpha(j) - \alpha(i) - 1 - \#\{\text{pairings of } \alpha \text{ inside } [\alpha(i) + 1, \alpha(j) - 1]\} \]
\[+ \alpha(i) - i + 1 - \#\{\text{pairings of } \alpha \text{ inside } [i, \alpha(i)]\} \]
\[= p_\alpha(i,\alpha(i)) + p_\alpha(\alpha(i) + 1, \alpha(j) - 1) + p_\alpha(\alpha(j),j) \]
where in the last line, if \( \alpha(i) + 1 = \alpha(j) \) then conventionally \( p_\alpha(\alpha(i) + 1, \alpha(j) - 1) = 0 \). Indeed the configuration does not allow for mixed pairings between the three intervals \([i, \alpha(i)]\), \([\alpha(i) + 1, \alpha(j) - 1]\), \([\alpha(j),j]\). So we can write
\[ (x^a)_{i,j} = \sum_{i < k < \ell < j} (x^{p_\alpha(i,\alpha(i))})_{i,k} (x^{a-p_\alpha(i,\alpha(i)) - p_\alpha(\alpha(j),j)})_{k,\ell} (x^{p_\alpha(\alpha(j),j)})_{\ell,j}. \]

The first factor is zero if \( k \leq \alpha(i) \) by the induction hypothesis, noting that \( p_\alpha(i,k) \leq p_\alpha(i,\alpha(i)) \) and similarly the third factor is zero if \( \ell \geq \alpha(j) \). If \( \alpha(i) + 1 = \alpha(j) \) the proof is finished; otherwise note that \( p_\alpha(k,\ell) \leq \alpha_\alpha(\alpha(i) + 1, \alpha(j) - 1) \leq a - p_\alpha(i,\alpha(i)) - p_\alpha(\alpha(j),j) \) and the second factor vanishes for the same reason.}

Taking \( a = N \geq p_\alpha(i,j) \) in Lemma 4.6 we find that \( \hat{\mathcal{O}}_\alpha \subset \mathcal{O} \) (set-theoretically).
Now observe that \( \hat{\mathcal{O}}_\alpha \) is defined by \( p \) equations, but \( \mathcal{O} \) is equidimensional of codimension \( p \), so that \( \hat{\mathcal{O}}_\alpha \) is (a complete intersection) of pure codimension \( p \), and so is a union of irreducible components of \( \mathcal{O} \).

One could with more effort conclude geometrically that \( \mathcal{O}_\alpha = \hat{\mathcal{O}}_\alpha \), but instead we shall use multidegrees and the uniqueness property of Lemma 3.3. Define \( \hat{J}_\alpha := \text{mdeg}_n \hat{\mathcal{O}}_\alpha \). Since the \( \hat{\mathcal{O}}_\alpha \) are complete intersections, one can calculate directly

\[
\hat{J}_\alpha = \prod_{i < j = \alpha(i)} \left( \frac{j - i + 1}{2} h + z_i - z_j \right).
\]

Note that an identical formula (really, at multidegrees and the uniqueness property of Lemma 3.3. Define \( \hat{J}_\alpha := \text{mdeg}_n \hat{\mathcal{O}}_\alpha \). Since the \( \hat{\mathcal{O}}_\alpha \) are complete intersections, one can calculate directly

\[
\hat{J}_\alpha = \prod_{i < j = \alpha(i)} \left( \frac{j - i + 1}{2} h + z_i - z_j \right).
\]

We can then check

**Lemma 4.7.** \( \hat{J}_\lambda = \sum_\alpha \hat{J}_\alpha e_\alpha \) satisfies Eq. (4.6).

**Proof.** Recall that Eq. (4.6) amounts to saying that the entries \( \hat{J}_\alpha \) of \( \hat{J}_\lambda \) must satisfy Eq. (4.5). Taking into account that we use the dual action defined above, we find the same dichotomy as in Section 4.3.3, but inverted:

1. If \( \alpha(i) = i + 1 \), then according to Section 4.3.3 case (2), \( m_{i;i',\alpha'} = +1 \), so \( m_{i;\alpha,\alpha} = -1 \), i.e. we are in case (1) of Section 4.3.3, so Eq. (4.5) can be rewritten as Eq. (4.3), which is trivially satisfied by \( \hat{J}_\alpha = C(h + z_i - z_{i+1}) \) where \( C \) does not depend on \( z_i, z_{i+1} \).
2. If \( \alpha(i) \neq i + 1 \), then according to Section 4.3.3 case (2), \( m_{i;i',\alpha'} = -1 \), so \( m_{i;\alpha,\alpha} = +1 \), i.e. we are in case (2) of Section 4.3.3, so Eq. (4.5) can be rewritten as Eq. (4.4), or more explicitly,

\[
-(h + z_i - z_{i+1}) \partial_i \hat{J}_\alpha = \begin{cases} 0 & \alpha(i) = \alpha(i + 1) = \emptyset \\ \hat{J}_{E(i,i+1)\alpha} & \text{otherwise} \end{cases}
\]

where again we have identified standard Young tableaux and link patterns when we write “\( E(i,i+1)\alpha \)”.

