Chapter 1

Affine and Euclidean Geometry

1.1 Points and vectors

We recall coordinate plane geometry from Calculus.

The set \( \mathbb{R}^2 \) will be called the plane. Elements of \( \mathbb{R}^2 \), that is ordered pairs \((x, y)\) of real numbers, are called points.

Consider directed segments (also called “arrows”) between points of the plane. We allow the start point and the end point of an arrow to coincide. Arrows up to translation are called (plane) vectors. That is, the arrow from \( A = (1, 3) \) to \( B = (5, 6) \) represents the same vector as the arrow from \( C = (-4, -4) \) to \( D = (0, -1) \). We write \( \vec{AB} = \vec{CD} \). The vector \( \vec{AA} \) is called the zero-vector and denoted by \( 0 \).

We can represent a vector by an ordered pair of real numbers as well: the vector \( \vec{AB} \) where \( A = (a_1, a_2) \) and \( B = (b_1, b_2) \) will be represented by \( \langle b_1 - a_1, b_2 - a_2 \rangle \). This is a fair definition, because if \( \vec{AB} = \vec{CD} \) then \( b_1 - a_1 = d_1 - c_1 \) and \( b_2 - a_2 = d_2 - c_2 \). The vector \( \vec{AB} \) of the paragraph above is \( \langle 4, 3 \rangle \).

1.2 Linear operations on vectors

The sum and difference of two vectors are defined geometrically in Figure 1.1. In this context real numbers will also be called scalars. A scalar multiple of a vector is defined in Figure 1.2.

The operations above (addition, subtraction, multiplication by a scalar) are called the linear operations on vectors. The geometric definitions above translate to the following algebraic expressions.
CHAPTER 1. AFFINE AND EUCLIDEAN GEOMETRY

Figure 1.1: Sum and difference

- \( \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \)
- \( \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle \)
- \( \lambda \cdot \langle a_1, a_2 \rangle = \langle \lambda a_1, \lambda a_2 \rangle \)

Proposition 1.2.1 (Vector space “axioms”). The linear operations on vectors satisfy the following properties.

- \( \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \)
- \( (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \)
- \( \mathbf{a} + \mathbf{0} = \mathbf{a} \)
- \( \mathbf{a} + (-\mathbf{a}) = \mathbf{0} \)
- \( \lambda \cdot (\mathbf{a} + \mathbf{b}) = \lambda \cdot \mathbf{a} + \lambda \cdot \mathbf{b} \)
- \( (\lambda + \mu) \cdot \mathbf{a} = \lambda \cdot \mathbf{a} + \mu \cdot \mathbf{a} \)
- \( \lambda \cdot (\mu \cdot \mathbf{a}) = (\lambda \mu) \cdot \mathbf{a} \)
1.3. CONVENTION ON IDENTIFYING POINTS WITH VECTORS

Figure 1.2: Scalar multiple

- $1 \cdot \mathbf{a} = \mathbf{a}$

Proof. The properties follow from the algebraic expressions for the linear operations.

Proposition 1.2.2 (2-dimensionality). Let $\mathbf{a}$ and $\mathbf{b}$ be non-parallel vectors (algebraically $a_1b_2 - a_2b_1 \neq 0$). For a vector $\mathbf{c}$ there are unique $\lambda, \mu$ real numbers such that $\mathbf{c} = \lambda \cdot \mathbf{a} + \mu \cdot \mathbf{b}$.

1.3 Convention on identifying points with vectors

To a point $A \in \mathbb{R}^2$ we can associate its “position vector” $\overrightarrow{OA}$ where $O = (0, 0)$ is the origin. To a vector $\mathbf{v}$ we can associate a point $P$ by considering an arrow $\overrightarrow{OP}$ representing $\mathbf{v}$.

The above two associations are inverses of each other, they define a one-to-one correspondence between points and vectors. Algebraically this one-to-one correspondence is $(a, b) \leftrightarrow \langle a, b \rangle$.

Throughout this text we will build in this identification in our notation without further explanation. For example, if $A$ is a point, and we write $5A$ then we really mean either the vector $5\overrightarrow{OA}$ or its endpoint. Or, if we say $A/2 + B/2$ is the midpoint of the segment $AB$ then here is how to read it precisely: the midpoint of the segment $AB$ is the endpoint of the vector $\frac{1}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OB}$.

1.4 Algebraic conditions expressing collinearity

The word “collinear” is a shorthand expression for “on the same line”. The word “concurrent” is a shorthand expression for “intersecting in one point”.

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Proposition 1.4.1. Let $A$ and $B$ be two different points. Point $C$ is on the line through $A$ and $B$ if and only if there is a real number $t$ such that

$$C = (1 - t)A + tB. \quad (1.1)$$

Moreover, $t$ and $C$ uniquely determine each other (i.e. for any $C$ on the $AB$ line there is a unique real number $t$, and for any real number $t$ there is a unique point $C$ on the $AB$ line satisfying (1.1).)

Proof. To obtain the position vector of a point $C$ on the line $AB$ we need to add the vector $\vec{A}$ and a multiple of $\vec{AB}$, see Figure 1.3.

Conversely, the end point of any such vector is obviously on the line $AB$. Observe that $\vec{A} + t \cdot \vec{AB} = (1 - t)A + tB$, which proves the proposition.

The proof that $C$ and $t$ determine each other is left as an exercise.

If $A = B$ then for any $t$ the point $(1 - t)A + tB$ obviously coincides with $A$ and $B$.

A useful rephrasing of Proposition 1.4.1 is that if $A \neq B$ then $C$ is on their line if and only if there exist numbers $x$ and $y$ such that

$$C = xA + yB, \quad x + y = 1.$$ 

Proposition 1.4.2. The points $A, B, C$ are collinear if and only if there exist real numbers $x, y, z$ not all 0, such that

$$xA + yB + zC = 0, \quad x + y + z = 0.$$
1.5. THE RATIO ($\vec{AC} : \vec{CB}$) FOR COLLINEAR POINTS $A \neq B, C$

Proof. Suppose $A, B, C$ are collinear.

If $A$ and $B$ are different points, then $C$ is on their line. According to Proposition 1.4.1 then there is a $t$ such that $C = (1 - t)A + tB$. After rearrangement we obtain $(1 - t)A + tB + (-1)C = 0$ and hence $1 - t, t, -1$ serve as $x, y, z$.

If $A = B$ then $x = 1, y = -1, z = 0$ satisfies the requirements.

To prove the opposite direction let us now assume that $xA + yB + zC = 0$, $x + y + z = 0$, and not all $x, y, z$ are 0. Then pick one non-zero among $x, y, z$. Without loss of generality we may assume it is $z$. Rearrangement gives $C = (-x/z)A + (-y/z)B$. The condition $x + y + z = 0$ translates to $(-x/z) + (-y/z) = 1$. If $A$ and $B$ are different then Proposition 1.4.1 implies that $C$ is on the $AB$ line. If $A$ and $B$ coincide then the remark after Proposition 1.4.1 implies that all $A, B, C$ are the same point, so they are collinear.

An important logical consequence of Proposition 1.4.2 is the following Corollary.

Corollary 1.4.3. If $A, B, C$ are not collinear (i.e. they form a triangle), and $x, y, z$ are real numbers with
\[ xA + yB + zC = 0, \quad x + y + z = 0, \]
then $x = y = z = 0$.

1.5 The ratio ($\vec{AC} : \vec{CB}$) for collinear points $A \neq B, C$

Let $A, B \neq C$ be collinear points. We will write that $(\vec{AC} : \vec{CB}) = \lambda$ if $\vec{AC} = \lambda \vec{CB}$. Such a $\lambda$ exists (and is unique) since $\vec{AC}$ and $\vec{CB}$ are collinear vectors with $\vec{CB} \neq 0$. Sometimes it is useful to extend this notion to the case when $B = C$ (but $A$ is not equal to them): in this case we define $(\vec{AC} : \vec{CB}) = \infty$.

Lemma 1.5.1. Let $A, B, C$ be collinear, $A \neq B$, and write $C = xA + yB$ with $x + y = 1$ (cf. Proposition 1.4.1). Then we have $(\vec{AC} : \vec{CB}) = y/x$.

Proof. From $C = xA + yB$ we obtain $\vec{AC} = y\vec{AB}$ and $\vec{CB} = x\vec{AB}$. Hence $(\vec{AC} : \vec{CB}) = y/x$ (note the $(\vec{AC} : \vec{CB}) = \infty$ convention if $C = B$, that is if $x = 0$).

In fact we can interpret the ratio $(\vec{AC} : \vec{CB})$ without mentioning vectors. It is the ratio of the length of the segment $AC$ over the length of the segment $CB$, with a sign convention. The sign convention is that if $C$ is in between $A$ and $B$, then $(\vec{AC} : \vec{CB})$ is positive, and if $C$ is outside of the segment $AB$ then $(\vec{AC} : \vec{CB})$ is negative.

