EQUIVARIANT AND INVARIANT THEORY OF NETS OF CONICS

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1. Introduction

Two parameter families of plane conics are called nets of conics. There is a natural group action on the vector space of nets of conics, namely the product of the group reparametrizing the underlying plane, and the group reparametrizing the parameter space of the family. In Section ?? we develop the equivariant cohomology theory of nets of conics with respect to this action.

We give several applications of the calculations. In Section ??, using a theorem of Kazarian, we determine Thom polynomials of contact singularities of type (3,3).

In Section ?? we develop the invariant theory of nets of conics. While this theory is mostly known, we emphasize that the equivariant point of view gives a conceptional way to approach invariant theory. This section however is relatively independent of the rest of the paper, it does not use cohomology theory. We give a formula for the degree 6 invariant (already known to Salmon and Sylvester) in terms of the Plücker coordinates as a pull-back of a degree 2 invariant (??) of the Plücker space. The formula (??) for this degree 2 invariant appears to be new.

In Section ?? we describe the hierarchy of the orbits of the nets of conics using the cohomological data obtained in Section ??. The method of *incidences* is quite general and may be applied to other representations with small GIT quotients.

In Section ?? we show how enumerative problems—in particular the intersection multiplicities of the determinant map from nets of conics to plane cubics—can be solved studying the equivariant cohomology classes of the orbits.

Throughout the paper we work in the complex algebraic category, hence in particular, $\mathrm{GL}(U)$ means the group of complex linear transformations of the complex vector space U, and GL_n denotes $GL(\mathbb{C}^n)$. Cohomology will be considered with integer coefficients.

The authors are grateful to M. Kazarian for several useful discussions on iterated residue forms of Thom polynomials and to C. T. C. Wall for very valuable comments on nets of conics. Additionally the authors would like to thank I. Dolgachev, J. Chipalkatti and P. Frenkel for useful conversations on the topics in this paper.

2. Classification of orbits of nets and their equivariant classes

2.1. **Orbits of nets of conics.** Let S^2U denote the second symmetric power of the vector space (or representation) U. Consider the vector spaces $U = \mathbb{C}^3$ and $V = \mathbb{C}^3$. The main object of this paper is the $\mathrm{GL}(U) \times \mathrm{GL}(V)$ representation $\mathbf{Noc} = \mathrm{Hom}(S^2U, V)$. Our motivation of

The first named author is supported by OTKA 81203. The second author is supported by the OTKA grants 72537 and 81203. The third author is supported by NSA grant CON:H98230-10-1-0171.

studying this representation is the algebraic interpretation in Section ??. However, it has a natural geometric interpretation, as follows.

Through the natural isomorphism $\mathrm{Hom}(S^2U, V) = \mathrm{Hom}(V^*, S^2U^*)$, elements of this vector space are families of homogeneous degree 2 polynomials on U parameterized by the vector space V^*. Lines in S^2U^* determine conics in U, hence elements of $\mathrm{Hom}(V^*, S^2U^*)$ are 2-parameter families (*nets*) of plane conics (**Noc** stands for *nets of conics*). The $\mathrm{GL}(U)$ action reparametrizes the underlying plane $\mathbf{P}(U)$, and the $\mathrm{GL}(V)$ action reparametrizes the parameter space V^*.

There is a natural stratification $\Sigma^2\cup\Sigma^1\cup\Sigma^0$ of $\mathbf{Noc}-\{0\}$, according to corank. Geometrically the strata correspond to conics, pencils of conics (ie. 1-parameter families of conics), and (proper) nets of conics, respectively. The orbit structure of conics and pencils of conics is widely known. The classification of orbits of proper nets of conics is given in [?] for the codimension>1 cases and in [?] for the family of codimension 1 orbits.

The list of codimension>1 orbits is given in the first 3 columns of Table ?? with the following conventions. Column 1 is the name of the orbit, column 2 is its codimension, and column 3 names three plane conics that span the image of $\phi \in \mathrm{Hom}(V^*, S^2U^*)$. Here we used the letters x, y, z for the coordinates on U.

2.2. **Equivariant classes.** G-invariant subvarieties (eg. orbit closures) represent cohomology classes in the equivariant cohomology ring of a G-representation. We want to determine the equivariant classes $[\overline{\eta}] \in H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)}(\mathbf{Noc}) \cong \mathbb{Z}[u_1,u_2,u_3,v_1,v_2,v_3]$ for the orbits $\eta \subset \mathbf{Noc}$. Here u_1,u_2,u_3 and v_1,v_2,v_3 denote the Chern classes of the groups $\mathrm{GL}(U)$ and $GL(V)$ respectively. These classes contain a lot of geometric information as we will show later.

For the codimension>1 orbits we use the method of *restriction equations* of [?, Thm.2.4], see also [?, Sect.3], so we need the symmetries of these orbits. More precisely, we need only a maximal torus of their stabilizer subgroups. These elementary calculations can be reduced to the level of Lie algebras. The results are summarized in the "symmetry" column of Table ??.

To explain the notation consider the orbit C represented by the net of conics $(y^2{+}2xz, 2yz, -x^2)$. The pair of matrices

$$\left(\begin{pmatrix} a^2 & 0 & 0\\ 0 & ab & 0\\ 0 & 0 & b^2\end{pmatrix}, \begin{pmatrix} a^2b^2 & 0 & 0\\ 0 & ab^3 & 0\\ 0 & 0 & a^4\end{pmatrix}\right) \in \mathrm{GL}(U)\times\mathrm{GL}(V), \qquad a,b\in\mathrm{GL}_1 \tag{1}$$

stabilize this net of conic. Since the codimension of C is 2, the dimension of the stabilizer subgroup has to be 2 as well ($\dim G = \dim V = 18$), so we determined the maximal torus. This is the data that is encoded as $(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, \alpha+3\beta, 4\alpha)$ in Table ??.

Theorem 2.1. *Consider the* $\mathrm{GL}(U)\times\mathrm{GL}(V)$ *representation* **Noc**. *The Theorem of Restriction Equations* [?, Thm. 3.5] *determines all the* $\mathrm{GL}(U)\times\mathrm{GL}(V)$ *equivariant classes of the codimension*>1 *orbit closures, eg. we have*

- $[\overline{C}] = 8(v_1-2u_1)^2$,
- $[\overline{D}] = -3u_2+3v_2-16u_1v_1+3v_1^2+17u_1^2$,
- $[\overline{D^*}] = 12u_2-3v_2-20u_1v_1+6v_1^2+16u_1^2$,
- $[\overline{E}] = 3u_3+3v_3-3u_1u_2+u_2v_1-6u_1v_1^2+13u_1^2v_1-2u_1v_2-8u_1^3+v_1^3$

TABLE 1. Codimension> 1 orbits and symmetries

	cd	representative	symmetry	Poincaré	δ
Σ^0					
C	2	$y^2+2xz, 2yz, -x^2$	$(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, \alpha+3\beta, 4\alpha)$	1,1	ν
D	2	x^2, y^2, z^2+2xy	$(2\alpha, 2\beta, \alpha+\beta), (4\alpha, 4\beta, 2\alpha+2\beta)$	1,2	θ
D^*	2	$2xz, 2yz, z^2+2xy$	$(2\alpha, 2\beta, \alpha+\beta), (3\alpha+\beta, \alpha+3\beta, 2\alpha+2\beta)$	1,2	θ
E	3	x^2, y^2, z^2	$(\alpha, \beta, \gamma), (2\alpha, 2\beta, 2\gamma)$	1,2,3	A
E^*	3	$2xy, 2yz, 2zx$	$(\alpha, \beta, \gamma), (\alpha+\beta, \beta+\gamma, \gamma+\alpha)$	1,2,3	A
F	3	$x^2+y^2, 2xy, 2yz$	$(\alpha, \alpha, \beta), (2\alpha, 2\alpha, \alpha+\beta)$	1,1	$\neq$
F^*	3	x^2+y^2, xz, z^2	$(\alpha, \alpha, \beta), (2\alpha, \alpha+\beta, 2\beta)$	1,1	Ω
G	4	x^2, y^2, yz	$(\alpha, \beta, \gamma), (2\alpha, 2\beta, \beta+\gamma)$	1,1,1	$\neq$
G^*	4	xy, xz, z^2	$(\alpha, \beta, \gamma), (\alpha+\beta, \alpha+\gamma, 2\gamma)$	1,1,1	$\neq$
H	5	$x^2, 2xy, y^2+2xz$	$(2\alpha, \alpha+\beta, 2\beta), (4\alpha, 3\alpha+\beta, 2\alpha+2\beta)$	1,1	Ξ
I	7	x^2, xy, y^2	$(\alpha, \beta, \gamma), (2\alpha, \alpha+\beta, 2\beta)$	1,1,2	0
I^*	7	xz, yz, z^2	$(\alpha, \beta, \gamma), (\alpha+\gamma, \beta+\gamma, 2\gamma)$	1,1,2	0
Σ^1					
(1^4)	4	$x^2-xz, y^2-yz, 0$	$(\alpha, \alpha, \alpha), (2\alpha, 2\alpha, \beta)$	1,1	Ж
(21^2)	5	$xy, xz+yz, 0$	$(\alpha, \alpha, \beta), (2\alpha, \alpha+\beta, \gamma)$	1,1,1	$\neq$
(31)	6	$xz, x^2-yz, 0$	$(\alpha+\beta, 2\alpha, 2\beta), (\alpha+3\beta, 2\alpha+2\beta, \gamma)$	1,1,1	Ξ
(22)	6	$x^2, yz, 0$	$(\alpha, \beta, \gamma), (2\alpha, \beta+\gamma, \delta)$	1,1,2	$\neq$
(4)	7	$xz+y^2, x^2, 0$	$(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, 4\alpha, \gamma)$	1,1,1	Ξ
K	8	$y^2, z^2, 0$	$(\beta, \alpha, \gamma), (2\alpha, 2\gamma, \delta)$	1,1,1,2	0
L	8	$xy, xz, 0$	$(\alpha, \beta, \gamma), (\alpha+\beta, \alpha+\gamma, \delta)$	1,1,1,2	0
M	9	$yz, y^2, 0$	$(\alpha, \beta, \gamma), (\beta+\gamma, 2\beta, \delta)$	1,1,1,1	0
Σ^2					
S	10	$xy-z^2, 0, 0$	$(2\alpha, 2\beta, \alpha+\beta), (2\alpha+2\beta, \gamma, \delta)$	1,1,2,2	0
PL	11	$xy, 0, 0$	$(\alpha, \beta, \gamma), (\alpha+\beta, \delta, \epsilon)$	1,1,1,2,2	0
DL	13	$x^2, 0, 0$	$(\alpha, \beta, \gamma), (2\alpha, \delta, \epsilon)$	1,1,1,2,2	0
0	18	$0, 0, 0$	$(\alpha, \beta, \gamma), (\delta, \epsilon, \kappa)$	1,1,2,2,3,3	0