This equation can also be checked directly case by case:

- If \( \alpha(i) = \alpha(i + 1) = \emptyset \), \( \hat{J}_\alpha \) does not depend on \( z_i, z_{i+1} \), so \( \partial_i \hat{J}_\alpha = 0 \).
- If \( \alpha(i) = \emptyset, \alpha(i + 1) = j > i + 1 \), \( \hat{J}_\alpha = C(\frac{j - i + 1}{2} h + z_{i+1} - z_j) \), so \( -(h + z_i - z_{i+1}) \partial_i \hat{J}_\alpha = C(h + z_{i+1} - z_i) \) which is indeed \( \hat{J}_{E(i,i+1)\alpha} \) since \( E(i,i+1)\alpha \) differs from \( \alpha \) only in pairing \( i, i + 1 \) and having \( j \) unpaired.
- The case \( \alpha(i) = j < i \), \( \alpha(i + 1) = \emptyset \) can be treated similarly.
- If \( \alpha(i) = j < i \), \( \alpha(i + 1) = k > i \), \( \hat{J}_\alpha = C(\frac{j - i + 1}{2} h + z_{i+1} - z_k) \), so \( -(h + z_i - z_{i+1}) \partial_i \hat{J}_\alpha = C(h + z_i - z_{i+1}) \frac{k - i + 1}{2} h + z_j - z_k \) which again coincides with \( \hat{J}_{E(i,i+1)\alpha} \), since \( E(i,i+1)\alpha \) pairs \( i, i + 1 \) and \( j, k \).
- The other two cases \( i < i + 1 < \alpha(i + 1) < \alpha(i) \) and \( \alpha(i + 1) < \alpha(i) < i + 1 \) can be treated similarly.

\( \square \)
Applying the intertwiner \( \phi \) and then Theorem 4.3 and Lemma 3.3, we conclude that the \( \hat{J}_\alpha \) coincide up to normalization with the multidegrees \( J_\alpha \) of the orbital varieties: 
\[
\hat{J}_\alpha = cJ_\alpha
\]
for some \( c \neq 0 \).

Now according to the above, \( \hat{O}_\alpha \) is a union of a certain certain subset of \( O_\beta \), so we can write at the level of multidegrees 
\[
\hat{J}_\alpha = \text{mdeg} \hat{O}_\alpha = \sum k_{\alpha,\beta} J_\beta
\]
where \( k_{\alpha,\beta} \) is the multiplicity of \( O_\beta \) in \( \hat{O}_\alpha \) (or zero if \( O_\beta \not\subset \hat{O}_\alpha \)). In order to conclude, we only need to note that according to Eq. (4.5) and Theorem 4.1.3, the \( J_\beta \), \( \beta \in \text{SYT}(\lambda) \), generate a subspace of \( \mathbb{C}[z_1, \ldots, z_n, h] \) which is an irreducible representation of the symmetric group under the action \( \hat{s}_i \) (associated to the conjugate partition \( \lambda' \), with our sign convention), and so in particular are linearly independent. So 
\[
cJ_\alpha = \sum k_{\alpha,\beta} J_\beta
\]
implies 
\[
k_{\alpha,\beta} = c\delta_{\alpha,\beta}
\]
and \( \hat{O}_\alpha = O_\alpha \) (set-theoretically). In other words, we have proved:

**Theorem 4.8.** \( O_\alpha \) is defined by the equations

\[
\left( x_{\frac{i-j+1}{2}} \right)_{i,j} = 0, \quad i < j = \alpha(i).
\]

In fact, since all coefficients of \( \hat{J}_\alpha = k_{\alpha,\alpha} J_\alpha \) at \( h = 0 \) are \( \pm 1 \) and \( J_\alpha \in \mathbb{Z}[z_1, \ldots, z_n, h] \), we have \( k_{\alpha,\alpha} = c = 1 \), i.e., \( \hat{J}_\lambda = J_\lambda \) (and \( \hat{O}_\alpha \) being a complete intersection, the equations above define \( O_\alpha \) as a reduced scheme). In particular,

\[
J_\alpha = \prod_{i<j=\alpha(i)} \left( \frac{j-i+1}{2} h + z_i - z_j \right).
\]

**4.5.3. Change of basis.** We can use again duality (conjugation of partition and Young tableaux) to find the intertwiner:

**Lemma 4.9.** The intertwiner is:

\[
\phi(e_\alpha) = (-1)^{p(p-1)/2} \varepsilon_\alpha \sum_{L=(l_1, \ldots, l_n)} J_{\alpha'} \mid_{z_1^{l_1-1} \cdots z_n^{l_n-1}} v_L
\]

where \( \mid_{z_1^{l_1-1} \cdots z_n^{l_n-1}} \) denotes the given coefficient of a polynomial.

**Proof.** We need to check that this \( \phi \) intertwines the symmetric group action. Note that \( \deg J_{\alpha'} = p(p-1)/2 + (n-p)(n-p-1)/2 \) so \( z_1^{l_1} \cdots z_n^{l_n} \) exhausts the degree and we might
as well set \( h = 0 \) in \( J_{\alpha'} \).

\[
P^{(i,i+1)}(\phi(\varepsilon_\beta)) = (-1)^{p(p-1)/2} \sum_{\xi_\beta} J_{\beta'} \bigg|_{z_1^{i-1} \cdots z_n^{i+1}} v \bigg|_{i+1,i,i+1,i+1,\ldots,i+1} \\
= (-1)^{p(p-1)/2} \sum_{\xi_\beta} (\xi_i J_{\beta'}) \bigg|_{z_1^{i-1} \cdots z_n^{i+1}} v_L \\
= (-1)^{p(p-1)/2} \sum_{\xi_\beta} \sum_{\alpha'} m_{i;\beta',\alpha'} J_{\alpha'} \bigg|_{z_1^{i-1} \cdots z_n^{i+1}} v_L \
\text{by Eq. (3.5) at } h = 0 \\
= \sum_{\alpha'} \xi_\beta \varepsilon_\alpha m_{i;\beta',\alpha'} \phi(\varepsilon_\alpha) \\
= -\sum_{\alpha} m_{i;\alpha,\beta} \phi(\varepsilon_\alpha) \\
= \phi(\rho^{(i,i+1)}(\varepsilon_\beta)) \text{ by Eq. (3.2)}
\]  