Proposition 1.5.2. Let $A \neq B$ be fixed. The ratio $(\vec{AC} : \vec{CB})$ uniquely determines $C$. 
Remark 1.5.3. We may be sloppy in notation and decide to write $\vec{AC}/\vec{CB}$ instead of $(\vec{AC} : \vec{CB})$, but we must be careful that this ratio is only defined in the very special situation where $A, B, C$ are collinear (and some coincidences do not happen). In general there is no such operation where we divide a plane vector by another plane vector!

1.6 First applications

A quadrilateral $ABCD$ is a parallelogram if $\vec{AB} = \vec{DC}$. This condition can be phrased as $B - A = C - D$, or rearranged to $D - A = C - B$, which means $\vec{AD} = \vec{BC}$ also holds.

**Proposition 1.6.1.** The diagonals of a parallelogram bisect each other.

![Figure 1.4: The diagonals of a parallelogram bisect each other.](image)

**Proof.** Let $ABCD$ be a parallelogram. Since it is a parallelogram, we have $\vec{AB} = \vec{DC}$ (denote this vector by $x$), $\vec{AD} = \vec{BC}$. These equalities imply that $(A + C)/2 = (B + D)/2$. Indeed, $B = A + x$, $D = C - x$, and hence $(B + D)/2 = ((A + x) + (C - x))/2$.

Now consider the point $P = (A + C)/2 = (B + D)/2$. The first defining expression implies that $P$ is the midpoint of $A$ and $C$. The second expression implies that $P$ is the midpoint of $B$ and $D$. Since they agree, $P$ is the intersection of $AC$ and $BD$, and it bisects both diagonals.

The median of a triangle is a segment connecting a vertex to the midpoint of the opposite side. A triangle has three medians.

**Proposition 1.6.2.** The medians of a triangle are concurrent. Moreover they divide each other by 2:1.
1.7. MENELAUS’ THEOREM

Proof. Let $ABC$ be a triangle. Consider the point $P = (A + B + C)/3$, and its equivalent expressions

$$P = \frac{2}{3} \cdot \frac{A + B}{2} + \frac{1}{3} \cdot C = \frac{2}{3} \cdot \frac{B + C}{2} + \frac{1}{3} \cdot A = \frac{2}{3} \cdot \frac{C + A}{2} + \frac{1}{3} \cdot B.$$  

The first expression claims that $P$ is on the segment connecting the midpoint of $A$ and $B$ with $C$, that is, on the median corresponding to $C$. The second expression claims that $P$ is on the median corresponding to $A$, and the third expression claims that $P$ is on the median corresponding to $B$. Since they are all equal, there is a point, namely $P$, that is on all three medians; and we proved that the medians are concurrent.

A byproduct of the above argument is that the intersection $P$ of the three medians is expressed as $2/3$ of the midpoint of a side plus $1/3$ the opposite vertex. According to Section 1.5 this proves that $P$ cuts the median $2 : 1.$

Remark 1.6.3. In the above two propositions we needed to make arguments about the intersections of certain lines. In our proofs we used a trick: we did not “compute” the intersections, but rather we “named” a point and then proved that this point is on the lines, and hence this point must be the intersection. You will find this trick useful when solving exercises.

PROJECT 1. Invent and prove the $3D, 4D, \ldots$ versions of Proposition 1.6.2.

1.7 Menelaus’ theorem

Theorem 1.7.1 (Menelaus’ theorem). Let $ABC$ be a triangle and let a transversal line $\ell$ intersect the lines of the sides $AB$, $BC$, $CA$ in $M$, $K$, $L$, respectively. We assume that none
The sum of the coefficients in this last expression is
\[ (rz + qy') + (px + rz') + (qy + px') = p(x + x') + q(y + y') + r(z + z') = p + q + r = 0 \] (1.3)
According to Corollary 1.4.3 (1.2) and (1.3) can only hold if all three
\[ rz + qy' = px + rz' = qy + px' = 0. \]
Therefore we have
\[ y'/z = -r/q, \quad z'/x = -p/r, \quad x'/y = -q/p, \]
and hence
\[ (A\vec{M} : M\vec{B}) \cdot (B\vec{K} : K\vec{C}) \cdot (C\vec{L} : L\vec{A}) = \frac{z'}{z} \cdot \frac{x'}{x} \cdot \frac{y'}{y} = \frac{z'}{x} \cdot \frac{x'}{y} \cdot \frac{y'}{z} = \frac{-p}{r} \cdot \frac{-q}{p} \cdot \frac{-r}{q} = -1 \]
what we wanted to prove.
1.8. BARYCENTRIC COORDINATES

Theorem 1.7.2 (reverse Menelaus’ theorem). Let $ABC$ be a triangle and let $M, K, L$ be points on the lines $AB, BC, CA$ such that

$$(A\tilde{M} : \tilde{M}B) \cdot (B\tilde{K} : \tilde{K}C) \cdot (C\tilde{L} : \tilde{L}A) = -1. \quad (1.4)$$

Then $K, L, M$ are collinear.

Proof. Let $\ell$ be the line connecting $L$ and $K$. In Problem 15 you will prove that $\ell$ intersects the $AB$ line. Let the intersection point be $M'$. According to Menelaus’ theorem we have

$$(A\tilde{M}' : \tilde{M}'B) \cdot (B\tilde{K} : \tilde{K}C) \cdot (C\tilde{L} : \tilde{L}A) = -1.$$  

Comparing this with the assumption (1.4) on $K, L, M$ we conclude that

$$(A\tilde{M}' : \tilde{M}'B) = (A\tilde{M} : \tilde{M}B).$$

Proposition 1.5.2 then implies that $M = M'$, hence the fact that $K, L, M$ are collinear. \qed

1.8 Barycentric coordinates

Theorem 1.8.1. Let $A, B$ and $C$ be non-collinear points in the plane. For any point $P$ we may write

$$P = xA + yB + zC$$

where the real coefficients $x, y, z$ satisfy

$$x + y + z = 1.$$  

Moreover, $x, y, z$ are uniquely determined by the point $P$.

We call $x, y, z$ the barycentric coordinates of $P$ with respect to the triangle $ABC$. 
Proof. The vectors $\vec{AB}$ and $\vec{AC}$ are not parallel. Hence any vector can be written as a linear combination of them, for example $\vec{AP} = p\vec{AB} + q\vec{AC}$. Using that $\vec{AP} = P - A$, $\vec{AB} = B - A$, $\vec{AC} = C - A$ we can rearrange it to

$$P = (1 - p - q)A + pB + qC,$$

and hence $x = 1 - p - q, y = p, z = q$ satisfy the requirements.

To prove the uniqueness of barycentric coordinates assume that $x, y, z$ and $x', y', z'$ are such that

$$P = xA + yB + zC, \quad x + y + z = 1,$$

$$P = x'A + y'B + z'C, \quad x' + y' + z' = 1.$$

Then we have

$$0 = (x - x')A + (y - y')B + (z - z')C, \quad (x - x') + (y - y') + (z - z') = 0.$$

Corollary 1.4.3 implies that $x - x' = y - y' = z - z' = 0$ which proves uniqueness. \qed

**PROJECT 2.** Observe the similarity between Proposition 1.4.1 and Theorem 1.8.1. They are the 1D and 2D cases of a general $n$-dimensional theorem. If you learned linear algebra (specifically the notions of linear independence, generating set, basis) then find and prove this general $n$-dimensional theorem.
1.9 Ceva’s theorem

Theorem 1.9.1 (Ceva’s Theorem\textsuperscript{1}). Let $ABC$ be a triangle and let $P$ be a point in the plane which does not lie on any of the sides of $\triangle ABC$. Suppose the lines $AP$, $BP$ and $CP$ meet the opposite sides of $ABC$ at $D$, $E$ and $F$, respectively. Then

$$(\overrightarrow{AF} : \overrightarrow{FB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = 1.$$ 

Note that $P$ does not need to lie inside the triangle.

Proof. Using barycentric coordinates, we write $P$ as

$$P = xA + yB + zC,$$

where $x + y + z = 1$. Let us consider the point

$$V = \frac{x}{x+y}A + \frac{y}{x+y}B.$$ \hspace{1cm} (1.5)

This expression implies that $V$ is on the line $AB$. Calculation shows that

$$V = \frac{1}{x+y}P + \frac{-z}{x+y}C,$$

\textsuperscript{1}Geometer Giovanni Ceva (1647–1734) is credited with this theorem.
and the sum of the coefficients $1/(x+y)+(-z)/(x+y) = (1-z)/(x+y) = (x+y)/(x+y) = 1$. Hence $V$ is also on the line $CP$. We conclude that the point $V$ is the intersection of $AB$ and $CP$, hence $V = F$. Moreover, from 1.5 we obtain that

$$(\vec{AF} : \vec{FB}) = \frac{y}{x+y} = \frac{y}{x}.$$ 

Similarly, we find that

$$(\vec{BD} : \vec{DC}) = \frac{z}{y}, \quad (1.6)$$
$$(\vec{CE} : \vec{EA}) = \frac{x}{z}. \quad (1.7)$$

Hence

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = \frac{y}{x} \frac{z}{y} \frac{x}{z} = 1.$$ 

\[\square\]

**Theorem 1.9.2** (Reverse Ceva’s theorem). Suppose $ABC$ is a triangle, $D, E, F$ are points on the lines of the sides (but none of them coincide with a vertex) such that

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = 1.$$ 

Then $AD, BE, CF$ are either concurrent, or they are pairwise parallel.