- $[\overline{E^*}] = -24u_3 + 3v_3 - 24u_1u_2 + 16u_2v_1 - 16u_1v_1^2 + 20u_1^2v_1 - 6v_1v_2 + 10u_1v_2 - 8u_1^3 + 4v_1^3$
- $[\overline{F}] = 2(v_1 - 2u_1)(6u_1^2 - 4u_1v_1 - 6u_2 + 3v_2)$,
- $[\overline{F^*}] = 2(v_1 - 2u_1)(5u_1^2 - 8u_1v_1 + 9u_2 - 3v_2 + 3v_1^2)$.

Proof. The proof does not follow from any general principle we are aware of, it is just an experimental fact. The symmetry data of the table put constraints on the classes $[\overline{\eta}]$. One can write down all these constraints for each codimension> 1 orbit η. A computer program shows that for each codimension> 1 orbit there is only one equivariant class in $H^*\big(B(\mathrm{GL}(U)\times\mathrm{GL}(V))\big)$ satisfying the constraints. □

For the family of codimension 1 orbits we look at the Wall-DuPlessis classification [?] from an equivariant point of view.

The affine plane $N_C = \{(y^2 + 2xz, 2yz, -x^2 + 2g(xz - y^2) + cz^2) : c, g \in C\}$ is normal to the orbit C at the point $(y^2+2xz, 2yz, -x^2)$. This plane is invariant under the action of the complex 2-torus T_C of (**??**). The T_C action on N_C has weights $2\alpha - 2\beta$ and $4\alpha - 4\beta$, corresponding to the weight vectors $(0, 0, xz - y^2)$ and $(0, 0, z^2)$. Hence, the orbits of T_C on N_C correspond to the parabolas with $\mu = (c : g^2) \in \mathbf{P}^1$ constant.

According to [**?**] these parabolas are exactly the intersections of the codimension 1 **Noc**-orbits (what we will denote by A_μ) with the normal slice N_C. We will refer to a **Noc**-orbit representative lying in T_C as a *c-g-form*. Recall the following *Incidence Theorem*.

Theorem 2.2. [**?**] *Consider a Lie group G acting on a vector space V complex linearly. For $v \in V$ let G_v denote the stabilizer subgroup of G. Let S be a subgroup of G_v and N_v an S-invariant normal slice to the orbit Gv at v. Suppose that $\eta \subset V$ is a G-invariant subvariety. Then*

$$[\eta \subset V]_S = [(\eta \cap N_v) \subset N_v]_S.$$

Theorem 2.3. *The equivariant classes of the A_μ orbits are $4(v_1-2u_1)$ for $\mu \neq \infty$ and $2(v_1-2u_1)$ for $\mu = \infty$.*

Proof. If $\mu \neq \infty$, then the class $[(\mu g^2 = c) \subset N_C]$ is equal to the weight of the $g = 0$ direction. Hence we have $[(\mu g^2 = c) \subset N_C] = 4\alpha - 4\beta$. For the curve $g = 0$ we have $[(g = 0) \subset N_C] = 2\alpha - 2\beta$. For the restriction homomorphism $r : H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)} \to H^*_{T_C} = \mathbb{Z}[\alpha, \beta]$ we have $r(u_1) = 2\alpha+(\alpha+\beta)+2\beta = 3\alpha+3\beta$ and $r(v_1) = (2\alpha+2\beta)+(\alpha+3\beta)+4\alpha = 7\alpha+5\beta$ by (**??**). Hence, if $[A_\mu] = Au_1+Bv_1$ $(\mu \neq \infty)$, then according to Theorem **??**, we have $A(3\alpha+3\beta)+B(7\alpha+5\beta) = 4\alpha - 4\beta$. The only solution is $A = -8, B = 4$. For $\mu = \infty$ the calculation is similar. □

Remark 2.4. From the equivariant classes of (cone) varieties one can calculate their degrees, see eg. [**?**, Sec. 6] and Section **??**. Therefore, Theorem **??** implies that the degree of the hypersurfaces given by the closures of the A_μ orbits is $4(1+1+1-2(0+0+0)) = 12$ in general, and 6 for $\mu = \infty$. This implies that the ring of invariants $R(\textbf{Noc})$ on **Noc** is generated by a degree 6 and a degree 12 polynomial. We will give a full description of $R(\textbf{Noc})$ in Section **??**.

3. Thom series of contact singularities corresponding to nets of conics

3.1. **Thom series of contact singularities.** Let Q be a finite dimensional commutative local $\mathbb{C}$-algebra. The corresponding contact singularity of relative codimension ℓ is the set of germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+\ell}, 0)$ (for any n) whose associated algebra $\mathbb{C}[x_1, x_2, \ldots]/(f_1, f_2, \ldots)$ is isomorphic to Q. The global behavior of this singularity is governed by an associated polynomial, the Thom polynomial $\mathrm{tp}_Q(\ell) \in \mathbb{Z}[c_1, c_2, \ldots]$. For the precise meaning of $\mathrm{tp}_Q(\ell)$ see for example [**?**] and references therein. One of the main goals of this paper is to determine the Thom polynomials $\mathrm{tp}_Q(\ell)$ for Q's that have close relations with nets of conics.

3.2. **The Kazarian iterated residue formula.** Denote the maximal ideal of Q by $\mathfrak{m}$. Let $d_k = \dim \mathfrak{m}^k / \mathfrak{m}^{k+1}$, and call $(d_1, d_2, \ldots, d_r)$ the dimension vector if $d_{>r} = 0$. We have $\sum d_i = \dim \mathfrak{m} = \dim Q - 1$, that we denote by μ. Define the vectors

$$\begin{aligned} w &= (\underbrace{1,\ldots,1}_{d_1},\underbrace{2,\ldots,2}_{d_2},\ldots,\underbrace{r,\ldots,r}_{d_r}),\\ e &= (\underbrace{d_1-1,d_1-2,\ldots,1,0}_{d_1},\underbrace{d_2-1,d_2-2,\ldots,1,0}_{d_2},\ldots,\underbrace{d_r-1,d_r-2,\ldots,1,0}_{d_r}). \end{aligned}$$

Theorem 3.1 ([?]). *There is a polynomial $k_Q(z_1,\ldots,z_\mu)$ (discussed below in Section **??**) such that*

$$\mathrm{tp}_Q(\ell)=\Phi\left(\frac{k_Q(z_1,\ldots,z_\mu)\prod_{i=1}^{\mu} z_i^{e(i)+\ell+1}\prod_{1\le i<j\le\mu}(z_j-z_i)}{\prod_{\substack{1\le i\le j<k\le\mu\\ w(i)+w(j)\le w(k)}}(z_k-z_i-z_j)}\right), \tag{2}$$

where the operation $\Phi(f(z))$ means expanding $f(z)$ as a Laurent series in the $z_1\ll z_2\ll\ldots\ll z_\mu$ region, and then changing the monomial $z_1^{p(1)}z_2^{p(2)}\ldots z_\mu^{p(\mu)}$ to $c_{p(1)}c_{p(2)}\ldots c_{p(\mu)}$ in each term (with the usual Chern-class convention of $c_0=1$, $c_{<0}=0$).

To illustrate the operation Φ here is an example (actually, $\mathrm{tp}_{\mathbb{C}[x]/(x^3)}(1)$)

$$\Phi\left(\frac{z_1^2z_2^2(z_2-z_1)}{z_2-2z_1}\right)=\Phi\left(z_1^2z_2^2(1-\frac{z_1}{z_2})(1+\frac{2z_1}{z_2}+\frac{4z_1^2}{z_2^2}+\ldots)\right)=$$

$$\Phi\left(z_1^2z_2^2+z_1^3z_2+2z_1^4z_2^0+4z_1^5z_2^{-1}+\ldots\right)=c_2^2+c_3c_1+2c_4c_0+4c_5c_{-1}+\ldots=c_2^2+c_3c_1+2c_4.$$

We call (**??**) an iterated residue formula, because the operation Φ can be formally written as an iterated residue operation (iterated residue forms of Thom polynomials were first discovered in [?]).