Next we check the normalization, which is fixed by Theorem 4.3. Consider the same tableau \( \alpha_0 \) that was used in the proof of Proposition 4.5, i.e., \( \alpha_0 = \begin{array}{c|ccccc}
1 & & & & & n-p \\
\hline
i+1 & \cdots & n-p & & & \\
\end{array} \) which is a tableau of \( \lambda' \). Then we have as before \( J_{\alpha_0} = \prod_{1 \leq i < j \leq n-p} (h + z_i - z_j) \prod_{n-p+1 \leq i < j \leq n} (h + z_i - z_j) \), so that \( \phi(\varepsilon_{\alpha_0}) = (-1)^{(n-p)(n-p-1)/2} A_{n-p} \otimes A_n \) where \( A_k \) is the antisymmetrizer \( \sum_{\sigma \in S_k} (-1)^{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \).

Now consider the multi-index \( L = (n - p, \ldots, 1, 1, \ldots, p) \). According to Eq. (3.6), \( I_L = (z_n - z_{n-2p+1})(z_{n-1} - z_{n-2p+2}) \cdots (z_{n-p+1} - z_{n-p}) \). But this is also \( J_{\alpha_0} \big|_{h=0} \) according to Eq. (4.12). Recalling that Theorem 4.1.3 implies that the \( J_{\alpha} \big|_{h=0} \) are linearly independent, we conclude that the coefficient of \( v_L \) in \( \phi(\varepsilon_{\alpha_0}) \) must be 1, which is consistent with the formula above.

4.5.4. Cyclicity. In order to simplify the discussion, we assume now that \( n = 2p = 2N \), i.e., the Young diagram is rectangular, and use again the rotation \( \rho \) of Section 4.4.3 (in principle the content of the present section is valid for \( p < n/2 \) but more work would be needed to define \( \rho \)).

Then it is obvious from the explicit form (4.12) that the \( J_{\alpha} \) satisfy the extra “cyclicity” relation:

\[
J_{\rho \alpha}(z_1, \ldots, z_n) = -J_{\alpha}(z_2, \ldots, z_n, z_1 - (N + 1)h)
\]

Indeed if \( n \) is paired to say \( i \), there is a factor \( \frac{n+i+1}{2} h + z_n - z_i \), but once rotated, the pairing \( i, i + 1 \) produces a factor \( \frac{i+1}{2} h + z_{i+1} - z_1 \) which corresponds to the substitution \( z_i \mapsto z_{i+1}, \ z_n \mapsto z_1 - \frac{n+2}{2} h \) and a change of sign.

This together with Eq. (4.3) implies the \( qKZ \) equation.

We shall see in next section the condition on \( \lambda \) for \( I_{\lambda} \) (and therefore \( J_{\lambda} \)) to satisfy \( qKZ \), a condition which is satisfied if \( \lambda \) has two columns.

5. \( I_{\lambda} \) is a \( q \)-conformal block, \( I_{\lambda} \) satisfies the \( qKZ \) equations if \( d(\lambda) \leq 1 \).

**Theorem 5.1.** Let \( \lambda \in \mathbb{N}^N \) be a partition of \( n \). If \( d(\lambda) > 0 \) then \( I_{\lambda} \) is a \( q \)-conformal block at level \( d(\lambda) \). If \( d(\lambda) = 0 \) then \( I_{\lambda} \) is a \( q \)-conformal block at level 1.
Proof. First we prove that $I_\lambda$ is a singular vector. Observe that the $e_{i,j}$-image of a $V[\lambda]$-valued function is a $V[\mu]$-valued function with $\mu_k = \lambda_k$ except $\mu_1 = \lambda_1 + 1$ and $\mu_j = \lambda_j - 1$. The action of $e_{i,j}$ and the deformed action of $S_n$ commute, hence $e_{i,j}I_\lambda$ is skew-symmetric. If $i < j$, then the degree $k(\lambda)$ of $e_{i,j}I_\lambda$ is strictly less than $k(\mu) = k(\lambda) + \lambda_i - \lambda_j + 1$. Hence $e_{i,j}I_\lambda$ must be 0 by Lemma 3.3.

The operation $e(z)$ also commutes with the deformed action of $S_n$, hence we can argue for the $q$-conformal block property similarly.

First let $d(\lambda) > 0$. The function $e(z)I_\lambda$ is a skew-symmetric $V[\mu]$-valued function of degree $k(\lambda) + 1$, where $\mu_k = \lambda_k$ except $\mu_1 = \lambda_1 + 1$, $\mu_N = \lambda_N - 1$. Calculation shows that $k(\mu) = k(\lambda) + d(\lambda) + 1$ which is strictly greater than the degree $k(\lambda) + 1$. Hence $e(z)I_\lambda = 0$ by Lemma 3.3.

Now let $d(\lambda) = 0$. The function $(e(z)^2)I_\lambda$ is a skew-symmetric $V[\mu]$-valued function of degree $k(\lambda) + 2$, where $\mu_k = \lambda_k$ except $\mu_1 = \lambda_1 + 2$, $\mu_N = \lambda_N - 2$. Calculation shows that $k(\mu) = k(\lambda) + 4$ which is strictly greater than the degree $k(\lambda) + 2$. Hence $e(z)^2I_\lambda = 0$ by Lemma 3.3.