**Proof.** If $AD, BE, CF$ are pairwise parallel, then the theorem is proved. Assume that two of these three lines intersect. Without loss of generality we assume that it is $AD$ and $BE$. Let $P = AD \cap BE$, and assume that $CF$ intersects $AB$ is the point $F'$. By Ceva’s Theorem 1.9.1 we have

$$(\vec{AF'} : \vec{F'B}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = 1.$$ 

Comparing this with the condition in the Theorem we obtain that

$$(\vec{AF'} : \vec{F'B}) = (\vec{AF} : \vec{FB}).$$

Proposition 1.5.2 then implies that $F = F'$, hence the fact that $AD, BE, CF$ are concurrent. \[\square\]

For fun, let us include here another “high-school style” proof of Ceva’s Theorem 1.9.1. This proof does not use vectors at all. Instead it uses the notion of area, and the obvious fact that the area of a triangle is half the product of base and height.
Proof. For simplicity let $D, E, F$ be on the sides (and not outside) of the triangle $ABC$, and let $P = AB \cap CF = BC \cap AD = CA \cap BE$. The triangles $AFC$ and $FBC$ have “bases” $AF$ and $FB$ and they share the same height $m_C$. Hence the ratio of their areas equals the ratio of their bases:

$$\frac{\text{Area}(AFC)}{\text{Area}(FBC)} = \frac{AF}{FB}.$$  

Similar argument for the triangles $AFP$ and $FBP$ gives

$$\frac{\text{Area}(AFP)}{\text{Area}(FBP)} = \frac{AF}{FB}.$$  

From the two equations above simple algebra implies

$$\frac{\text{Area}(AFC) - \text{Area}(AFP)}{\text{Area}(FBC) - \text{Area}(FBP)} = \frac{AF}{FB}.$$  

The difference of the triangles $AFC$ and $AFP$ is the triangle $CAP$. The difference of the triangles $FBC$ and $FBP$ is the triangle $BCP$. Hence we obtained

$$\frac{\text{Area}(CAP)}{\text{Area}(BCP)} = \frac{AF}{FB}. \quad (1.8)$$  

We obtained (1.8) by considering the $AB$ side of the triangle the “base”. Repeating the same argument but now considering the $BC$ side or the $CA$ side to be the “base” we obtain the equations.

$$\frac{\text{Area}(ABP)}{\text{Area}(CAP)} = \frac{BD}{DC},$$  

$$\frac{\text{Area}(BCP)}{\text{Area}(ABP)} = \frac{CE}{EA}.$$  

From the last three equations we obtain

$$(\bar{A}F : F\bar{B}) \cdot (\bar{B}D : D\bar{C}) \cdot (\bar{C}E : E\bar{A}) = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} =$$

$$= \frac{\text{Area}(CAP)}{\text{Area}(BCP)} \cdot \frac{\text{Area}(ABP)}{\text{Area}(CAP)} \cdot \frac{\text{Area}(BCP)}{\text{Area}(ABP)} = 1,$$

which proves Ceva’s theorem in the case when $P$ is inside the triangle. Similar arguments work when $P$ is outside. \qed
1.10 Desargues’ theorem—a few affine versions

Desargues’ theorem is a remarkable theorem on incidences of certain lines and points involving two triangles. The key notions are as follows.

- For $ABC\triangle$ and $A'B'C'\triangle$ we may consider the three lines connecting the corresponding vertexes: $AA'$, $BB'$, and $CC'$. We will consider the condition that these three lines are concurrent (or are pairwise parallel). If concurrent, we call the intersection point the center of perspectivity.

- For $ABC\triangle$ and $A'B'C'\triangle$ we may consider the intersections of the corresponding sides $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$. We will consider the condition that these three points (exist and) are collinear—or none of the three exist. If collinear, we call the obtained line the axis of perspectivity.

**Theorem 1.10.1.** Let $ABC$ and $A'B'C'$ be triangles such that $AB\parallel A'B'$, $AC\parallel A'C'$, $BC\parallel B'C'$. Then the three lines $AA'$, $BB'$, $CC'$ are either concurrent or pairwise parallel.

**Proof.** Because of the conditions on parallel lines we can write

$$B - C = k_1(B' - C'), \quad C - A = k_2(C' - A'), \quad A - B = k_3(A' - B'), \quad (1.9)$$

for some real numbers $k_1, k_2, k_3$. Adding together these three equalities (and rearranging the right hand side) we obtain

$$0 = (k_2 - k_3)A' + (k_3 - k_1)B' + (k_1 - k_2)C'. \quad (1.10)$$

The coefficients of (1.10) add up to 0, hence Corollary 1.4.3 implies that

$$k_2 - k_3 = k_3 - k_1 = k_1 - k_2 = 0, \quad \text{and hence} \quad k_1 = k_2 = k_3.$$

Let $k$ be the common value of $k_1, k_2$ and $k_3$. Then from (1.9) we can deduce

$$A - kA' = B - kB' = C - kC'. \quad (1.11)$$

We can consider two cases. If $k = 1$ then (1.11) implies that the lines $AA'$, $BB'$, $CC'$ are pairwise parallel. If $k \neq 1$ then (1.11) can be rearranged to

$$P = \frac{1}{1-k}A + \frac{-k}{1-k}A' = \frac{1}{1-k}B + \frac{-k}{1-k}B' = \frac{1}{1-k}C + \frac{-k}{1-k}C'$$

showing that the point $P$ is on all three lines $AA'$, $BB'$, $CC'$—proving that $AA'$, $BB'$, $CC'$ are concurrent.
Theorem 1.10.2. Let \(ABC\) and \(A'B'C'\) be triangles such that \(AA', BB', CC'\) are concurrent. Assume that the point \(K = AB \cap A'B, L = BC \cap B'C', M = CA \cap C'A'\) exist. Then the points \(K, L, M\) are collinear.

Proof. The intersection point on \(AA', BB', CC'\) can be written as

\[ k_1A + (1-k_1)A' = k_2B + (1-k_2)B' = k_3C + (1-k_3)C'. \]  \((1.12)\)

Rearranging, say, the first equality we obtain

\[ k_1A - k_2B = (1-k_2)B' - (1-k_1)A'. \]

If \(k_1 = k_2\) then this formula says \(k_1(A - B) = (1-k_1)(B' - A')\), meaning that \(AB || A'B'\) which is not the case. So we know that \(k_1 \neq k_2\). Hence, we may divide by \(k_1 - k_2\) and write

\[ \frac{k_1}{k_1-k_2}A + \frac{-k_2}{k_1-k_2}B = \frac{1-k_2}{k_1-k_2}B' + \frac{-(1-k_1)}{k_1-k_2}A'. \]

The sum of the coefficients on the left hand side is 1, and the sum of the coefficients on the right hand side is also 1 (check it!). Therefore the left hand side expression is a point on the \(AB\) line, and the right hand side expression is a point on the \(A'B'\) line. Hence the common value must be the intersection \(AB \cap A'B'\). We obtained that

\[ K = \frac{k_1}{k_1-k_2}A + \frac{-k_2}{k_1-k_2}B, \]

equivalently

\[ (k_1 - k_2)K = k_1A - k_2B. \]  \((1.13)\)

We deduced (1.12) from the fact that the first expression and the second expression in (1.12) are equal. Similarly, the fact that the second and third, as well as the first and third expressions in (1.12) are equal we obtain

\[ (k_2 - k_3)L = k_2B - k_3C, \quad (k_3 - k_1)M = k_3C - k_1A. \]  \((1.14)\)

Adding together all three equalities in (1.13) and (1.14) we get

\[ 0 = (k_1 - k_2)K + (k_2 - k_3)L + (k_3 - k_1)M. \]

Observe that none of the three coefficients are 0, and they add up to 0. According to Proposition 1.4.2 this means that \(K, L, M\) are collinear, what we wanted to prove. \(\square\)
Theorem 1.10.3. Let $ABC$ and $A'B'C'$ be triangles such that $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist and are collinear. Then $AA'$, $BB'$, $CC'$ are either concurrent or are pairwise parallel.

It is possible to prove this theorem with the techniques we used in the last two proofs—and it may be a good practice for students to write down such a proof. However, for a change we are going to prove it by reduction to Theorem 1.10.2—showing that in some sense Desargues’ theorem and its reverse are the same, in other words, Desargues’ theorem is “self-dual”.

Proof. If $AA'$, $BB'$, $CC'$ are pairwise parallel, then we are done. If not, then two of them intersect, say, $AA'$ intersects $BB'$.