3.3. **The polynomial k_Q is an equivariant class in the vector space of non-associative algebra structures.** The polynomial k_Q in Theorem **??** is not unique. However there is a choice for it with geometric meaning. Let us fix complex vector spaces

$$N=N_1\ge N_2\ge\ldots\ge N_r\ge N_{r+1}=0$$

with $\dim N_i/N_{i+1}=d_i$. Let Alg_d be the vector space of not necessarily associative but commutative, filtration-preserving products on N, that is

$$\mathrm{Alg}_d=\{\alpha\in\mathrm{Hom}(S^2N,N):\alpha(N_i\otimes N_j)\subset N_{i+j}\}.$$

Consider the subset

$$\eta_Q=\{\alpha:\alpha\text{ associative, and the obtained algebra }(N,\alpha)\text{ is isomorphic to }\mathfrak{m}\}\subset\mathrm{Alg}_d. \tag{3}$$

After picking a basis on N which respects the filtration the group $\mathrm{GL}_{d_1}\times\mathrm{GL}_{d_2}\times\ldots\times\mathrm{GL}_{d_r}$, with maximal torus T^μ, acts on N. Let the Chern roots of T^μ be called z_i (with $z_1,\ldots,z_{d_1}$ corresponding to GL_{d_1}, etc). The subset η_Q is invariant under the induced action on Alg_d. Observe that the denominator in (**??**) is the Euler class of this action on Alg_d. The final part of Kazarian's Theorem **??** is that for k_Q one can choose the torus equivariant cohomology class represented by the closure of η_Q:

$$k_Q = [\overline{\eta_Q}] \in H^*_{T^\mu}(\mathrm{Alg}_d) = \mathbb{Z}[z_1, \dots, z_\mu].$$

Remark 3.2. Observe that we detected the set η_Q in the vector space Alg_d where d is the "canonical grading" of Q, that is, $d_k = \dim \mathfrak{m}^k / \mathfrak{m}^{k+1}$. However, definition (**??**) of η_Q makes sense for any other d with $\sum d_i = \dim Q - 1$. If the corresponding η_Q is non-empty for such an "exotic grading" d, then $k_Q = m_Q[\overline{\eta_Q}]$ where m_Q is an extra z-monomial factor, which can be calculated fairly easily.

3.4. **Algebras with dimension vector** $(3,3)$ **and nets of conics.** For the dimension vector $d = (3,3)$ the space of algebra-structures is $\mathrm{Alg}_{(3,3)} \cong \mathrm{Hom}(S^2\mathbb{C}^3, \mathbb{C}^3)$ with the action of $\mathrm{GL}_3 \times \mathrm{GL}_3$. That is, this representation is the same as **Noc** of Section **??**.

In this point of view, orbits of **Noc** correspond to isomorphism classes of local algebras of dimension $6+1$, that possess the (canonical or exotic) grading $(3,3)$ (cf. Remark **??**). The dimension vector $(3,3)$ is the canonical grading of the algebra, if and only if the corresponding homomorphism in $\mathrm{Hom}(S^2\mathbb{C}^3, \mathbb{C}^3)$ has full rank, i.e. it is a Σ^0 net of conics (see Section **??**).

3.5. **Examples of Thom series and Thom polynomials.** Let $Q = \mathbb{C}[[x,y,z]] \,/\, (y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2)+\mathfrak{m}^3$. The results of Section **??**, **??** and Theorem **??** imply

$$\mathrm{tp}_Q(1) = \Phi\left(\frac{(-8(z_1+z_2+z_3)+4(z_4+z_5+z_6))z_1^4z_2^3z_3^2z_4^4z_5^3z_6^2\prod_{1\le i<j\le 6}(z_j-z_i)}{\prod_{k=4}^6\prod_{1\le i\le j\le 3}(z_k-z_i-z_j)}\right) = \tag{4}$$

$$8\Delta_{(544111)} + 4\Delta_{(444211)} + 16\Delta_{(844)} + 20\Delta_{(6442)} + 32\Delta_{(64411)} + 120\Delta_{(6541)} + 160\Delta_{(655)} +$$
$$+16\Delta_{(54421)} + 32\Delta_{(55411)} + 40\Delta_{(5542)} + 80\Delta_{(5551)} + 80\Delta_{(664)} + 40\Delta_{(7441)} + 112\Delta_{(754)},$$

if $g \neq 0$ and half of this expression for $g=0$. The Schur polynomials Δ are defined by $\Delta_{(\lambda_1,\dots,\lambda_r)} = \det(c_{\lambda_i+j-i})_{r\times r}$, for example $\Delta_{(31)} = \det\begin{pmatrix} c_3 & c_4 \\ c_0 & c_1 \end{pmatrix} = c_3c_1 - c_4$.

Remark 3.3. Since

$$(y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2)+\mathfrak{m}^3 = (y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2, xyz),$$

the polynomial (**??**) is the Thom polynomial of the singularity

$$(x,y,z) \mapsto (y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2, xyz),$$

or any of its unfoldings. It is proved in [**?**, §12] that "lowering" this Thom polynomial calculates the Thom polynomial of any member of the family of contact singularities

$$h_{c,g}(x,y,z) := (y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2).$$

(In [**?**] a different representation of $h_{c,g}$ is used, namely $(x^2-\lambda yz, y^2-\lambda xz, z^2-\lambda xy)$. This representation is less adapted to our purposes—in general 12 different values of λ correspond to the same orbit, and the singularities corresponding to B and B^* cannot be written in this form.) Hence we have

$$\text{Thom polynomial of } h_{c,g} = \text{tp}_Q(0) = 8\Delta_{(433)} + 4\Delta_{(3331)},$$

if $g \neq 0$ and half of it for $g = 0$. The singularities $h_{c,g}$ form the smallest codimensional example of a family of non-equivalent contact singularities $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ ([**?**]). Because of the presence of this continuous modulus in the classification of singularities, the method of [**?**] calculating Thom polynomials broke down at codimension 9.

4. Invariant theory of nets of conics

Most of the results in this section are known (see [**?**], [**?**] and [**?**]), but we also would like to show how equivariant theory leads to these results with the hope that it can be applied in a more general context.

4.1. **The ring of (semi) invariants.** Suppose that the Lie group G acts on the vector space W and $\hat{G}$ is the character group of G. We say that $f \in \mathbb{C}[W]$ is a *relative invariant corresponding to the character* $\chi \in \hat{G}$ (i.e. $f \in R_\chi(W)$) if for all $v \in W$ and $g \in G$

$$f(gv) = \chi(g)f(v). \tag{5}$$

The ring of *semi-invariants* is $R(W) := \bigoplus_{\chi \in \hat{G}} R_\chi(W)$.

Note that an element of $R(W)$ is not necessarily a relative invariant for any $\chi \in \hat{G}$. Semi-invariants of G are always invariants of the commutator subgroup G', but in general the ring of invariants of G' can be bigger. In the following two examples, however, they coincide.

(a) The $\mathrm{GL}(U) \times \mathrm{GL}(V)$-action on $\mathbf{Noc}$: Any character of $\mathrm{GL}(U) \times \mathrm{GL}(V)$ is of the form $\chi_{a,b}(g,h) := \det^a(g)\det^b(h)$. If $f \in R(\mathbf{Noc})$ is homogeneous of degree $l = 3d$, then

$$f\big((\lambda I, \mu I)v\big) = f(\lambda^{-2}\mu v) = \lambda^{-2l}\mu^l f(v) = \det{}^{-2d}(\lambda I)\det{}^d(\mu I)f(v),$$

therefore f is a relative invariant corresponding to the character $\chi_{-2d,d}$. (In GIT language we see that there is a unique linearization for GIT-quotient.) In other words

$$R(\mathbf{Noc}) = \bigoplus_{d \in \mathbb{N}} R_{\chi_{-2d,d}}(\mathbf{Noc}).$$

In this case all characters are determinants, so $R(\mathbf{Noc})$ coincides with the ring of absolute invariant polynomials for the $\mathrm{SL}_3 \times \mathrm{SL}_3$-action on $\mathbf{Noc}$.

(b) As we discussed before, the maximal torus of the stabilizer of the net $(y^2 + 2xz, 2yz, -x^2)$ is $\mathrm{GL}_1 \times \mathrm{GL}_1$. It acts on the normal space N_C to its orbit with weights $4\alpha - 4\beta$ and $2\alpha - 2\beta$. With respect to this action we have

$$R(N_C) = \bigoplus_{d \in \mathbb{N}} R_{\chi_{2d,-2d}}(N_C) = \mathbb{C}[N_C] = \mathbb{C}[c, g],$$

where $c \in R_{\chi_{4,-4}}(N_C)$ and $g \in R_{\chi_{2,-2}}(N_C)$. Consider the restriction map $i^* : R(\mathbf{Noc}) \to R(N_C)$. From the first line of Table **??** (equivalently, from (**??**)) one sees that i^* maps $R_{\chi_{-2d,d}}(\mathbf{Noc})$ into $R_{\chi_{d,-d}}(N_C)$.

We claim that $i^*: R(\mathbf{Noc}) \to R(N_C)$ is injective. Indeed, according to the splitting above it is enough to show that it is injective on $R_{\chi_{-2d,d}}(\mathbf{Noc})$ for any given d, which follows from the fact that the values of a relative invariant f on N_C determine its values on $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \cdot N_C$ via (**??**) and $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \cdot N_C$ is dense in **Noc**. Now we prove that the homomorphism i^* is also surjective, by finding relative invariants of **Noc** mapped to c and g.