Corollary 5.2. Let $d(\lambda) \leq 1$. For generic $z \in \mathbb{C}^n$ the space of $q$-conformal blocks at level 1 is one-dimensional.

Proof. We recalled in Lemma 2.2 that for generic $z$ the dimension of $q$-conformal blocks is at most 1. We proved in Theorem 5.1 that $I_\lambda$ is a (generically nonzero) $q$-conformal block. Hence, for generic $z$ this space is one-dimensional.

Consider $z \in \mathbb{C}^n$ for which Corollary 5.2 holds and for which the $q$KZ operators have no singularities, e.g. $z_1 - z_2 + h \neq 0$ etc. Over the configuration space of these $z$’s one may consider the bundle of singular vectors. The $q$-conformal blocks form a rank 1 subbundle. It is proved in [MV98] that the subbundle of $q$-conformal blocks is preserved by the $q$KZ connection. In our language this means the following theorem.

Theorem 5.3. [MV98] Theorem 2] If, for the $z$’s defined above, the space of $q$-conformal blocks at level 1 is spanned by a $V[\lambda]$-valued function $I$, then a scalar function multiple of $I$ satisfies the $q$KZ difference equations (2.3).

Theorem 5.4. Let $d(\lambda) \leq 1$. Then $I_\lambda$ satisfies the $q$KZ equations (2.3).

The rest of this section is the proof of this theorem.

Proof. The function $I_\lambda$ is a $q$-conformal block at level 1. By Theorem 5.3, there is a scalar function $f(z_1, \ldots, z_n)$ such that $fI_\lambda$ satisfies the $q$KZ equations. We obtain

$$f(\ldots, z_i - (N + 1)h, \ldots)I_\lambda(\ldots, z_i - (N + 1)h, \ldots) = K_i f I_\lambda, \quad i = 1, \ldots, n.$$  

After rearrangement we have

$$f(\ldots, z_i - (N + 1)h, \ldots)I_\lambda(\ldots, z_i - (N + 1)h, \ldots) = K_i I_\lambda, \quad i = 1, \ldots, n. \tag{5.1}$$

Since $I_\lambda(\ldots, z_i - (N + 1)h, \ldots)$ and $K_i I_\lambda$ are rational functions of $z_1, \ldots, z_n$, the ratios

$$g_i = \frac{f(\ldots, z_i - (N + 1)h, \ldots)}{f}, \quad i = 1, \ldots, n,$$

are rational functions of $z_1, \ldots, z_n$ of degree 0.
Our first goal is to show that \( g_1 \) is a constant function equal to 1. We start with two lemmas.

**Lemma 5.5.** Let \( I \) be a \( V[\lambda] \)-valued skew-symmetric function (for example \( I = I_\lambda \)). Then
\[
R^{(i,i+1)}(z_i - z_{i+1}) I = - P^{(i,i+1)} I(z_i \leftrightarrow z_{i+1}).
\]

**Proof.** Skew symmetry with respect to the transposition \( s_i \) implies the formula by direct calculation.

**Lemma 5.6.** \( K_1I_\lambda \) is a polynomial.

**Proof.** By Lemma 5.5 \( K_1I_\lambda = (-1)^{n-1} P^{(1,n)} \cdots P^{(1,2)} I_\lambda(z_2, \ldots, z_n, z_1) \).

We claim that the components of \( I_\lambda \) do not have a common polynomial factor of degree \( \geq 1 \). Indeed, if a polynomial, necessarily homogeneous, divides all components of \( I_\lambda \), then in the quasiclassical limit \( h = 0 \) a polynomial would divide all components of \( I^{h=0}_\lambda \). One sees from the explicit form of \( I^{h=0}_\lambda \) in the Remark in Section 3 that this is not the case.

This claim together with Lemma 5.6 implies that the denominator of \( g_1 \) is a constant function, and since \( g_1 \) is a rational function of degree 0, \( g_1 \) must be a constant function. The \( h = 0 \) limit of equation (5.1) then implies that \( g_1 = 1 \). In other words, we proved that \( I_\lambda \) satisfies the first \( qKZ \) equation.

Our next goal is to show that the first \( qKZ \) equation implies the others for skew-symmetric functions.

For brevity we will write \( p \) for \( (N+1)h \). Assume that the \( i \)-th \( qKZ \) equation holds. At the right end of the formula we can use Lemma 5.3 to obtain
\[
I(z_i \rightarrow z_i - p) = R^{(i,i-1)}(z_i - z_{i-1} - p) \cdots R^{(i,1)}(z_i - z_1 - p) R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}) I
\]
Applying \( -P^{(i,i+1)} \) to this equation, together with the iterated application of
\[
P^{(i,i+1)} R^{(i,m)}(u) = R^{(i+1,m)}(u) P^{(i,i+1)}
\]
we get
\[
-P^{(i,i+1)} I(z_i \rightarrow z_i - p) = R^{(i+1,i-1)}(z_i - z_{i-1} - p) \cdots R^{(i+1,1)}(z_i - z_1 - p) \times R^{(i+1,n)}(z_i - z_n) \cdots R^{(i+1,i+2)}(z_i - z_{i+2}) I(z_i \leftrightarrow z_{i+1}).
\]

To this equation we substitute \( z_i \leftrightarrow z_{i+1} \) and obtain
\[
-P^{(i,i+1)} I(z_i \rightarrow z_{i+1} - p, z_{i+1} \rightarrow z_i) = R^{(i+1,i-1)}(z_{i+1} - z_{i-1} - p) \cdots R^{(i+1,1)}(z_{i+1} - z_1 - p) \times R^{(i+1,n)}(z_{i+1} - z_n) \cdots R^{(i+1,i+2)}(z_{i+1} - z_{i+2}) I.
\]
We can use Lemma 5.5 to write the left hand side in the form
\[
R^{(i,i+1)}(z_i - z_{i+1} + p) I(z_i \rightarrow z_{i+1} - p).
\]