Consider the triangles $AA'M$ and $BB'L$. By looking at the picture one can see that the lines $AB$, $A'B'$, $ML$ connecting the corresponding vertexes are concurrent. Theorem 1.10.2 can be applied to the triangles $AA'M$ and $BB'L$, and we obtain that $C = AM \cap BL$, $C' = MA' \cap LB'$, and $AA' \cap BB'$ are collinear. That is, $CC'$ passes through the intersection of $AA' \cap BB'$, and hence $AA'$, $BB'$, and $CC'$ are concurrent.
1.11. DESARGUES TRIANGLES IN INTERSECTING PLANES

In a later chapter we will see a simple and elegant way of phrasing Desargues’ theorem—in projective geometry. All of the three theorems above (and more) are some special cases of that projective Desargues’ theorem.

**PROJECT 3.** *We can connect two points \(A\) and \(B\) of the plane if we have a straightedge. Now suppose that \(B\) is “hidden”, it is only given by portions of two intersecting lines, but we cannot go close to the intersection point \(B\); for example it is outside of the margin of our paper. How can we connect \(A\) and \(B\) with a straightedge? How to connect two hidden points?*

### 1.11 Desargues triangles in intersecting planes

In this section we go out of our way again and show an interesting “high-school style” argument in relation with Desargues’ theorem.

Consider two planes \(P_1\) and \(P_2\) in 3 dimensions, intersecting in the line \(\ell\). Let \(ABC\Delta\) be in \(P_1\) and let \(A'B'C'\Delta\) be in \(P_2\). We will analyse the conditions and the claim of Theorem 1.10.3 for these two triangles.

The condition is about the three points \(K = AB \cap A'B', L = BC \cap B'C', M = CA \cap C'A'\). Observe that \(AB \subset P_1, A'B' \subset P_2\), hence \(K \in P_1 \cap P_2 = \ell\). Similarly, \(L\) and \(M\) must also
lie on \( \ell \). So the assumption of Theorem 1.10.3 that \( K, L, M \) are collinear does not even have to be assumed! It automatically holds.

The statement of Theorem 1.10.3 is about the three lines \( AA', BB', CC' \). It will be useful to consider three more planes. Let \( S_K \) be the plane containing the intersecting lines \( AB, A'B' \). Let \( S_L \) be the plane containing the intersecting lines \( BC, B'C' \). Let \( S_M \) be the plane containing the intersecting lines \( AC, A'C' \).

Observe that both \( A \) and \( A' \) are contained in \( S_K \) and in \( S_M \). If two points are contained in two planes, then their connecting line must be the intersection of the two planes. We have \( AA' = S_K \cap S_M \). Similarly \( BB' = S_K \cap S_L \), and \( CC' = S_L \cap S_M \).

We obtained that the three lines \( AA', BB', CC' \) are the pairwise intersections of three planes in space. Let's see how can three (pairwise intersecting) planes look like in three space. There are two possibilities: (i) either the third one is parallel with the intersection line of the first two, or (ii) the third one intersects the intersection line of the first two. Theses two configurations are illustrated in the picture.

In the first case the three intersection lines are pairwise parallel: \( AA', BB', CC' \) are pairwise parallel. In the second case the three intersection lines are concurrent: \( AA', BB', CC' \) are concurrent.
CC’ are concurrent.

What we found is that the 3D version of Desargues theorem 1.10.3 is a tautology.

**PROJECT 4.** Analyses 3D versions of the other two versions of Desargues’ theorem above.

**PROJECT 5.** Find a no-calculation proof of e.g. Theorem 1.10.3, by first moving one of the triangles out of plane into 3D.

### 1.12 Dot product: algebra and geometry

Let us recall the notion of dot product from Calculus. The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a number denoted by $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$.

Geometrically $\mathbf{ab} = |\mathbf{a}| |\mathbf{b}| \cos \phi$, where $|\mathbf{x}|$ denotes the length of a vector $\mathbf{x}$ and $\phi$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Especially, $\mathbf{ab} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Algebraically $(a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$.

Problem: recall from calculus why the above geometric and algebraic definition agree.

The following properties are easily verified from the algebraic definition.

- $\mathbf{ab} = \mathbf{ba}$
- $(\mathbf{a} + \mathbf{b}) \mathbf{c} = \mathbf{ac} + \mathbf{bc}$
- $(\lambda \mathbf{a}) \mathbf{b} = \lambda (\mathbf{ab})$
- $\mathbf{aa} \geq 0$, and $\mathbf{aa} = 0$ if and only if $\mathbf{a} = \mathbf{0}$
- $\mathbf{ab} = 0$ for all $\mathbf{b}$ implies that $\mathbf{a} = \mathbf{0}$.

The power of dot product that we will repeatedly use in geometry is the duality: its clear geometric meaning and its simple algebraic properties. (What we will not use any further is the $a_1b_1 + a_2b_2$ expression.)

**Definition 1.12.1.** The length of a vector $\mathbf{a}$ is defined to be $|\mathbf{a}| = \sqrt{\mathbf{aa}}$. (The square root makes sense because of the the non-negativity property above.) The length of a segment $\overline{AB}$ is defined to be $d(\overline{AB}) = |\overrightarrow{AB}|$. The distance of two sets $P, Q \subset \mathbb{R}^2$ is defined to be $\inf\{d(A, B) : A \in P, B \in Q\}$. 
1.13 Altitudes of a triangle are concurrent

Let $ABC$ be a triangle. A line passing through the vertex and perpendicular to the opposite side is called an altitude. A triangle has three altitudes.

**Theorem 1.13.1.** The three altitudes of a triangle are concurrent.

**Proof.** Let $D$ be the intersection of the altitudes containing the vertexes $A$ and $B$. Then $AD \perp BC$ and $BD \perp AC$. Hence we have

$$(D - A)(B - C) = 0, \quad (D - B)(C - A) = 0.$$ 

Adding these two equations together, and using the algebraic properties of dot product we obtain

$$0 = (D - A)(B - C) + (D - B)(C - A) = ... = (D - C)(A - B).$$

Therefore $D - C \perp A - B$, that is, the line $DC$ is the altitude containing $C$. All three altitudes pass through $D$.

The intersection of the three altitudes is called the orthocenter of the triangle.

Consider the vertexes and the orthocenter. It is remarkable that each of these four points is the orthocenter of the triangle formed by the other three points. Such a set of four points will be called an orthocentric tetrad.

1.14 Feuerbach circle

**Lemma 1.14.1.** If $A, B, C, D$ is an orthocentric tetrad then

$$(A + B - C - D)^2 = (A - B + C - D)^2 = (A - B - C + D)^2 =$$
1.14. FEUERBACH CIRCLE

\((-A + B + C - D)^2 = (-A + B - C + D)^2 = (-A - B + C + D)^2.\)

**Proof.** The six numbers above are in fact three pairs, e.g. \((A + B - C - D)^2\) and \((-A - B + C + D)^2\) are clearly equal, because they are length squares of a vector and its opposite vector. What we need to prove is that two numbers not in the same pair are also equal. Without loss of generality let us choose the first two. Calculation shows that

\[(A + B - C - D)^2 - (A - B + C - D)^2 = 4(A - D)(B - C).\]

Since \(A, B, C, D\) is an orthocentric tetrad \(A - D\) is orthogonal to \(B - C\), and hence \((A - D)(B - C) = 0\), showing that \((A + B - C - D)^2 = (A - B + C - D)^2.\)

**Theorem 1.14.2** (Feuerbach circle). Let \(D\) be the orthocenter of the \(ABC\) triangle. Consider the following nine points (a) the midpoints of the sides, (b) the midpoints of the segments connecting vertexes to the orthocenter, (c) the feet of the altitudes. These nine points are on one circle.

**Proof.** We will only prove that points (a) and (b) are on one circle. The fact that points (c) are also on the same circle is left as an exercise.

Observe that the six points in (a) and (b) are the midpoints of the six segments connecting two of \(A, B, C, D\) where \(A, B, C, D\) form an orthocentric tetrad. Hence these points are

\[(A + B)/2, (A + C)/2, (A + D)/2, (B + C)/2, (B + D)/2, (C + D)/2.\]
Let $N = (A + B + C + D)/4$. The vectors connecting $N$ to the six points are

\[ (-A - B + C + D)/4, (-A + B - C + D)/4, (-A + B + C - D)/4, \]
\[ (A - B - C + D)/4, (A - B + C - D)/4, (A + B - C - D)/4. \]

These six vectors have the same length because of Lemma 1.14.1. Therefore all six points are of the same distance from $N$: they are on one circle.

\[ \square \]

### 1.15 Angle sum of a triangle

**Lemma 1.15.1.** Let the line $m$ intersect a pair of parallel lines $\ell$, $\ell'$; and let $\alpha$ and $\alpha'$ be the angles obtained as in Figure ? (a). Then $\alpha = \alpha'$.

**Proof.** Take unit vectors $x$, $-x$ and $u$ in the lines $\ell$, $\ell'$, and $m$ as in the picture. Then

\[ \alpha = \arccos(x \cdot u), \quad \alpha' = \arccos((-x) \cdot (-u)), \]

so they obviously agree. \[ \square \]

**Theorem 1.15.2.** The sum of the angles of a triangle is $\pi$.

**Proof.** Let the line $\ell'$ be parallel to $AB$ and pass through the point $C$, see Picture ? (b). According to Lemma 1.15.1 $\alpha = \alpha'$ and $\beta = \beta'$ hence we have

\[ \alpha + \beta + \gamma = \alpha' + \beta' + \gamma = \pi. \]

\[ \square \]
1.16 Law of cosines, law of sines

For a triangle $\triangle ABC$ the sides opposite to $A, B, C$ will be denoted by $a, b, c$ respectively, and the angles at $A, B, C$ will be called $\alpha, \beta, \gamma$ respectively.