4.2. **The determinant map.** Composing nets with the determinant $S^2U^* \to \mathbb{C}$ (well defined upto a scalar factor) we get a degree 3 polynomial map $\delta : \mathbf{Noc} \to S^3V$. This map is $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant if we let $\mathrm{GL}(U)$ act on S^3V as scalars by the second tensor power of the determinant representation. Consequently we have a homomorphism $\delta^* : R(S^3V) \to R(\mathbf{Noc})$. Let us review now the invariant theory of the plane cubics $S^3\mathbb{C}^3$ which was one of the first achievements of the early invariant theory.

Theorem 4.1. [?] *There are invariants a, b of $S^3\mathbb{C}^3$ of degree 4 and 6, respectively such that $R(S^3\mathbb{C}^3) \cong \mathbb{C}[a, b]$ and every smooth plane cubic γ can be transformed (using the GL_3-action) into the Weierstrass-form:*

$$y^2z + x^3 + a(\gamma)xz^2 + b(\gamma)z^3.$$

Remark 4.2. The Weierstrass-form is analogous to the c-g-form of nets of conics from Section **??**. The subset $\{y^2z + x^3 + axz^2 + bz^3 : a, b \in \mathbb{C}\}$ is a normal slice to the orbit of the cuspidal cubic $y^2z + x^3$. This observation was used in [?] to calculate the equivariant classes for plane cubics. The choice of these orbits is not accidental. Their closure is the nullcone, so the normal slice intersects all invariant hypersurfaces.

Consider now the determinant of the c-g-form $\nu_{c,g} = (y^2 + 2xz, 2yz, -x^2 + 2g(xz - y^2) + cz^2)$. Considering the three 3×3 matrices of the three components of $\nu_{c,g}$ we obtain

$$\begin{aligned} \delta(\nu_{c,g}) &= \det \left((-x) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + (-y) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + z \begin{pmatrix} -1 & 0 & g \\ 0 & -2g & 0 \\ g & 0 & c \end{pmatrix} \right) \\ &= y^2z + x^3 + (c - 3g^2)xz^2 + 2g(c + g^2)z^3. \end{aligned}$$

Here we parameterized $\mathbb{C}^3$ with $-x, -y, z$ to obtain our result, the Weierstrass form, without sign changes. Therefore $i^*\delta^*a = c - 3g^2$ and $i^*\delta^*b = 2g(c + g^2)$. We denote the degree 12 invariant $-48\delta^*a$ by J_{12}.

To complete the calculation of $R(\mathbf{Noc})$ we need to find the degree 6 invariant which restricts to g on the normal slice N_C. This is a straightforward job with a computer, but using a geometric idea it can be done by hand.

4.3. **The Plücker map and the invariant I_2.** We have a degree 3 $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant map

$$\psi : \mathbf{Noc} = \mathrm{Hom}(S^2U, V) \xrightarrow{\Lambda^3} \mathrm{Hom}(\bigwedge^3 S^2U, \bigwedge^3 V) \cong \bigwedge^3 S^2U^* \otimes \bigwedge^3 V.$$

We will call $\mathbf{Pl} := \bigwedge^3 S^2U^* \otimes \bigwedge^3 V$ the *Plücker space.*

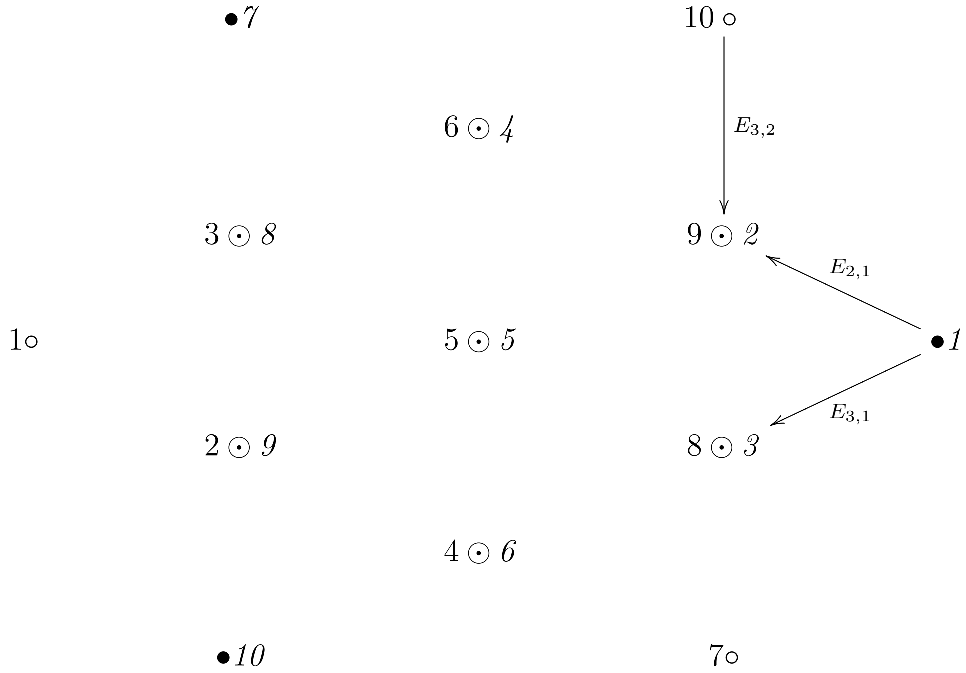

FIGURE 1. The 20 weights of **Pl**

Picking a basis for V^*, to each net we can associate a triple $M = (M_1, M_2, M_3)$ of conics, and ψ sends M to $M_1 \wedge M_2 \wedge M_3 \in \bigwedge^3 S^2U^*$. The image of ψ is the cone of the Grassmannian $\mathrm{Gr}_3(S^2U^*)$. Our next goal is to show that the representation **Pl** has a degree 2 invariant I_2 which pulls back to a degree 6 invariant J_6 of **Noc**.

Let ξ, η, ν be a basis in U^* and x, y, z the corresponding dual basis in U. Write E_{ij} for the element of $\mathfrak{sl}(U)$ mapping the ith basis vector of U to the jth basis vector and annihilating the other two basis vectors.

Consider the basis $e_1 := \xi^2$, $e_2 := \xi\eta$, $e_3 := \xi\nu$, $e_4 := \eta^2$, $e_5 := \eta\nu$, $e_6 := \nu^2$ in S^2U^*, and set $e_{ijk} := e_i \wedge e_2 \wedge e_3 \in \bigwedge^3 S^2(U^*)$. Under the natural identification of S^2U with the dual of $S^2(U^\star)$, the basis in $S^2(U)$ dual to $e_1, \dots, e_6$ is $t_1 := x^2$, $t_2 := 2xy$, $t_3 := 2xz$, $t_4 := y^2$, $t_5 := 2yz$, $t_6 := z^2$. Then $t_{ijk} := t_i \wedge t_j \wedge t_k \in \bigwedge^3 S^2(U)$ is the basis dual to e_{ijk}. In particular, $t_{ijk}(\pi(M))$ is the 3×3 minor corresponding to the (i, j, k) columns of the 3×6 matrix of the net M viewed as a linear map from V^* to S^2U^*, with respect to the chosen bases.

As an $\mathrm{SL}(U)$-representation $\mathbf{Pl} \cong S^3(U) \oplus S^3(U^*)$ (this follows for example from the calculations below or by calculating the weights of **Pl** (see Figure **??**).).

Denoting by W the 10-dimensional $\mathrm{SL}(U)$-module $S^3(U)$ we have $S^2\,\mathbf{Pl}^* \cong S^2W \oplus S^2W^* \oplus W \otimes W^*$. The first two summands do not contain the trivial $\mathrm{SL}(U)$-module (say by the theorem of Aronhold on the invariants of ternary cubic forms), and the third summand contains one copy of the trivial representation by Schur's Lemma, spanned by $w_1w_1^* + \cdots + w_{10}w_{10}^*$, where $w_1, \dots, w_{10}$ is a basis of W and $w_1^*, \dots, w_{10}^*$ is the corresponding dual basis in W^*. Table **??** contains explicit elements $w_1, \dots, w_{10} \in \bigwedge^3 S^2(U)$ spanning an $\mathfrak{sl}(U)$-summand isomorphic to W. Each weight space in $\bigwedge^3 S^2 U$ is 1-dimensional. Therefore $x^2 \wedge xy \wedge xz$ is a highest weight vector generating an $\mathfrak{sl}(U)$-module isomorphic to W. The $w_2, \dots, w_{10}$ are obtained by applying successively the operators $E_{21}, E_{31} \in \mathfrak{sl}(U)$, as indicated in Table **??**.