Then applying the \( R^{(i,i+1)}(z_i - z_i - p) \) operator to both sides results in the \( i + 1 \)-st \( qKZ \) equation. This finishes the proof of Theorem 5.4.
6. A $q$-Selberg type integral

According to the general principle in [MV00], if a KZ-type equation has a one-dimensional space of solutions, then the hypergeometric or $q$-hypergeometric integrals representing the solutions can be calculated explicitly, see demonstrations of that principle in [FSV03, RSV10, TV97, TV03, Var10, Var09, Var10]. In this section we give another example of this type.

In the rest of the paper we fix $N = 2$ and consider the bundle of the $\mathfrak{gl}_2$ $q$-conformal blocks at level 1 over $(\mathbb{C}^2)^{\otimes n}$. That bundle is of rank 1. The $q$KZ operators define a discrete flat connection on that bundle. In Section 3 we constructed a vector-valued polynomial on $\mathbb{C}^n$ that generates the space of flat sections of the discrete connection. Moreover, in Section 4 we identified that polynomial with the generating function of extended Joseph polynomials of orbital varieties associated with nilpotent $n \times n$-matrices. On the other hand in [TV97, MV98, MV99] flat sections of the same connection were constructed as multidimensional $q$-hypergeometric integrals. In this section we identify the $q$-hypergeometric flat sections constructed in [TV97, MV98, MV99] with the polynomial section constructed in Section 3 and obtain a $q$-Selberg type identity that a multidimensional $q$-hypergeometric integral equals a polynomial.

6.1. Quantized conformal blocks at level 1 and $q$KZ equations. First we recall some earlier definitions specialized for $\mathfrak{gl}_2$, and with the substitution $h = 1$. We consider the vector representation $\mathbb{C}^2$ of $\mathfrak{gl}_2$ with the standard basis $v_1, v_2$ and denote $V = (\mathbb{C}^2)^{\otimes n}$. The space $V$ has a basis of vectors $v_{i_1} \otimes \cdots \otimes v_{i_n}$, where $i_j \in \{1, 2\}$. Every such a sequence $(i_1, \ldots, i_n)$ defines a decomposition $L = (L_1, L_2)$ of $\{1, \ldots, n\}$ into disjoint subsets, $L_j = \{i \mid i_1 = j\}$. The basis vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$ is denoted by $v_L$. We have $V = \bigoplus_{\lambda=(\lambda_1, \lambda_2)} V[\lambda]$, where $\lambda_1 + \lambda_2 = n$, and $V[\lambda] = \{v \in V \mid e_{i_1} v = \lambda_1 v, i = 1, 2\}$. Denote by $\mathcal{L}_\lambda$ the set of all indices $L$ with $|L_j| = \lambda_j$, $j = 1, 2$. The vectors $\{v_L \mid L \in \mathcal{L}_\lambda\}$ form a basis of $V[\lambda]$.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define an operator $e(z) : V \to V$, by the formula

$$e(z) = \sum_{j=1}^{n} (e_{j,1}^z + \sum_{s=j+1}^{n} (e_{1,1} - e_{2,2})^{(s)} e_{1,2}^{(j)}) .$$

For $\lambda = (\lambda_1, \lambda_2)$ with $1 \geq \lambda_1 - \lambda_2 \geq 0$, we define the space of quantized conformal blocks at level 1 as

$$CB_\lambda(z) = \{v \in V[\lambda] \mid e_{1,2} v = 0, e(z)^{2+\lambda_2-\lambda_1} v = 0\}.$$  

Note that for any $n$, the space $V$ has a unique subspace $V[\lambda]$ with $1 \geq \lambda_1 - \lambda_2 \geq 0$. If $n = 2\ell$, then $\lambda = (\ell, \ell)$ and if $n = 2\ell + 1$, then $\lambda = (\ell + 1, \ell)$. That $\lambda$ will be called the middle weight. The middle weight $\lambda$ is determined by $n$ and we will denote the subspace $CB_\lambda(z)$ just by $CB(z)$.

Corollary 5.2 claims that for generic $z \in \mathbb{C}^n$, $\dim CB(z) = 1$.

For $i = 1, \ldots, n$, the KZ operators at level 1 on $V$ are

$$K^i(z_1, \ldots, z_n) = R^{(i,i)}(z_i - z_{i-1} - 3) \cdots R^{(i,1)}(z_i - z_1 - 3) \times$$

$$\times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}) .$$

The KZ operators define on $V$ a discrete flat connection,

$$K^i(z_1, \ldots, z_i - 3, \ldots, z_n) K^i(z_1, \ldots, z_n) = K^i(z_1, \ldots, z_j - 3, \ldots, z_n) K^i(z_1, \ldots, z_n) .$$
for all \(i, j\), see [FR92]. A \(V\)-valued function \(I(z)\) is a flat section if it satisfies the \(qKZ\) equations,

\[
I(z_1, \ldots, z_i - 3, \ldots, z_n) = K_i(z_1, \ldots, z_n)I(z_1, \ldots, z_i, \ldots, z_n), \quad i = 1, \ldots, n.
\]

The subbundle of conformal blocks at level 1 is invariant with respect to the \(qKZ\) connection,

\[
K_i(z_1, \ldots, z_n) : CB(z_1, \ldots, z_n) \to CB(z_1, \ldots, z_i - 3, \ldots, z_n)
\]

for all \(i\), see [MV98 MV99].