**Theorem 1.16.1** (Law of Cosines). We have

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$  

*Proof.* We have $c^2 = (a - b)^2 = a^2 + b^2 - 2ab = a^2 + b^2 - 2ab \cos \gamma$.  

**Corollary 1.16.2** (Pythagorean theorem). In a right triangle with hypothenuse $c$ we have $c^2 = a^2 + b^2$.

*Proof.* This is the Law of Cosines for $\gamma = \pi/2$.

**Corollary 1.16.3** (Triangle inequality). For three points $A, B, C$ in the plane we have

$$d(AB) \leq d(BC) + d(CA).$$

The proof follows from the Law of Cosines, details are left as an exercise.

**Theorem 1.16.4** (Law of Sines). We have

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$  

We will give two proves: the first one shows that the Law of Sines is a formal consequence of the Law of Cosines. The second proof is geometric.

*Proof.* Proof1. From the Law of Cosines we get $\cos \gamma = (a^2 + b^2 - c^2)/(2ab)$.

Using $\sin \gamma = \sqrt{1 - \cos^2 \gamma}$ we have

$$\sin \gamma = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} = \sqrt{\frac{4a^2b^2 - (a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2)}{4a^2b^2}} = \sqrt{\frac{-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2}{2ab}}.$$

Dividing both sides by $c$ we obtain

$$\frac{\sin \gamma}{c} = \text{an expression symmetric in } a, b, c.$$

Therefore we will get the same expression on the right hand side, if we start with $\alpha$ or $\beta$, not $\gamma$. This proves that $\sin \gamma/c = \sin \alpha/a = \sin \beta/b$.  

*Proof.* Proof2. The altitude $m$ passing through vertex $C$ is a side of two right triangles (See picture) yielding the two expressions: $m = b \sin \alpha, m = a \sin \beta$. Putting the right hand sides equal and rearranging gives $\sin \alpha/a = \sin \beta/b$.  

\[\]
1.17 Angle bisectors, perpendicular bisectors

Consider an angle $\alpha$ less than $\pi$. The ray inside the angle that cuts $\alpha$ into two angles of measure $\alpha/2$ is called the angle bisector.

The distance $d(P, \ell)$ of a point $P$ to a line $\ell$ is the the infimum of the distances between $P$ and $A$ where $A \in \ell$.

**Lemma 1.17.1.** The distance of a point to a line is obtained on the perpendicular segment dropped from the point to the line.

**Proof.** The Pythagorean theorem proves that $x < x'$ on picture (a).

**Lemma 1.17.2.** The points in an angle that are of the same distance from the two rays of the angle are exactly the points of the angle bisector.

**Proof.** For a point $P$ as in Picture (b) its distance to the two sides is $u$ and $v$ according to Lemma 1.17.1. We have $u = c \sin(\beta)$, $v = c \sin(\gamma)$. Hence $u = v$ holds if and only if $\sin(\beta) = \sin(\gamma)$. Well known properties of the sin function imply that this holds if and only if $\beta = \gamma$.

**Theorem 1.17.3.** The three angle bisectors of a triangle are concurrent.
Proof. Let \( x_a, x_b, x_c \) be the angle bisectors through the vertexes \( A, B, C \). Let \( P \) be the intersection of the \( x_a \) and \( x_b \). Then

\[
\begin{align*}
P \in x_a & \implies d(P, b) = d(P, c) \\
P \in x_b & \implies d(P, a) = d(P, c)
\end{align*}
\]

where three of the four \( \implies \) implications above use Lemma 1.17.2. Since the intersection of \( x_a \) and \( x_b \) is on \( x_c \), we have that \( x_a, x_b, x_c \) are concurrent.

A byproduct of the theorem is that the intersection of the angle bisectors has the same distance to the sides. In other words there is a circle with this center that touches the sides of the triangle: the so-called circle inscribed in the triangle.

For a segment \( AB \), the line passing through the midpoint of \( AB \) and perpendicular to \( AB \) is called the perpendicular bisector.

**Theorem 1.17.4.** The three perpendicular bisectors of the sides of a triangle are concurrent.

The proof is obtained by solving the first two of the following problems:

1. Find and prove a lemma analogous to Lemma 1.17.2 but it it about the perpendicular bisector of a segment.
2. Using your lemma from Problem 1 find a proof of Theorem 1.17.4 (logically similar to the proof of Theorem 1.17.3.
3. Find a byproduct of your proof in Problem 2, analogous to the byproduct of the proof of Theorem 1.17.3.

### 1.18 Rotation, applications

For \( \alpha \) an angle and \( v = \vec{OA} \) a plane vector let \( R_\alpha(v) \) denote the vector obtained from rotating \( v = \vec{OA} \) around the origin in the counterclockwise direction, see Picture ? (a). Thus \( R_\alpha \) is
a map from vectors to vectors. Calculation shows that algebraically

\[ R_\alpha : \langle x, y \rangle \mapsto \langle x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha \rangle. \]

If a vector is given by an arrow \( v = \vec{PQ} \) where \( P \) is not the origin, then to get \( R_\alpha(v) \) we formally need to translate \( \vec{PQ} \) to \( \vec{OA} \), then rotate this \( \vec{OA} \) by \( \alpha \). Picture (b) shows that this procedure is not necessary: \( R_\alpha(v) \) is also obtained by rotating \( \vec{PQ} \) around \( P \) by \( \alpha \).

**Proposition 1.18.1.** The rotation operator on vectors is consistent with vector operations as follows,

- \( R_\alpha(a + b) = R_\alpha(a) + R_\alpha(b) \),
- \( R_\alpha(a - b) = R_\alpha(a) - R_\alpha(b) \),
- \( R_\alpha(\lambda a) = \lambda R_\alpha(a) \),
- \( R_\alpha(a)R_\alpha(b) = ab \).

**Proof.** All three can be calculated from the algebraic description of the operations. \( \square \)

**Theorem 1.18.2.** Let \( ABC \triangle \) be a triangle, and let \( T_1 \) and \( T_2 \) squares on the sides \( AC \) and \( BC \) outside the triangle. Let \( K \) and \( L \) be the centers of \( T_1 \) and \( T_2 \), and \( X \) is the midpoint of \( AB \). Then \( XK \) and \( XL \) have the same length.
Proof. Let \( \mathbf{a} = \vec{AC} \), \( \mathbf{b} = \vec{BC} \), and let \( R = R_{\pi/2} \) be the rotation by \( \pi/2 \) operator. Observe that we can express all relevant vectors in our picture using \( \mathbf{a}, \mathbf{b}, R \): for example \( \vec{AC}'' = R(\mathbf{a}), \vec{BC}'' = -R(\mathbf{b}) \).

We have

\[
\vec{XK} = \vec{XA} + \vec{AK} = \frac{\vec{BA}}{2} + \frac{\vec{AC} + \vec{AC}''}{2} = \frac{-\mathbf{a} + \mathbf{b}}{2} + \frac{\mathbf{a} + R(\mathbf{a})}{2} = \frac{R(\mathbf{a}) + \mathbf{b}}{2},
\]

\[
\vec{XL} = \vec{XB} + \vec{BL} = \frac{\vec{AB}}{2} + \frac{\vec{BC} + \vec{BC}''}{2} = \frac{-\mathbf{b} + \mathbf{a}}{2} + \frac{\mathbf{b} - R(\mathbf{b})}{2} = \frac{\mathbf{a} - R(\mathbf{b})}{2}.
\]

The idea of the proof is that we suspect that not only \( XK \) and \( XL \) are of the same length but one is the \( \pi/2 \) rotation of the other. Hence we calculate

\[
R(\vec{XL}) = R \left( \frac{\mathbf{a} - R(\mathbf{b})}{2} \right) = \frac{R(\mathbf{a}) - R(R(\mathbf{b}))}{2}.
\]

Observe that applying \( R \) twice on a vector is the same as multiplication by \(-1\). Indeed, \( R_{\pi/2} R_{\pi/2} = R_{\pi} \) = multiplication by \((-1)\). Hence

\[
R(\vec{XL}) = \frac{R(\mathbf{a}) + \mathbf{b}}{2} = \vec{XK},
\]

what we wanted to prove.

The proof above is not the shortest or most elegant proof, but illustrates the main point: naming sufficient vectors and operations (but not more) that determine the picture we can express any other vectors in terms of the named ones. Then we can make comparisons among any two. A more “elegant” version of the same proof will be given in the exercises.
Theorem 1.18.3 (Napoleon Bonaparte\textsuperscript{2}). Let $ABC\triangle$ be an arbitrary triangle and let $T_a$, $T_b$, $T_c$ by regular (a.k.a. equilateral) triangles on the sides of $a,b,c$, outside of $ABC$. Let $A', B', C'$ be the centers of $T_a$, $T_b$, $T_c$. Then $A'B'C'\triangle$ is a regular triangle.