Recall that the action of $E_{i,j}$ is the sum of the replacements of each occurrence of the jth basis vector to the ith one, e.g.

$$E_{3,1}(x^2 \wedge xy \wedge xz) = 2xz \wedge xy \wedge xz + x^2 \wedge yz \wedge xz + x^2 \wedge xy \wedge z^2 = x^2 \wedge xy \wedge z^2 - x^2 \wedge xz \wedge yz.$$

Up to non-zero scalars w_{10}^* in Table **??** is the only weight vector whose weight is opposite to the weight of the lowest weight element w_{10} in W, therefore it must be a highest weight vector generating a submodule isomorphic to W^* in $\bigwedge^3 S^2 U^*$. Applying successively appropriate elements of $\mathfrak{sl}(U)$ to w_{10}^* one computes $w_2^*, \dots, w_{10}^* \in \bigwedge^3 S^2 U^*$. For example, by the left column of the table we have $w_{10} = E_{31} w_6$, hence $w_6^* = -E_{31} w_{10}^*$. In addition to the information in the left column of the table we need also the relations

$$w_8 = \frac{1}{3} E_{32} w_7, \quad w_9 = \frac{1}{2} E_{32} w_8, \quad \text{and} \quad w_{10} = E_{32} w_9.$$

With the w_i, w_i^* given in Table **??** (for example, $w_2 = -x^2 \wedge xz \wedge y^2 + x^2 \wedge xy \wedge yz = -\frac{1}{2} t_{134} + \frac{1}{4} t_{125}$) we have the equality

$$\begin{aligned} -8I_2 := 8 \sum_{i=1}^{10} w_i w_i^* = {} & t_{235}^2 - 8t_{146}^2 \\ & - 8t_{134}t_{346} + 8t_{126}t_{246} + 8t_{145}t_{156} \\ & + 6t_{123}t_{456} - 6t_{136}t_{245} + 6t_{124}t_{356} \\ & - 4t_{125}t_{256} + 4t_{135}t_{345} - 4t_{234}t_{236} \\ & + 2t_{134}t_{256} - 2t_{125}t_{346} + 2t_{135}t_{246} - 2t_{126}t_{345} \\ & + 2t_{145}t_{236} + 2t_{156}t_{234} - 2t_{146}t_{235}. \end{aligned} \tag{6}$$

Remark 4.3. (i) Set $u_{ijk} := t_{ijk} \circ \psi$, so u_{ijk} is an element of the coordinate ring of **Noc**. Recall a classical result of Sylvester (see page 365 in [?] or [?]), asserting (after a change to our coordinate system) that

$$\begin{aligned} -8\theta := {} & u_{235}^2 - 8u_{146}^2 \\ & + 4u_{146}u_{235} + 4u_{135}u_{345} - 4u_{125}u_{256} - 4u_{234}u_{236} \\ & + 8u_{145}u_{156} - 8u_{134}u_{346} + 8u_{126}u_{246} \\ & + 8u_{123}u_{456} - 8u_{136}u_{245} + 8u_{124}u_{356} \end{aligned}$$

is an SL(U)-invariant on **Noc**. We thank I. Dolgachev for bringing this reference to our attention. One can easily verify using the straightening algorithm (cf. Section 13.2.2 in [?]) that $J_6 := I_2 \circ \psi$ coincides with Salmon's θ. Notice that one cannot reconstruct I_2 from θ since ψ^* has a kernel, generated by the *Plücker relations*. To the best of our knowledge formula (**??**) for I_2 is new.

Table 2. Un-normalized generators

w_1	$= x^2 \wedge xy \wedge xz$	w_{10}^*	$= x^2 \wedge xy \wedge y^2$
$w_2 = E_{21}w_1$	$= -x^2 \wedge xz \wedge y^2 + x^2 \wedge xy \wedge yz$	$w_9^* = -E_{32}w_{10}^*$	$= -x^2 \wedge xz \wedge y^2 - 2x^2 \wedge xy \wedge yz$
$w_3 = E_{31}w_1$	$= x^2 \wedge yz \wedge xz + x^2 \wedge xy \wedge z^2$	$w_8^* = -\frac{1}{2}E_{32}w_9^*$	$= 2x^2 \wedge xz \wedge yz + x^2 \wedge xy \wedge z^2$
$w_4 = E_{21}w_2$	$= 2xy \wedge y^2 \wedge xz + 2x^2 \wedge y^2 \wedge yz$	$w_7^* = -\frac{1}{3}E_{32}w_8^*$	$= -x^2 \wedge xz \wedge z^2$
$w_5 = E_{31}w_2$	$= x^2 \wedge y^2 \wedge z^2 + 2xz \wedge xy \wedge yz$	$w_6^* = -E_{31}w_{10}^*$	$= 2xy \wedge xz \wedge y^2 + x^2 \wedge y^2 \wedge yz$
$w_6 = E_{31}w_3$	$= 2x^2 \wedge yz \wedge z^2 + 2xz \wedge xy \wedge z^2$	$w_5^* = -E_{31}w_9^*$	$= -4xy \wedge xz \wedge yz - x^2 \wedge y^2 \wedge z^2$
$w_7 = E_{21}w_4$	$= 6xy \wedge y^2 \wedge yz$	$w_4^* = -E_{21}w_7^*$	$= 2xy \wedge xz \wedge z^2 + x^2 \wedge yz \wedge z^2$
$w_8 = E_{31}w_4$	$= 2xy \wedge y^2 \wedge z^2 + 2xz \wedge y^2 \wedge yz$	$w_3^* = -E_{31}w_6^*$	$= 2xy \wedge y^2 \wedge z^2 - 4xz \wedge y^2 \wedge yz$
$w_9 = E_{31}w_5$	$= 2xy \wedge yz \wedge z^2 + 2xz \wedge y^2 \wedge z^2$	$w_2^* = -E_{21}w_4^*$	$= 2xz \wedge y^2 \wedge z^2 - 4xy \wedge yz \wedge z^2$
$w_{10} = E_{31}w_6$	$= 6xz \wedge yz \wedge z^2$	$w_1^* = -E_{21}w_2^*$	$= 6y^2 \wedge yz \wedge z^2$

(ii) A different construction of the invariant J_6 (not as a pullback from **Pl**) can be obtained from [?] by specializing a degree 6 invariant there to symmetric matrix triples.

One can calculate

$$\psi(\nu_{c,g}) = (y^2 + 2xz) \wedge 2yz \wedge (x^2 + 2g(xz - y^2) + cz^2) = 12ge_{345} + 2c(e_{456} + 2e_{356}) - 2e_{145} - 4e_{135},$$

and hence, for $J_6 = I_2 \circ \psi$ we have $J_6(\nu_{c,g}) = I_2(\psi(\nu_{c,g})) = 24g$. This concludes the proof of the following theorem.

Theorem 4.4. *The ring of invariants of* **Noc** *is freely generated by* J_6 *and* J_{12}.

4.4. **A geometric interpretation of the splitting of Pl.** Following C.T.C. Wall [?] we can interpret the projection maps from **Pl** to its irreducible factors.

4.4.1. *The Jacobi map.* A net φ is a linear map from S^2U to V, alternatively a quadratic map from U to V. Its derivative at $u \in U$ is a linear map $d_u\varphi : T_uU \to T_uV$. Since tangent spaces of a vector space can be canonically identified with the vector space itself and $d_u\varphi$ is linear in u, the derivative $d\varphi$ defines a linear map from U to $\mathrm{Hom}(U,V)$. We also have the degree 3 determinant map

$$\det : \mathrm{Hom}(U,V) \overset{\bigwedge^3}{\to} \mathrm{Hom}(\bigwedge^3 U, \bigwedge^3 V) \cong \bigwedge^3 U^* \otimes \bigwedge^3 V,$$

which can be composed with $d\varphi$ to obtain a degree 3 map $\mathrm{Jac}(\varphi)$ from U to $\bigwedge^3 U^* \otimes \bigwedge^3 V$. We can also consider Jac as a map

$$\mathrm{Jac} : \mathbf{Noc} \to S^3U^* \otimes \bigwedge^3 U^* \otimes \bigwedge^3 V.$$

The map Jac factors through the Plücker map providing a linear projection

$$\pi_1 : \mathbf{Pl} \to S^3U^* \otimes \bigwedge^3 U^* \otimes \bigwedge^3 V.$$

Picking a basis in V^* we can identify Jac with the Jacobian covariant $\mathrm{Jac} : \bigoplus^3 S^2(U^*) \to S^3(U^*)$, which is a joint covariant of triples of conics defined as

$$\mathrm{Jac}(M) := \det \begin{pmatrix} \partial_\xi M_1 & \partial_\xi M_2 & \partial_\xi M_3 \\ \partial_\eta M_1 & \partial_\eta M_2 & \partial_\eta M_3 \\ \partial_\nu M_1 & \partial_\nu M_2 & \partial_\nu M_3 \end{pmatrix}$$

(recall that ξ, ν, η is our basis in U^*, and the M_i are homogeneous quadratic polynomials in ξ, η, ν). Now Jac is an alternating trilinear function in M_1, M_2, M_3, hence it factors through an $\mathrm{SL}(U)$-equivariant linear map

$$\pi_1 : \mathbf{Pl} \to S^3(U^*).$$

It maps the basis vector $e_{ijk} \in \mathbf{Pl}$ to the 3×3 minor corresponding to the i, j, k columns of the matrix

$$\begin{pmatrix} 2\xi & \eta & \nu & 0 & 0 & 0 \\ 0 & \xi & 0 & 2\eta & \nu & 0 \\ 0 & 0 & \xi & 0 & \eta & 2\nu \end{pmatrix}$$

(the columns contain the partial derivatives for each of $\xi^2, \xi\eta, \xi\nu, \nu^2, \eta\nu, \nu^2$). Recall that $x^3, 3x^2y, \dots, 6xyz, \dots$ is the basis in $S^3(U)$ dual to the basis $\xi^3, \xi^2\eta, \dots, \xi\eta\nu, \dots$ of $S^3(U^*)$. Now π_1^* embeds $S^3(U^*)^* \cong S^3(U)$ into $\mathbf{Pl}^*$. For example, $\pi_1(e_{123}) = 2\xi^3$, and no other $\pi_1(e_{ijk})$ contains the monomial ξ^3. This shows that $\pi_1^*(x^3) = 2t_{123} = 8w_1$ (where w_1 is the element given in Table **??**). Similarly $\xi^2\eta$ is contained only in $\pi_1(e_{125}) = 2\xi^2\eta$ and $\pi_1(e_{134}) = -4\xi^2\eta$. It follows that $\pi_1^*(3x^2y) = 2t_{125} - 4t_{134} = 8w_2$. One checks in the same way that the basis $x^3, 3x^2y, \dots 6xyz, \dots, z^3$ of $S^3(U)$ is mapped under π_1^* onto $8w_1, \dots, 8w_{10}$ (cf. Table **??**).