Recall the polynomial \(I_\lambda\) from Section \(3\), with the substitution \(h = 1\). In notation we will not indicate the \(h = 1\) substitution. Hence in the rest of the paper we have e.g.

\[
I_{(2,1)} = (z_1 - z_2 + 1)v_{121} + (z_3 - z_1 - 2)v_{121} + (z_2 - z_3 + 1)v_{211}.
\]

Results of the first part of the paper, in our present conventions, are as follows.

- \(I_\lambda\) is skew symmetric with respect to the \(S_n\)-action (2.1) with \(h = 1\).
- \(I_\lambda\) has degree \(k(\lambda) = \frac{1}{2} \lambda_1(\lambda_1 - 1) + \frac{1}{2} \lambda_2(\lambda_2 - 1)\) (the minimal degree skew symmetric polynomial).
- Given \(\lambda\), let \(L = (L_1, L_2)\) be the partition of \(\{1, \ldots, n\}\) with \(L_1 = \{1, \ldots, \lambda_1\}\). Then \(I_\lambda\) is normalized in such a way that its \(L\)-th coordinate is

\[
\prod_{1 \leq a < b \leq \lambda_1} (z_a - z_b + 1) \prod_{\lambda_1 < a < b \leq n} (z_a - z_b + 1).
\]

That polynomial \(I_\lambda\) will be called minimal.
- If \(\lambda = (\lambda_1, \lambda_2)\) is the middle weight, i.e. \(1 \geq \lambda_1 - \lambda_2 \geq 0\), then \(I_\lambda \in CB(z)\) and \(I_\lambda\) satisfies the \(qKZ\) equations (6.1).

6.2. An integral representation for quantized conformal blocks at level 1. In this section \(\lambda\) is the middle weight for \(n\), \(\lambda = (\ell, \ell)\) if \(n = 2\ell\) and \(\lambda = (\ell + 1, \ell)\) if \(n = 2\ell + 1\).

Define the master function

\[
\Phi(t_1, \ldots, t_\ell, z_1, \ldots, z_n) = \prod_{n \geq j > i \geq 1} \frac{\Gamma((z_j - z_i + 1)/3)}{\Gamma((z_j - z_i - 1)/3)} \prod_{\ell \geq j > i \geq 1} \frac{\Gamma((t_j - t_i + 1)/3)}{\Gamma((t_j - t_i - 1)/3)} \times \prod_{i=1}^{n} \prod_{j=1}^{\ell} \frac{\Gamma((z_i - t_j - 1)/3)}{\Gamma((z_i - t_j)/3)}.
\]

For \(L = (L_1, L_2) \in \mathcal{L}_\lambda\) with \(L_2 = \{i_1 < \cdots < i_\ell\}\), define the function \(w_L(t_1, \ldots, t_\ell, z_1, \ldots, z_n)\) by the formula

\[
w_L = \sum_{\sigma \in S_\ell} \prod_{j=1}^{\ell} \frac{1}{t_{\sigma_j} - z_{i_j}} \prod_{m=1}^{i_j-1} \frac{t_{\sigma_j} - z_m + 1}{t_{\sigma_j} - z_m} \prod_{1 \leq i < j < \ell, \sigma(j) > \sigma(i)} \frac{t_{\sigma_i} - t_{\sigma_j} + 1}{t_{\sigma_i} - t_{\sigma_j} - 1}.
\]

Define the \(V[\lambda]\)-valued weight function by the formula

\[
w(t_1, \ldots, t_\ell, z_1, \ldots, z_n) = \sum_{L \in \mathcal{L}_\lambda} w_L(t_1, \ldots, t_\ell, z_1, \ldots, z_n)v_L.
\]
Define the *trigonometric weight function* \( W(t_1, \ldots, t_\ell, z_1, \ldots, z_n) \) by the formula

\[
W = \pi^\ell \prod_{j=1}^{\ell} \frac{\sin((z_2 j - z_{2j-1} + 1)/3)}{\sin((t_j - z_{2j-1})/3) \sin((t_j - z_{2j})/3)} \prod_{m=1}^{2j-2} \frac{\sin((t_j - z_m + 1)/3)}{\sin((t_j - z_m)/3)}.
\]

Using the formula \( \Gamma(1 - x) \Gamma(x) = \frac{\pi}{\sin(\pi x)} \), we can write

\[
\Phi W = \prod_{n \geq j' \geq i \geq 1} \frac{\Gamma((z_j - z_i + 1)/3)}{\Gamma((z_j - z_i - 1)/3)} \prod_{\ell(j' > i \geq 1} \frac{\Gamma((t_j - t_i + 1)/3)}{\Gamma((t_j - t_i - 1)/3)} \times
\]

\[
\times \prod_{j=1}^{\ell} \prod_{m=0}^{1} \Gamma((z_{2j-m} - t_j - 1)/3) \Gamma(1 - (z_{2j-m} - t_j)/3) \times
\]

\[
\times \prod_{j=1}^{\ell} \prod_{i=1}^{2j-2} \frac{\Gamma(1 - (z_i - t_j)/3)}{\Gamma(1 - (z_i - t_j - 1)/3)} \prod_{i=2j+1}^{n} \frac{\Gamma((z_i - t_j - 1)/3)}{\Gamma((z_j - t_j)/3)} \times
\]

\[
\times \pi^{-\ell} \prod_{j=1}^{\ell} \sin((z_{2j} - z_{2j-1} + 1)/3).
\]