Proof. Let $b = \overrightarrow{AB}$ and $c = \overrightarrow{AC}$, and let $R = R_{\pi/3}$ be the rotation by $\pi/3$ operator. Our goal is to express relevant vectors, namely $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ in terms of these.

First observe that if $x$ is a side vector of a regular triangle then the vector pointing from a vertex to its center as drawn in Picture ? (b) is $(x + R(x))/3$. Therefore we have

$$A'B' = \frac{c + R(c)}{3}, \quad C'A' = \frac{b - c + R(b - c)}{3}, \quad B'C' = \frac{-b + R(-b)}{3},$$

see Picture ? (c). Now we can express

$$A'B' = -\frac{b - c + R(b - c)}{3} - c + \frac{c + R(c)}{3} = -\frac{1}{3}b - \frac{1}{3}c - \frac{1}{3}R(b) + \frac{2}{3}R(c),$$

$$A'C' = -\frac{b - c + R(b - c)}{3} - c + b + \frac{-b + R(-b)}{3} = \frac{1}{3}b - \frac{2}{3}c - \frac{2}{3}R(b) + \frac{1}{3}R(c).$$

\textsuperscript{2}yes, him
1.19. WEDGE PRODUCT OF TWO PLANE VECTORS

What we want to prove is \( R(\vec{A}'\vec{B}') = \vec{A}'\vec{C}' \) so let us calculate

\[
R(\vec{A}'\vec{B}') = R \left( -\frac{1}{3} \vec{b} - \frac{1}{3} \vec{c} - \frac{1}{3} R(\vec{b}) + \frac{2}{3} R(\vec{c}) \right) = -\frac{1}{3} R(\vec{b}) - \frac{1}{3} R(\vec{c}) - \frac{1}{3} R(R(\vec{b})) + \frac{2}{3} R(R(\vec{c})).
\]

Looking at Picture ? (d) we see that \( R(\vec{x}) = \vec{x} + R(R(\vec{x})) \), and hence \( R(R(\vec{x})) = R(\vec{x}) - \vec{x} \). We further have

\[
R(\vec{A}'\vec{B}') = -\frac{1}{3} R(\vec{b}) - \frac{1}{3} R(\vec{c}) - \frac{1}{3} R(R(\vec{b})) + \frac{2}{3} R(R(\vec{c})).
\]

This last expression is the same as the expression for \( \vec{A}'\vec{C}' \) above, hence \( R(\vec{A}'\vec{B}') = \vec{A}'\vec{C}' \) what we wanted to prove.

1.19 Wedge product of two plane vectors

In Calculus we learn the geometry and the algebra of the notion of cross product \( \vec{a} \times \vec{b} \) of two vectors \( \vec{a}, \vec{b} \in \mathbb{R}^3 \) in 3-space. Geometrically \( \vec{a} \times \vec{b} \) has length \( |\vec{a}||\vec{b}| \sin \theta \) (where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \)), it lies in the line orthogonal to the plane spanned by \( \vec{a} \) and \( \vec{b} \) and its direction satisfies the right-hand rule. Algebraically

\[
\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.
\]

Plane vectors \( \langle a_1, a_2 \rangle \) can be considered space vectors by \( \langle a_1, a_2, 0 \rangle \). If we take the cross product of two such plane-space vectors then we obtain a vector of the form \( \langle 0, 0, * \rangle \). We do not want to keep carrying the \( (0, 0) \)-part, hence we give a new definition capturing only the third coordinate.

**Definition 1.19.1.** The wedge product \( \vec{a} \wedge \vec{b} \) of two plane vectors \( \vec{a} = \langle a_1, a_2 \rangle, \vec{b} = \langle b_1, b_1 \rangle \) is the third coordinate of the cross product \( \langle a_1, a_2, 0 \rangle \times \langle b_1, b_1, 0 \rangle \).

From the arguments above we obtain that

- (algebra) \( \langle a_1, a_2 \rangle \wedge \langle b_1, b_1 \rangle = a_1b_2 - a_2b_1 \),

- (geometry) \( \vec{a} \wedge \vec{b} = \pm |\vec{a}||\vec{b}| \sin \theta \). Since \( |\vec{a}||\vec{b}| \sin \theta \) is the area of a the parallelogram spanned by \( \vec{a} \) and \( \vec{b} \) (see Picture ?), we have

\[
\vec{a} \wedge \vec{b} = \pm \text{Area(parallelogram spanned by \vec{a} and \vec{b})}.
\]

Analysing the right-hand rule mentioned above we can determine wether + or - stand in the formula above: if the direction of \( \vec{b} \) is obtained from the direction of \( \vec{a} \) by a counterclockwise rotation by not more than \( \pi \) then the sign is positive, otherwise negative.
The algebraic interpretation easily proves the following properties.

- (antysymmetry) \( a \land b = -b \land a \), \( a \land a = 0 \).

- (bilinearity) \((a + b) \land c = a \land c + b \land c\), \((\lambda \cdot a) \land b = \lambda \cdot a \land b\).

- \( a \land b = 0 \) if and only if \( a \) and \( b \) are parallel. (*)

Again, the power of this operation is the duality between the properties just listed (proved by algebra) and the geometric interpretation. Here is an application of the the wedge product.

**Proposition 1.19.2.** The points \( A, B, C \) are collinear if and only if \( A \land B + B \land C + C \land A = 0 \).

**Proof.** The points \( A, B, C \) are collinear if and only if the vectors \( \vec{BA} \) and \( \vec{CB} \) are in one line. They are in in line if and only if their spanned parallelogram degenerates to a segment, i.e. has area 0. Hence \( A, B, C \) are in one line if and only if

\[
0 = (A - B) \land (B - C) = A \land B - A \land C - B \land B + B \land C = A \land B + B \land C + C \land A.
\]

Before Proposition 1.19.2 our algebraic interpretations of collinearity were Propositions 1.4.1 and 1.4.2. All the theorems proved using those two propositions have alternative proofs using our new algebraic interpretation Proposition 1.19.2. We will not reprove earlier theorems though (student may find it a good exercise), but rather give some new incidence theorem, and as a change we will prove them using Proposition 1.19.2.

**Theorem 1.19.3** (Newton-Gauss line). Let \( a, b, c, d \) be four pairwise intersecting lines (this configuration is called a complete quadrilateral). The pairwise intersections are six points. Three pairs of these six points are not connected by the lines \( a, b, c, d \), these are called diagonals (\( PS, RQ, UV \) in the picture). The midpoints of the three diagonals are collinear.
1.19. WEDGE PRODUCT OF TWO PLANE VECTORS

**Proof.** Using the notation of the picture consider the sum of the following twelve terms

\[
P \wedge R \quad P \wedge Q \quad S \wedge R \quad S \wedge Q \\
R \wedge U \quad Q \wedge V \quad R \wedge V \quad Q \wedge U \\
U \wedge P \quad V \wedge P \quad V \wedge S \quad U \wedge S
\]

We will view this sum in two different ways.

First: The sum of the terms in each column is zero, because the triples of points \((P, R, U)\), \((P, Q, V)\), \((S, R, V)\), \((S, Q, U)\) are collinear, see Proposition 1.19.2. Hence the total sum is zero.

Second: The first row is 4 times \((P + S)/2 \wedge (R + Q)/2\), the second row is 4 times \((R + Q)/2 \wedge (U + V)/2\). The third row is 4 times \((U + V)/2 \wedge (P + S)/2\).

We conclude that

\[
\frac{P + S}{2} \wedge \frac{R + Q}{2} + \frac{R + Q}{2} \wedge \frac{U + V}{2} + \frac{U + V}{2} \wedge \frac{P + S}{2} = 0,
\]

and hence according to Proposition 1.19.2 \((P + S)/2\), \((R + Q)/2\), \((U + V)/2\) are collinear.

**Theorem 1.19.4** (parallel case of Pappus’ theorem). Let \(A, B, C\) be collinear points and let \(A', B', C'\) be collinear. Then two of

\[AB' || A'B, \quad BC' || B'C, \quad AC' || A'C\]

imply the third.
Proof. Consider the following three numbers

\[(A - B') \land (A' - B), \quad (B - C') \land (B' - C), \quad (C - A') \land (C' - A).\]

The vanishing of these three numbers is equivalent to the three parallelity conditions of the theorem, according to (*) above. Hm...better way of referencing there is needed.

However, one can distribute the sum of these three numbers, use the antisymmetry property of \( \land \) and conclude that the total sum is 0. Hence the vanishing of two of them indeed implies the vanishing of the third one.

We will learn more on Pappus’ theorem in Section ??.