4.4.2. *The dual Jacobi map.* We have a degree 2 map

$$\mathrm{Hom}(\mathbb{C}^2, U) \overset{S^2}{\to} \mathrm{Hom}(S^2\mathbb{C}^2, S^2U).$$

Composing with a net φ and choosing an element in $\mathrm{Hom}(\mathbb{C}^2, U)$ we get a linear map in $\mathrm{Hom}(S^2\mathbb{C}^2, V)$. Taking its determinant we get a degree 6 map from $U \oplus U \cong \mathrm{Hom}(\mathbb{C}^2, U)$ to the one-dimensional vector space $L = \mathrm{Hom}(\bigwedge^3 S^2\mathbb{C}^2, \bigwedge^3 V) \cong \bigwedge^3 V$. (As a representation of $\mathrm{GL}(U)$ the line L is isomorphic to the trivial one-dimensional representation.) Notice that this map factors through $\bigwedge^2 U$, providing a degree 3 map from $\bigwedge^2 U$ to L. Varying φ we end up with a degree 3 map from $\mathbf{Noc} \to S^3 \bigwedge^2 U^* \otimes L$ which factors through the Plücker map ψ. Now notice that $\bigwedge^2 U^* \cong U \otimes \bigwedge^3 U^*$. Hence we defined a linear map

$$\pi_2 : \mathbf{Pl} \to S^3U \otimes \left(\bigwedge^3 U^*\right)^3 \otimes \bigwedge^3 V.$$

Notice that the $\mathrm{GL}(V)$-action played no active role in the projections π_1 and π_2 as it was expected from the abstract splitting of the representation $\bigwedge^3 S^2U^*$.

As an $\mathrm{SL}(U)$-equivariant linear map $\pi_2 : \mathbf{Pl} \to S^3(U)$ can be constructed as follows: Picking a basis in V^* we can identify a net with a triple $M = (M_1, M_2, M_3)$ where $M_i \in S^2U^*$. We may think of $S^3(U)$ as the space of cubic polynomial functions on U^*. Now $\pi_2(M_1 \wedge M_2 \wedge M_3)$ vanishes on a linear form $f \in U^*$ if the net M restricted to the zero locus of f does not have full rank.

More explicitly, eliminate the variable ν from the ternary quadratic forms M_i using the relation $x\xi + y\eta + z\nu = 0$; we obtain three binary quadratic forms in the variables ξ, η. Now $\pi_2(M_1 \wedge$

$M_2 \wedge M_3$) is the determinant of the 3×3 matrix whose columns contain the coefficients of these three binary quadratic forms. In particular, $\pi_2(e_{ijk})$ is the (i, j, k) minor of

$$\begin{pmatrix} z & 0 & -x & 0 & 0 & x^2/z \\ 0 & 0 & 0 & z & -y & y^2/z \\ 0 & z & -y & 0 & -x & 2xy/z \end{pmatrix}$$

(showing also that we end up with a cubic polynomial in x, y, z). The dual π_2^* embeds S^3U^* into $\mathbf{Pl}^*$, and in the same way as in the case of the Jacobi map one may check that the basis vectors $\xi^3, \xi^2\eta, \dots, \nu^3$ are mapped to $\frac{-1}{3}w_1^*, \dots, \frac{-1}{3}w_{10}^*$ from Table **??**.

4.5. **Stability.** A net of conics is in the nullcone if both J_6 and J_{12} are zero on it. The geometric quotient of **Noc** is $\mathbf{P}^1$ and the quotient map on the complement of the nullcone is given by $k := J_6^2/J_{12}$. An orbit η is strictly semistable if $k^{-1}(k(\eta)) \supsetneq \eta$. We can use formula (**??**) to calculate J_6 and the explicit form of the degree 4 invariant of the plane cubics to calculate J_{12}. Notice that Theorem **??** is not sufficient, since non-smooth cubics do not admit a Weierstrass form. Nevertheless these are simple calculations which show that the only codimension>1 orbits outside the nullcone are D, D^*, E, E^* with

	J_6	J_{12}	k
D	1	1	1
D^*	-8	16	4
E	1	1	1
E^*	-8	16	4

Since k is a bijection on the codimension 1 orbits it is enough to find values of c and g with the given k. Since $k(\nu_{c,g}) = \frac{12g^2}{3g^2-c}$, it is immediate that $k(B) = 1$ for $B := \nu_{-9,1}$ and $k(B^*) = 4$ for $B^* := \nu_{0,1}$ (B and B^* are notations from [?]). Consequently the complete list of semistable orbits are B, B^*, D, D^*, E, E^*.

4.5.1. *The discriminant.* For the representations **Noc** and $S^3\mathbb{C}^3$ the nonstable variety is a hypersurface and we call their defining equation the *discriminant* of the representation. It is a classical result that the discriminant of the plane cubics is $\Delta = 4a^3 + 27b^2$. Using the c-g-form we can quickly check that $-2^8 3^3 \delta^* \Delta = (J_6^2 - J_{12})^2 (J_6^2 - 4J_{12})$ (see Section **??** for the details), consequently a net is unstable if and only if its determinant cubic is unstable, but the δ^*-image of the discriminant is not the discriminant: the component $(J_6^2 - J_{12}) = \overline{B}$ is counted with multiplicity 2.

5. Hierarchy of the nets of conics

C. T. C. Wall's result on the classification of the **Noc**-orbits can be verified using the results of Sections **??** and **??**. The codimension 1 orbits are classified by their k-invariant. The fact stated in Theorem **??**, namely that the restriction equations determine the equivariant classes imply that no orbit is missed in Table **??**. Any missing orbit would cause an indeterminacy in the solution of the restriction equations, hence would contradict to Theorem **??**.

To determine the hierarchy we use that a cohomologically defined incidence class determines adjacency for *positive* orbits: Consider a Lie group G acting on a vector space V complex linearly. For $v \in V$ let T_v denote the maximal torus of the stabilizer subgroup of G.

Definition 5.1. *The orbit Gv is positive if there is a linear functional φ on the weight space of T_v such that for all weights w_i of the T_v-action on the normal space of the orbit Gv at v we have $\varphi(w_i) > 0$.*

Theorem 5.2. [**?**] *Let $\eta \subset V$ be a G-invariant subvariety and suppose that the orbit Gv is positive for some $v \in V$. Then $v \in \eta$ if and only if $[\eta \subset V]_{T_v} \neq 0$.*

Table **??** contains the description of normal slices to orbits, and their weights. The last column contains the values of the functional φ (that is, the values $\varphi(\alpha), \varphi(\beta), \ldots$) if such a functional—proving the positivity of the given orbit—exists. By inspection we obtain the following fact.

Proposition 5.3. *All unstable orbits of* **Noc** *are positive.*

Thus, Theorem **??** determines almost all adjacencies of the orbits, namely the ones involving unstable orbits. The missing adjacencies of the semistable orbits can be determined by calculating the k-invariant. As a result we obtain the complete hierarchy, depicted on Figure **??**.

Example 5.4. Consider the orbits F and F^*, and their adjacency with the orbit (1^4). Let v be the point in the (1^4) orbit given in the Table, and let $j_{(1^4)}$ be the restriction homomorphism $H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)}(\mathbf{Noc}) \to H^*_{T_v}(\mathbf{Noc})$. One can read from the table above that the homomorphism $j_{(1^4)}$ is the substitution

$$c_i = \sigma_i(\alpha, \alpha, \alpha), \qquad d_i = \sigma_i(2\alpha, 2\alpha, \beta),$$

where σ_i denotes the ith elementary symmetric polynomial. For the equivariant classes given in Theorem **??** we have

$$j_{(1^4)}([\overline{F}]) = 0, \qquad j_{(1^4)}([\overline{F^*}]) = -6(2\alpha - \beta)(4\alpha^2 - 4\alpha\beta + \beta^2) \neq 0.$$

Hence, we have that (1^4) is contained in the orbit closure of F^*, but is not contained in the orbit closure of F.

5.1. **The equivariant cohomology rings of the orbits.** With a little extra sudoku type calculations one can determine the equivariant cohomology rings $H^*(BG_x)$ of the orbits Gx. In column 5 of Table **??** we listed the degrees of a free generating set for these rings. This information can be used e.g. to define certain "higher" Thom polynomials. Since the equivariant cohomology spectral sequence of the codimension filtration (Kazarian spectral sequence, see [**?**], [**?**, Sect.10]) degenerates, the Poincaré series of the rings $H^*(BG_x)$ shifted by the codimension add up to the Poincaré series of $H^*(BG)$. For the open stratum $O = \bigcup_{\mu \in \mathbf{P}^1} A_\mu$ we have $H^*_G(O) = H^*_{\mathrm{GL}_1}(\mathbf{P}^1)$ for the trivial GL_1-action, so the Poincaré series of $H^*_G(O)$ is $\frac{1+t}{1-t}$, and we get that

$$\frac{1+t}{1-t} + \frac{t^2}{(1-t)^2} + \frac{2t^2}{(1-t)(1-t^2)} + \ldots + \frac{t^{18}}{(1-t)^2(1-t^2)^2(1-t^3)^2} = \frac{1}{(1-t)^2(1-t^2)^2(1-t^3)^2}.$$

Here the t-exponents of the numerator are the codimensions (column 2) and the t-exponents of the denominators are degrees of the generators of $H^*(BG_x)$ (column 5).