The function \( \Phi W w \) is a meromorphic function of \( t_1, \ldots, t_\ell \) with first order poles at the hyperplanes

\[
t_i - t_j = 1 + 3s, \quad i < j, \quad s = 1, 2, \ldots
\]
\[
t_j - z_m = -1 + 3s, \quad m \leq 2j, \quad s = 0, 1, \ldots
\]
\[
t_j - z_m = -3s, \quad m \geq 2j - 1, \quad s = 0, 1, \ldots
\]

Define an oriented unbounded one-chain \( C \subset \mathbb{C} \). It consists of the vertical line \(-\ell/2 + \sqrt{1-k}R\), oriented from \(-\ell/2 - \sqrt{1-k} \infty\) to \(-\ell/2 + \sqrt{1-k} \infty\), and \(2n\) circles \( C_1, \ldots, C_{2n} \) of radius \( \ell/2 \). The circle \( C_j \) for \( 1 \leq j \leq n \) is centered at \( j\sqrt{1-k} \) and oriented counterclockwise, while the circle \( C_j \) for \( n < j \leq 2n \) is centered at \(-1 + j\sqrt{1-k} \) and oriented clockwise. Define the integration cycle

\[
C_{\ell} = \{(t_1, \ldots, t_\ell) \in \mathbb{C}^\ell \mid t_i \in C_i \}.
\]

For \((z_1, \ldots, z_n) \in \mathbb{C}^n\) such that \(|z_j - j\sqrt{1-k}| < \frac{1}{4}\), define a \(V[\lambda]\)-valued *q-hypergeometric integral* by the formula

\[
\Psi_\lambda(z_1, \ldots, z_n) = \int_{C_{\ell}} \Phi(t, z) w(t, z) W(t, z) \, dt_1 \ldots dt_\ell.
\]

**Theorem 6.1 ([TV97]).** The function \( \Psi_\lambda(z_1, \ldots, z_n) \) is well-defined and extends to a meromorphic function on \( \mathbb{C}^n \). Moreover, the function \( \Psi_\lambda(z_1, \ldots, z_n) \) is a solution of the \(qKZ\) equations (6.1).

**Theorem 6.2 ([MV98]).** For generic \( z \in \mathbb{C}^n \), we have \( \Psi_\lambda(z_1, \ldots, z_n) \in CB(z) \).

**Theorem 6.3.** Let \( n = 2\ell, \lambda = (\ell, \ell) \) or \( n = 2\ell + 1, \lambda = (\ell + 1, \ell) \). Let \( I_\lambda(z_1, \ldots, z_n) \) be the minimal skew-symmetric \( V[\lambda]\)-valued polynomial as above. Then the \(q\)-hypergeometric
integral $\Psi_\lambda(z_1, \ldots, z_n)$ equals the polynomial $c_n I_\lambda(z_1, \ldots, z_n)$, where

\begin{equation}
(6.2) \quad c_2 = 2\pi \sqrt{-1} \frac{\Gamma(2/3) \Gamma(-1/3)}{\Gamma(1/3)}, \quad c_{2\ell} = 3^{-\ell(\ell-1)} c_2^\ell, \quad c_{2\ell+1} = (-1)^\ell 3^{-\ell^2} c_2^\ell.
\end{equation}

By Section 4, the polynomial $I_\lambda$ is the generating function of the extended Joseph polynomials of the orbital varieties associated with nilpotent $n \times n$-matrices. Hence, Theorem 6.3 gives an integral representation for those extended Joseph polynomials.

**Proof.** For $n = 2$ the middle weight is $(1, 1)$ and we have $I_{(1,1)} = v_{12} - v_{21}$, $\Psi_{(1,1)} = c_2 I_{(1,1)}$, where

\[
\begin{aligned}
c_2 &= \pi^{-1} \sin(\pi(z_{2j} - z_{2j-1} + 1)/3) \frac{\Gamma((z_{2} - z_{1} + 1)/3)}{\Gamma((z_{2} - z_{1} - 1)/3)} \\
&\quad \times \int c_2 \int_{m=0}^{1} \frac{\Gamma((z_{2j-m} - t_j - 1)/3) \Gamma(1 - (z_{2j-m} - t_j)/3)}{t - z_1} dt \\
&= 2\pi \sqrt{-1} \frac{\Gamma(2/3) \Gamma(-1/3)}{\Gamma(1/3)},
\end{aligned}
\]

see (1.1).

For arbitrary $n$ we have $\Psi_\lambda(z_1, \ldots, z_n) = c_n(z_1, \ldots, z_n) I_\lambda(z_1, \ldots, z_n)$, where $c_n(z_1, \ldots, z_n)$ is a scalar function 3-periodic with respect to every variable. Indeed, both $\Psi_\lambda(z_1, \ldots, z_n)$ and $I_\lambda(z_1, \ldots, z_n)$ are quantized conformal blocks at level 1 and both satisfy the $q$KZ equations. To check that $c_n$ is given by (6.2) we consider the asymptotic zone:

(i) $|z_{2i} - z_{2i-1}| \leq 1$, $|\text{Im}(z_{2i})| \leq 1$ for $i = 1, \ldots, \ell$,

(ii) $\text{Re}(z_{2i+2} - z_{2i}) \gg 1$ for $i = 1, \ldots, \ell - 1$ and $\text{Re}(z_n - z_{n-2}) \gg 1$ if $n$ is odd,

use the Stirling formula for the Gamma functions,

\[
\frac{\Gamma((x + \alpha)/p)}{\Gamma((x + \beta)/p)} = (x/p)^{\alpha - \beta}(1 + o(1)), \quad |\arg(x/p)| < \pi,
\]

and similarly to the proof of Theorem 6.7 in [TV97] observe that

\[
c_{2\ell} = 3^{-\ell(\ell-1)} c^\ell_2 (1 + o(1)), \quad c_{2\ell+1} = (-1)^\ell 3^{-\ell^2} c^\ell_2 (1 + o(1)).
\]

This proves the theorem. \hfill \Box

7. **An alternative integral for $N = 2$**

Finally, we formulate an integral representation in the two-row case $\lambda = (n - p, p)$, distinct from that of the previous paragraph, and which generalizes that of [RSZ07] (which was the case $n$ odd, $n = 2p + 1$).