1.20 A 3D view of plane geometry, triple product

In Calculus we learned the geometry and algebra of the triple product \((a \times b) \cdot c\) of three space vectors \(a, b, c\). It will be convenient to use the following notation \(a \land b \land c = (a \times b) \cdot c\), and we may call it the triple product, or triple wedge product of \(a, b, c\). In particular the following hold

- \(a \land b \land c = (a \times b) \cdot c = a \cdot (b \times c);\)

- \(a \land b \land c\) is plus or minus the volume of the parallelepiped spanned by the space vectors \(a, b, c;\)

- \((a_1, a_2, a_3) \land (b_1, b_2, b_3) \land (c_1, c_2, c_3) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix};\)
1.20. A 3D VIEW OF PLANE GEOMETRY, TRIPLE PRODUCT

- (3-linearity) $a \wedge b \wedge c$ is linear in each of the variables. Linearity in the first variable means

$$ (a \pm a') \wedge b \wedge c = a \wedge b \wedge c \pm a' \wedge b \wedge c, $$

and linearity in the second and third variables are similar;

- (antisymmetry) $a \wedge b \wedge c = b \wedge c \wedge a = c \wedge a \wedge b = -a \wedge c \wedge b = -c \wedge b \wedge a = -b \wedge a \wedge c.$

In view of the second property above it makes no sense of considering the triple product of vectors lying in the $(x, y, 0)$ plane. A useful tool, however is considering our plane as the $z = 1$ plane in 3-space. That is, if a point was $(a_1, a_2)$ earlier, now we consider it as $(a_1, a_2, 1)$.

What we gained is a new operation: we can form the triple product $A \wedge B \wedge C$ for three points. What we lost is that we partially lost our earlier operations: for example $A + B$ does not make sense any more since $(a_1, a_2, 1) + (b_1, b_2, 1) = (a_1 + b_1, a_2 + b_2, 2)$ is not in the $z = 1$ plane any more. However, for example $xA + yB$ makes sense if $x + y = 1$.

**Proposition 1.20.1.** For a triangle $ABC\triangle$ in the $z = 1$ plane we have

$$ \text{Area}(ABC\triangle) = \pm \frac{1}{2} A \wedge B \wedge C, $$

and the sign is positive if and only if going around the triangle in the order $A$, $B$, $C$ is counterclockwise.

The proof is left as an exercise.

**Corollary 1.20.2.** The three points $A, B, C$ of the $z = 1$ plane are collinear if and only if $A \wedge B \wedge C = 0$.

**Proof.** Both conditions are equivalent to the condition that the volume of the parallelepiped spanned by the space vectors $A, B, C$ is zero. 

Corollary 1.20.2 is now our 4th algebraic interpretation of collinearity of three points in the plane—however this in new settings. Again, all theorems that were proved using any of the earlier three interpretations (Propositions 1.4.1, 1.4.2, 1.19.2) can be proved with this new one too—just we need to be careful using operation that make sense in the $z = 1$ plane. Let us illustrate this with the following new proof Menelaus' theorem 1.7.1.
Proof. Consider our triangle in the $z = 1$ plane. We have $K = xB + x'C$, $L = yC + y'A$, $M = zA + z'B$ with $x + x' = y + y' = z + z' = 1$. According to Corollary 1.4.3 we have 

$$0 = K \wedge L \wedge M = (xB + x'C) \wedge (yC + y'A) \wedge (zA + z'B).$$

Using 3-linearity and antisymmetry of the triple product we can distribute the above expression and obtain

$$0 = (xyz + x'y'z')A \wedge B \wedge C.$$

Since $ABC$ is a triangle (with non-zero area) we have $A \wedge B \wedge C \neq 0$. Therefore $xyz + x'y'z' = 0$ which is a rearrangement of Menelaus’ theorem.  

1.21 Problems

1. Prove Proposition 1.2.2.

2. Prove Proposition 1.5.2.

3. In the formula $C = (1 - t)A + tB$, trace the position of $C$ on the line as $t$ varies from $-\infty$ to $\infty$.

4. In $ABC\triangle$ let $U$ be the midpoint of $AB$ and let $V$ be the midpoint of $AC$. Prove that $UV$ is parallel to $BC$ and has half the length.

5. In a quadrilateral let $U$ and $V$ be the midpoints of two opposite sides. Prove that the segment $UV$ and the segment connecting the midpoints of the diagonals bisect each other.

6. Let $S$ be the centroid of the $ABC\triangle$. Calculate $S\vec{A} + S\vec{B} + S\vec{C}$.

7. Let $R$ be an arbitrary point in the plane and $ABCD$ a parallelogram. Prove that $\vec{R}\vec{A} + \vec{R}\vec{C} = \vec{R}\vec{B} + \vec{R}\vec{D}$.

8. In the $ABCD$ quadrilateral let $\vec{AB} = \vec{a}$, $\vec{DC} = \vec{b}$. Let the points $A, X_1, X_2, X_3, D$ divide the $AD$ side into four equal parts. Let the points $B, Y_1, Y_2, Y_3, C$ divide the $BC$ segment into four equal parts. Express $X_1Y_1, X_2Y_2, X_3Y_3$ in terms of $\vec{a}$ and $\vec{b}$.

9. Let $ABCDA'B'C'D'$ be a cube ($ABCD$ is a square and $A'B'C'D'$ is a translated copy of $ABCD$). Let $\vec{a} = \vec{AB}, \vec{b} = \vec{AD}, \vec{c} = \vec{AA}'$. Let $P$ be the midpoint of $C'D'$. Let $Q$ be the center of the $BCC'B'$ square. Express $\vec{AP}, \vec{AQ}, \vec{AD'}, \vec{BD}$ in terms of $\vec{a}, \vec{b}, \vec{c}$. 

1.21. PROBLEMS

10. Let $ABCD$ be a parallelogram. Let the points $A, X_1, X_2, B$ divide $AB$ into three equal parts. Let $C, Y_1, Y_2, D$ divide $CD$ into three equal parts. Let $X_2, U, V, Y_2$ divide $X_2Y_2$ into three equal parts. Express $\overrightarrow{AV}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

11. Let $P_1, \ldots, P_n$ be points, and $\mu_1, \ldots, \mu_n$ be real numbers with $\sum_{i=1}^n \mu_i = 1$. For a point $O$ consider $v = \sum_{i=1}^n \mu_i \overrightarrow{OP}_i$. Let $S$ be the end point of the vector $v$ if it is measured from $O$. Prove that the point $S$ does not depend on the choice of $O$. (Hint: choose two different $O_1$ and $O_2$ and calculate the vector between the obtained “two” $S$ points. You should get $0$.)

12. Let $D$ be a point on the line of $BC$, and let $E$ be a point on the line of $AC$ of the $ABC$ triangle (but let $D,E$ be distinct from the vertices). Assume the lines $AD$ and $BE$ intersect in a point $P$. Prove that if $\overrightarrow{BD} \cdot \overrightarrow{DC} = -1$ then $CP$ is parallel with $AB$.

13. Let the reflection of the point $A$ over the point $B$ be $C$. Express $C$ in terms of $A$ and $B$.

14. (cont.) Let $ABC$ be a triangle. The reflection of $A$ over $B$ is $A'$. The reflection of $B$ over $C$ is $B'$. The reflection of $C$ over $A$ is $C'$. Prove that the centroid of $ABC$ and the centroid of $A'B'C'$ coincide.

15. In the proof of the reverse Menelaus’ theorem we claimed that $\ell$ intersects the line of $AB$ (that is, $\ell$ and $AB$ are not parallel). Prove this statement.

16 (P). Consider the $ABC\Delta$ and non-zero real numbers $k_1, k_2, k_3$. Let $P_{AB}$ and $P'_{AB}$ be on the $AB$ line, the first in the $AB$ segment, the second outside of the $AB$ segment, such that 

$$\frac{|AP_{AB}|}{|P_{AB}B|} = \frac{|AP'_{AB}|}{|P'_{AB}B|} = \frac{k_1}{k_2}.$$ 

Define $P_{BC}$ and $P'_{BC}$ on the $BC$ line similarly with the ratio $k_2/k_3$; and define $P_{CA}$ and $P'_{CA}$ on the $CA$ line similarly with the ratio $k_3/k_1$. Show that the lines $AP_{BC}$, $BP_{CA}$, $CP_{AB}$ are concurrent.

17 (P). (cont.) Prove that the line $P_{CA}P_{AB}$ contains the point $P'_{BC}$.

18 (P). (cont.) Prove that $P'_{AB}$, $P'_{BC}$, and $P'_{CA}$ are collinear.

19 (P). Let $P$ be different from the vertices of the triangle $ABC\Delta$. Let $PBLC$, $PCMA$, $PANB$ be parallelograms. Prove that the segments $AL$, $BM$, $CN$ bisect each other.

20 (P). Points $P, Q, R$ lie on the sides of the $ABC\Delta$ and are such that

$$(B \overrightarrow{P} : \overrightarrow{PC}) = (C \overrightarrow{Q} : \overrightarrow{QA}) = (A \overrightarrow{R} : \overrightarrow{RB}).$$

Prove that the centroids of the triangles $PQR$ and $ABC$ coincide.
21 (P). The triangles $A_1B_1C_1\triangle$, $A_2B_2C_2\triangle$, and $A_3B_3C_3\triangle$ have their corresponding sides parallel. Hence each pair of triangles has a center of perspectivity. Prove that the three centers of perspectivities are concurrent.