Table 3. Normal weights and positivity

Σ^0			normal weights	$\varphi(\alpha)$,$\varphi(\beta)$,...
C	2	$y^2+2xz, 2yz, -x^2$	$4\alpha-4\beta, 2\alpha-2\beta$	1,0
D	2	x^2, y^2, z^2+2xy	$3\alpha-3\beta, 3\beta-3\alpha$	-
D^*	2	$2xz, 2yz, z^2+2xy$	$3\alpha-3\beta, 3\beta-3\alpha$	-
E	3	x^2, y^2, z^2	$2\alpha-\beta-\gamma, 2\beta-\alpha-\gamma, 2\gamma-\alpha-\beta$	-
E^*	3	$2xy, 2yz, 2zx$	$-2\alpha+\beta+\gamma, -2\beta+\alpha+\gamma, -2\gamma+\alpha+\beta$	-
F	3	$x^2+y^2, 2xy, 2yz$	$2\alpha-2\beta, 2\alpha-2\beta, \alpha-\beta$	1,0
F^*	3	x^2+y^2, xz, z^2	$2\beta-2\alpha, 2\beta-2\alpha, \beta-\alpha,$	-1,0
G	4	x^2, y^2, yz	$2\alpha-2\gamma, \beta-\alpha, 2\beta-\alpha-\gamma, 2\beta-2\gamma$	1,2,0
G^*	4	xy, xz, z^2	$2\gamma-2\alpha, \alpha-\beta, -2\beta+\alpha+\gamma, 2\gamma-2\beta$	-1,-2,0
H	5	$x^2, 2xy, y^2+2xz$	$2\alpha-2\beta, 2\alpha-2\beta, 3\alpha-3\beta, 3\alpha-3\beta, 4\alpha-4\beta$	1,0
I	7	x^2, xy, y^2	$\alpha-\gamma, 2\alpha-\beta-\gamma, 2\alpha-2\gamma, \beta-\gamma, -2\gamma+\alpha+\beta, 2\beta-2\gamma$	0,0,-1
I^*	7	xz, yz, z^2	$-\alpha+\gamma, -2\alpha+\beta+\gamma, -2\alpha+2\gamma, -\beta+\gamma, 2\gamma-\alpha-\beta, -2\beta+2\gamma$	0,0,1
Σ^1				
(1^4)	4	$x^2-xz, y^2-yz, 0$	$\beta-2\gamma, \beta-2\gamma, \beta-2\gamma, \beta-2\gamma$	1,0
(21^2)	5	$xy, xz+yz, 0$	$2\alpha-2\beta, \gamma-2\alpha, \gamma-2\alpha, \gamma-\alpha-\beta, \gamma-2\beta$	1,0,3
(31)	6	$xz, x^2-yz, 0$	$3\beta-3\alpha, 2\beta-2\alpha, \gamma-3\alpha-\beta, \gamma-4\alpha, \gamma-2\alpha-2\beta, \gamma-4\beta$	0,1,5
(22)	6	$x^2, yz, 0$	$2\alpha-2\beta, 2\alpha-2\gamma, \delta-\alpha-\beta, \delta-\alpha-\gamma, \delta-2\beta, \delta-2\gamma$	1,0,0,2
(4)	7	$xz+y^2, x^2, 0$	$2\alpha-2\beta, 3\alpha-3\beta, 4\alpha-4\beta, \gamma-3\alpha-\beta, \gamma-2\alpha-2\beta, \gamma-4\beta$	1,0,5
K	8	$y^2, z^2, 0$	$2\alpha-2\beta, 2\alpha-\beta-\gamma, 2\gamma-2\beta, 2\gamma-\alpha-\beta, \delta-2\beta,$	
			$\delta-\alpha-\beta, \delta-\beta-\gamma, \delta-\alpha-\gamma$	0,-1,0,0
L	8	$xy, xz, 0$	$\alpha-\beta, \alpha-\gamma, \alpha+\beta-2\gamma, \alpha+\gamma-2\beta, \delta-2\alpha, \delta-2\beta,$	
			$\delta-2\gamma, \delta-\beta-\gamma$	1,0,0,3
M	9	$yz, y^2, 0$	$\beta+\gamma-2\alpha, \beta-\alpha, 2\beta-2\alpha, 2\beta-\alpha-\gamma, 2\beta-2\gamma, \delta-2\alpha,$	
			$\delta-\alpha-\beta, \delta-\alpha-\gamma, \delta-2\gamma$	0,1,0,2
Σ^2				
S	10	$xy-z^2, 0, 0$	$\gamma-2\alpha-2\beta, \gamma-3\alpha-\beta, \gamma-4\alpha, \gamma-\alpha-3\beta, \gamma-4\beta,$	
			$\delta-2\alpha-2\beta, \delta-3\alpha-\beta, \delta-4\alpha, \delta-\alpha-3\beta, \delta-4\beta$	0,0,1,1
PL	11	$xy, 0, 0$	$\delta-2\alpha, \delta-2\beta, \delta-\alpha-\gamma, \delta-\beta-\gamma, \delta-2\gamma$	
			$\epsilon-2\alpha, \epsilon-2\beta, \epsilon-\alpha-\gamma, \epsilon-\beta-\gamma, \epsilon-2\gamma, \alpha+\beta-2\gamma$	1,0,0,3,3
DL	13	$x^2, 0, 0$	$\delta-\alpha-\beta, \delta-2\beta, \delta-\alpha-\gamma, \delta-2\gamma, \delta-\beta-\gamma$	
			$\epsilon-\alpha-\beta, \epsilon-2\beta, \epsilon-\alpha-\gamma, \epsilon-2\gamma, \epsilon-\beta-\gamma$	
			$2\alpha-2\beta, 2\alpha-\beta-\gamma, 2\alpha-2\gamma$	1,0,0,3,3

6. Enumerative questions, multiplicities of the determinant map

Equivariant classes contain enumerative data in a compressed form. One of the simplest enumerative invariants is the degree. Using [**?**, Section 6] we see that the degree of a G-invariant subvariety $\eta \subset \mathbf{Noc}$ can be obtained by substituting 1 to the Chern roots of $\mathrm{GL}(V)$ and 0 to the

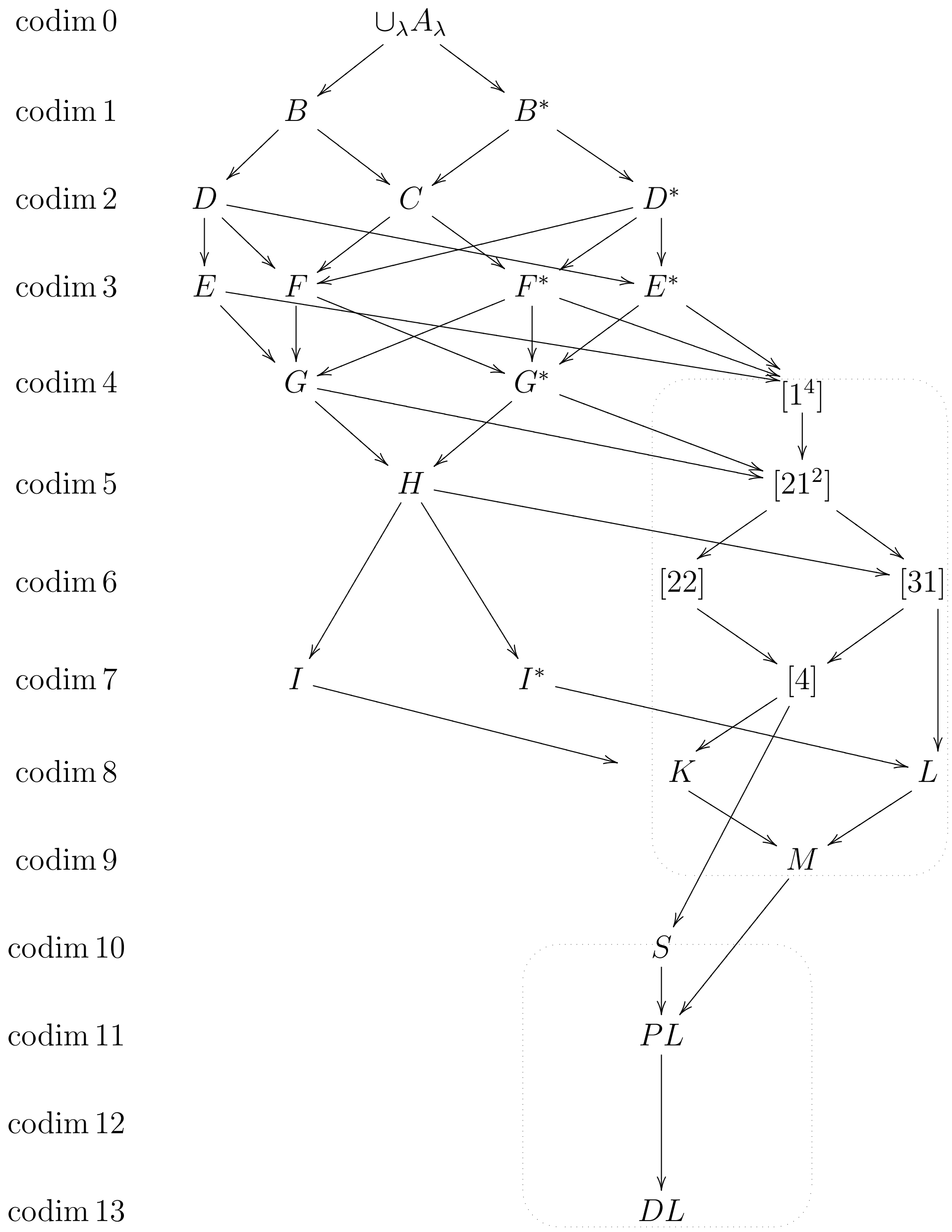

FIGURE 2. The hierarchy of **Noc** orbits

Chern roots of $\mathrm{GL}(U)$ in the polynomial $[\eta] \in H^*(B(\mathrm{GL}(U) \times \mathrm{GL}(V)))$. For example, we obtain

$$\deg(\overline{C}) = 72, \qquad \deg(\overline{D}) = 36, \qquad \deg(\overline{D^*}) = 45.$$

Consequently in a generic two-parameter linear family of nets of conics there are 72 of type C, 36 of type D and 45 of type D^*, etc.