Expanding $I_\lambda = \sum_L I_L v_L$, the multi-indices that contribute to the sum have $n - p$ 1s and $p$ 2s. Let us parameterize them as follows: denote $L(a)$ the multi-index whose 2s are located at indices $a = (a_1 < \cdots < a_p)$.

**Theorem 7.1.**

\[
I_{L(a)} = (-1)^{p(n-p+1)} p \prod_{1 \leq i < j \leq n} (h+z_i-z_j) \int \prod_{k=1}^{p} dw_k \prod_{1 \leq k < \ell \leq p} (w_\ell - w_k)(h + w_k - w_\ell) \int \prod_{k=1}^{p} \frac{\prod_{1 \leq k < \ell \leq p} (w_\ell - w_k)(h + w_k - w_\ell)}{\prod_{1 \leq i \leq p} (w_k - z_i) \prod_{j=1}^{p} (h + w_k - z_j)}.
\]
The integration cycle is the product of $p$ identical 1-dimensional cycles. The 1-dimensional cycle is any contour that surrounds once counterclockwise each of the $z_1, \ldots, z_n$ but none of the $z_1 - h, \ldots, z_n - h$.

Note that the integrals have no pole at infinity (the integrand behaves as $w_k^{2(p-1)-(n+1)}$ as $w_k \to \infty$) so we may as well consider that the contour surrounds clockwise the $z_1 - h, \ldots, z_n - h$ but none of the $z_1, \ldots, z_n$.

**Proof.** We are going to apply Lemma 3.3. Denote by $\hat{I}_{L(a)}$ the r.h.s. of the formula above.

First one needs to check that $\hat{I}_{L(a)}$ is a polynomial in $z_1, \ldots, z_n, h$. This is a routine calculation based on the application of the residue formula for the $w_k$ integrals and the check that would-be poles in the variables $z_i$ have vanishing residue (see a similar calculation in [FZJ08]; since the formula is homogeneous in $z_1, \ldots, z_n, h$ this leaves only a power of $h$ in the denominator which is cancelled by the factor $h^p$. The degree of $\hat{I}_{L(a)}$ is then (as a homogeneous polynomial in $z_1, \ldots, z_n, h$) $p + n(n - 1)/2 + 2p(p - 1)/2 - p(n + 1) = p(p - 1)/2 + (n - p)(n - p - 1)/2 = k(\lambda)$.

Next, we check that $\hat{I}_\lambda = \sum_L \hat{I}_{Lv_L}$ is skew-symmetric by use of Lemma 3.1. Fixing $i = 1, \ldots, n - 1$, there are four possibilities:

- If $L_i = L_{i+1} = 1$, the integrand (including the prefactor in front of the integral) is $h + z_i - z_{i+1}$ times a symmetric function of $z_i, z_{i+1}$. This implies that $\hat{s}_i I_L = -I_L$.
- If $L_i = L_{i+1} = 2$, say $a_k = i, a_{k+1} = i + 1$, then the integrand minus itself with $z_i \leftrightarrow z_{i+1}$ is skew-symmetric in $w_k, w_{k+1}$ and therefore its integral is zero. This implies again that $\hat{s}_i I_L = -I_L$.
- If $L_i = 2, L_{i+1} = 1$, say $a_k = i, a_{k+1} > i + 1$, then the integrand is $\frac{h + z_i - z_{i+1}}{w_k - z_i}$ times a symmetric function of $z_i, z_{i+1}$. Applying $\hat{s}_i$ results in $-\frac{(h + w_k - z_i)(h + z_i - z_{i+1})}{(w_k - z_i)(w_k - z_{i+1})}$ times the same function, which is nothing but minus the integrand with $a_k \to i + 1$, which is precisely $L \to s_i(L)$. That is, $\hat{s}_i I_L = -I_{s_i(L)}$.
- The case $L_i = 1, L_{i+1} = 2$ is treated similarly.

Therefore all the hypotheses of Lemma 3.3 are satisfied, and $\hat{I}_\lambda$ is proportional to $I_\lambda$. In order to fix the normalization, we consider the case $a = (n - p + 1, \ldots, n)$, i.e., $L(a) = L_0$. Then the integrals can be computed one by one as follows. The integral over $w_p$ has only one pole outside the contour, at $z_n - h$. Next, the integral over $w_{p-1}$ has two poles, at $z_n - h$ and $z_{n-1} - h$, but the first one is cancelled by the factor $w_p - w_{p-1}$ in the numerator (since we have taken the residue at $w_p = z_n - h$). So there is only one contribution, the residue at $w_{p-1} = z_{n-1} - h$; and so on. In the end, evaluating the residues at $w_k = z_{k+n-p} - h$ results in: $I_{L_0} = \prod_{1 \leq i < j \leq n-p} (h + z_i - z_j) \prod_{n-p+1 \leq i < j \leq n} (h + z_i - z_j)$, which coincides with $I_{L_0}$, so that $\hat{I}_\lambda = I_\lambda$. \qed

**References**


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