A more subtle enumerative invariant is the *intersection multiplicity*. Suppose that $f: X \to Y$ is a map of smooth varieties and $Z \subset Y$ a codimension d subvariety. Suppose also that the preimage $f^{-1}Z$ is pure d-codimensional. Then we can assign positive integers μ_i to every component Y_i of $f^{-1}Z$ called intersection multiplicities (for more details see [?]). If $X = Y = \mathbb{C}$ and $Z = \{0\}$, then the intersection multiplicities are the usual multiplicities of the roots of f.

An important property of the intersection multiplicity ([?, Sect. 7]) is that if all components Y_i of $f^{-1}Z$ are of codimension d, then

$$f^*([X]) = \sum \mu_i [Y_i]. \tag{7}$$

In general this equation does not determine the intersection multiplicities μ_i. However (??) generalizes to the equivariant setting where there is more chance that the classes $[Y_i]$ will be linearly independent.

Consider the determinant map $\delta: \mathbf{Noc} \to S^3V$ studied in Section ??. From the normal forms in Table ?? it is easy to calculate the image under the determinant map. E.g. for C the normal form is $y^2 + 2xz, 2yz, -x^2$ which means that the generic net in matrix form is

$$\begin{pmatrix} -\kappa & \cdot & \lambda \\ \cdot & \lambda & \mu \\ \lambda & \mu & \cdot \end{pmatrix},$$

with determinant $\mu^2\kappa - \lambda^3$ corresponding to the cuspidal curve ν. A table can be found in [?]. For the readers' convenience we included the images of the determinant map in the last column of Table ??. The notation tries to imitate the shape of the various classes of plane cubics, i. e. θ is the orbit (closure) of conic intersected by line, Ω the conic and tangent line, A the three nonconcurrent lines, $\neq$ is the double line intersected by a third line, Ξ is the triple line and ЖK is the three concurrent lines. The codimension 1 orbits will be treated in the next section.

The equivariant classes of the GL_3-representation $S^3\mathbb{C}^3$ (ie. plane cubics) were calculated by B. Kőműves [?]. The cases we need are

$$[\nu] = 24e_1^2,\ [\theta] = 18e_1^2 + 9e_2,\ [\Omega] = 36e_1^3 + 18e_1e_2,\ [\mathsf{A}] = 12e_1^3 + 6e_1e_2 + 27e_3,\ [\text{ЖK}] = e_1[\mathsf{A}], \tag{8}$$

where e_i denote the GL_3-Chern classes.

The map δ is $\mathrm{GL}_3 \times \mathrm{GL}_3$-equivariant, if we replace the GL_3-action on the plane cubics by the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-action as in Section ??. The effect of this change on (??) is replacing the GL_3-Chern roots ϵ_i ($\epsilon_1 + \epsilon_2 + \epsilon_3 = e_1$ etc) by some linear combination of the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-Chern roots. A comparison of the actions of the maximal tori of GL_3 and $\mathrm{GL}(U) \times GL(V)$ gives the substitution $\epsilon_i \mapsto \delta_i - 2/3u_1$ (where δ_i denote the $\mathrm{GL}(V)$-Chern roots). Consequently for the Chern classes we obtain the substitution $e_1 \mapsto v_1 - 2u_1$, $e_2 \mapsto v_2 - 4/3u_1v_1 + 4/3u_1^2$ and $e_3 \mapsto v_3 - 2/3v_2u_1 + 4/9v_1u_1^2 - 8/27u_1^3$.

After this substitution we have all the ingredients of (??) for the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant map δ. Then explicit calculation implies the following theorem.

Theorem 6.1. *In $GL(U) \times GL(V)$-equivariant cohomology we have*

$$\delta^*([\nu]) = 3[C],$$

$$\delta^*([\theta]) = 4[D] + [D^*],$$

$$\delta^*([\Omega]) = 9[F] + 6[F^*],$$

$$\delta^*([\mathsf{A}]) = 8[E] + [E^*] + 2[F],$$

and the coefficients on the right hand side are uniquely determined, therefore they are the intersection multiplicities.

For the three concurrent lines the codimension condition is *not* satisfied, but a similar calculation gives the equality

$$\delta^*([\mathrm{X\!\!K}]) = 12[(1^4)] + \big(4[G] + 1/2[G^*] + 2(d_1 - 2c_1)[F]\big),$$

and, like above, the coefficients are uniquely determined. The class in the bracket is supported on $\overline{F}$, so we can call 12 the intersection multiplicity of (1^4).

Remark 6.2. The intersection multiplicities are always at most the algebraic multiplicities by [?, Pr. 8.2] and they agree if the image is (locally) Cohen-Macaulay and the preimage has the same codimension by [?, p.108]. In [?] J. Chipalkatti determines which orbit closures are arithmetically Cohen-Macaulay for the plane cubics. Among the orbits of Theorem **??** ν is Cohen-Macaulay, since it is a complete intersection, Ω is arithmetically Cohen-Macaulay consequently Cohen-Macaulay, but θ and A are not arithmetically Cohen-Macaulay. Therefore the intersection multiplicities for ν and Ω are algebraic multiplicities as well. For θ and A we do not know if they are Cohen-Macaulay. If the algebraic multiplicities for θ and A differ from the intersection multiplicities, then they cannot be Cohen-Macaulay. Unfortunately we were not able to calculate these algebraic multiplicities.

For the codimension 1 orbits we study the induced map on the GIT quotients.

6.1. **The induced map on the GIT quotients of nets of conics to plane cubics.** To see that the $G = \mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant determinant map δ induces a map d of the corresponding GIT quotients we need to check that semistable orbits are mapped to semistable orbits. It follows from general principles but here is a direct verification. For codimension > 1 orbits of **Noc** we see this fact from Table **??**. (For the plane cubics the semistable orbits are the smooth curves which are also stable, together with the nodal curve and θ and A.) The codimension 1 orbits A_μ have a $\nu_{c,g}$ representative with $(c,g) \neq (0,0)$, so for $\delta(\nu_{c,g})$ either $a = c - 3g^2$ or $b = 2g(c + g^2)$ is not zero, therefore the image is semistable.

The traditional parametrization of the GIT quotient $S^3\mathbb{C}^3//G$ is the j-invariant $j = \frac{4a^3}{\Delta}$: $S^3\mathbb{C}^3//G \to \mathbf{P}^1$, where $\Delta = 4a^3 + 27b^2$ is the discriminant. On the other hand we saw in Section **??** that the invariant $k = \frac{J_6^2}{J_{12}} : \mathbf{Noc}//G \to \mathbf{P}^1$ parametrizes the GIT quotient $\mathbf{Noc}//G$. Using these parametrizations we can calculate $\tilde{d} := j \circ d \circ k^{-1} : \mathbf{P}^1 \to \mathbf{P}^1$:

Using the c-g-form we get (by some abuse of notation)

$$\Delta = \Delta(\nu_{c,g}) = 4a^3 + 27b^2 = 4a^3 + 27(4g^2)(a + 4g^2)^2 = 4(a + 3g^2)(a + 12g^2)^2,$$

and we have $J_6 = 24g$ and $J_{12} = -48a$, so $48^3\Delta = (-4J_{12} + J_6^2)(-J_{12} + J_6^2)^2$. Therefore $j \circ d = \frac{J_{12}^3}{(J_6^2-4J_{12})(J_6^2-J_{12})^2}$ and

$$\tilde{d}(x:y) = (4y^3 : (x-4y)(x-y)^2).$$

The branching points are at the singular points (take the affine chart $y = 1$, and find the zeroes of $3(x-1)(x-3)$—the derivative of $(x-4)(x-1)^2$—and an extra branching at $x = \infty$.):

$x = 1$ ($\Delta = 0$ i.e j-invariant is ∞) corresponds to the semistable point (class of the nodal curve). It has preimages B with $k(B) = 1$ and B^* with $k(B^*) = 4$. The point B has multiplicity two. Notice that B and B^* represent the two semistable points in $\mathbf{Noc}//G$ (in the GIT quotient $B \sim D \sim E$ and $B^* \sim D^* \sim E^*$).

$x = \infty$ (j-invariant is 0) corresponds to the degree 4 orbit of elliptic curves defined by $a = 0$. It has one preimage (of multiplicity 3): the orbit of nets of conics defined by $J_{12} = 0$.

$x = 3$ (j-invariant is 1) corresponds to the degree 6 orbit of elliptic curves defined by $b = 0$. It has one preimage with $k = 3$ and another one with multiplicity 2, the degree 6 orbit of nets of conics defined by $J_6 = 0$.

Remark 6.3. For hypersurfaces the intersection multiplicities agree with the algebraic (or scheme theoretic) multiplicities. To determine these multiplicities in our case it is enough to compare degrees. One obtains that all algebraic multiplicities are 1, except for B which has multiplicity 2.

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