22 (P). A line drawn through the vertex $A$ of a parallelogram $ABCD$ cuts $CB$ in $P$ and $CD$ in $Q$. A line through $C$ cuts $AB$ in $R$ and $AD$ in $S$. Prove that $PR$ and $QS$ are parallel.

23. Prove that for a parallelogram the sum of squares of the sides is equal to the sum of the squares of the diagonals.

24. Let $O$ be the center of the $ABCDEF$ regular hexagon whose side length is 1. Find

$$\vec{AB} \cdot \vec{AO}, \vec{AB} \cdot \vec{AC}, \vec{BC} \cdot \vec{EF}, \vec{FC} \cdot \vec{BD}, \vec{FC} \cdot \vec{EF}.$$ 

25 (P). Let $A, B, C, D$ be points in the plane, let $A'$ be the midpoint of $BC$, let $B'$ be the midpoint of $CA$, and let $C'$ be the midpoint of $AB$. Prove that

$$(D - A')(C - B) + (D - B')(A - C) + (D - C')(B - A) = 0.$$ 

26 (P). Suppose that the segment connecting the midpoints of $AB$ with $CD$, and the segment connecting the midpoints of $BC$ with $DA$ are of the same length. Prove that $AC$ is perpendicular to $BD$.

27. Prove Corollary 1.16.3, the triangle inequality.

28. (Thales’ theorem) Let $O$ be the midpoint of $AC$. Prove that $\angle(ABC) = \pi/2$ if and only if $d(AO) = d(BO)$. (That is $\angle(ABC) = \pi/2$ if and only if $B$ is on the circle with center $O$ and radius $d(AO)$.)

29. Finish the proof of Feuerbach’s Theorem 1.14.2, i.e. prove that points (c) are also on the same circle as points (a) and (b).

30. Reflect the orthocenter of $ABC\triangle$ over the midpoints of the sides. Prove that the obtained three points are on the circumscribed circle.

31. Solve problems 1, 2, 3 after Theorem 1.17.4.

32. Prove that a triangle is equilateral if two of its circumcenter, centroid, and orthocenter coincide.

33. Let $O$ be the center of circumscribed circle of $ABC\triangle$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vectors pointing from $O$ to the vertexes. Let $M$ be the endpoint of $\mathbf{a} + \mathbf{b} + \mathbf{c}$ measured from $O$. Prove that $M$ is the orthocenter of $ABC\triangle$. 

34. (cont.) Let \( O \) be the center of the circumscribed circle of \( ABC\). Let \( M \) be the orthocenter, and let \( S \) be the centroid of \( ABC\). Prove that \( O, M, S \) are collinear. (The obtained line is called the Euler-line of \( ABC\).) Find \( \overrightarrow{OS}/\overrightarrow{SM} \).

35. Let \( D \) be a point on the side \( BC \) of \( ABC\). Prove that
\[
\frac{BD}{DC} = \frac{|AB| \sin(DAB\angle)}{|AC| \sin(DAC\angle)}.
\]

36. (cont.) Prove the Angle Bisector Theorem: If the angle bisector from \( A \) intersects \( BC \) in \( D \), then
\[
\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}.
\]

37. (cont.) Reprove that angle bisectors are concurrent by using the Angle Bisector Theorem and Ceva’s theorem.

38. Let \( AB = 6 \), \( BC = 10 \) in the \( ABC\). The angle bisector at \( B \) intersects \( AC \) in \( D \). Connect \( D \) with the midpoint of \( AB \), let the intersection point with \( BC \) be \( E \). What is the length \( |BE| \)?

39. Let the lengths of the sides of \( ABC\) be \( a, b, c \) (\( a \) is opposite of \( A \), etc). If \( O \) is the center of the inscribed circle then prove that
\[
O = \frac{aA + bB + cC}{a + b + c}.
\]

40. Let \( P \) be an interior point of \( ABC\). The lines connecting \( P \) with the vertices cut \( ABC\) into six smaller triangles. We color every second of these six triangles with red, the rest with blue. Prove that the product of the areas of the red triangles is the same as the product of the areas of the blue triangles.

41. (cont.) In the problem above replace “area” with “radius of the circumscribed circle”.

42 (P). Let \( P, Q, R, S \) be the centers of the squares that are described externally on the sides of a quadrilateral (in this order). Prove that \( PR \) and \( QS \) are of the same length, and are perpendicular to each other.

43 (P). If \( A', B', C' \) are the midpoints of \( BC \), \( CA \), \( AB \) respectively, then show that
\[
4A' \land B' \land C' = A \land B \land C.
\]
Deduce that \( \text{Area}(A'B'C'\Delta) = \frac{1}{4}\text{Area}(ABC\Delta) \).
44 (P). Let the side lengths of the $ABC\triangle$ be $a,b,c$ ($a$ is opposite with $A$ etc), and let the angles be $\alpha,\beta,\gamma$. Let the foot of the altitude from $A$ be $D$. Prove that

$$aD = (b \cos \gamma)B + (c \cos \beta)C.$$  

Deduce that the area of the triangle formed by the feet of the altitudes is

$$2 \cos \alpha \cos \beta \cos \gamma \cdot \text{Area}(ABC\triangle).$$

45 (P). Let $D, E, F$ be points on the sides $AB, BC, CA$ of a triangle, dividing the sides in the ratios $k_1 : 1, k_2 : 1, k_3 : 1$. Show that

$$\frac{\text{Area}(DEF\triangle)}{\text{Area}(ABC\triangle)} = \frac{1 + k_1 k_2 k_3}{(1 + k_1)(1 + k_2)(1 + k_3)}.$$  

46. Give a proof of Proposition 1.20.1. Hint: Let $T$ be the tetrahedron with vertexes $(0,0,0), A, B, C$. Use the geometric interpretation of the triple product to conclude that the volume of $T$ is plus or minus one sixth of the triple product $ABC$. Finish the proof by observing that the volume of $T$ is one third of the area of the $ABC\triangle$.

47. Reprove Ceva’s theorem, using the triple wedge operation. Hint: Try to rephrase the “high-school style” proof from the end of Section 1.9.
Chapter 2

Projective Geometry

2.1 Projective plane as extended affine plane

In relation with some exercises in Chapter 1 we developed the intuitive idea that we would like to send a point in a configuration to “infinity”. This idea is made precise by defining a new object, the projective plane $\mathbb{P}^2$. Just like the affine plane has points and lines and the incidence relation between them, the projective plane will also have points and lines and incidence relation between them. However, the projective plane will have “more” points and lines. Moreover the incidence relation will be somewhat more “complete” and more “symmetric” than in the affine case.

Let $a$ be a line on the Euclidean plane. The collection of all lines parallel with $a$ will be called a “parallelism class” of lines on $\mathbb{R}^2$, and let’s call this object $I_a$. So, if $a \parallel b$ then $I_a = I_b$. One way of thinking of the $I_a$’s as “symbols” associated with lines in such a way that parallel lines have the same symbol.

**Definition 2.1.1.** Let points of $\mathbb{P}^2$ be

- points of $\mathbb{R}^2$ (these will be called affine points),
- all the $I_a$’s (these will be called ideal points).

Let the lines of $\mathbb{P}^2$ be

- extended lines of $\mathbb{R}^2$: extend each line in $\mathbb{R}^2$ with its own parallelism class, that is extend $a$ with $I_a$,
- the collection of all ideal points is declared to be on one new line, the ideal line.
CHAPTER 2. PROJECTIVE GEOMETRY

Hence we added to the points of the plane \( \mathbb{R}^2 \) infinitely many new points, one for each parallelism class of lines on \( \mathbb{R}^2 \). We also added one new line. Observe that the lines coming from lines in \( \mathbb{R}^2 \) are exactly one point “longer” on \( \mathbb{P}^2 \).

PROJECT 6. One way of imagining \( I_a \) is imagining it at the “end” of \( a \). Since \( a \) has two “ends”, we must visualize it at both ends of \( a \) at the same time. In this visualization, if a point “goes to infinity” in one direction of the projective line \( a \cup \{I_a\} \), then after it disappears in one direction, it comes back at infinity at the other “end” of \( a \). Let \( A, B, C \) be the vertices of a triangle in a counterclockwise order. Imagine that we send this triangle to “infinity” in one direction, and as described above, it reappears from the opposite direction. Will the reappearing triangle be named \( ABC \) clockwise or counterclockwise?

2.2 Incidence properties of \( \mathbb{P}^2 \)

The following observation can be established by checking all possible cases.

**Proposition 2.2.1.** For any two points there is a unique line passing through them. For any two lines there is a unique point contained in both of them.

2.3 Remarkable incidence theorems

Desargues’ theorem holds in \( \mathbb{P}^2 \).

**Theorem 2.3.1** (Desargues’ theorem—projective). Two triangles are perspective with respect to a point if and only if they are perspective with respect to an axis.

PROJECT 7. Learn the “vector calculus” of projective geometry—by searching for “projective coordinates”. Then prove the projective Desargues’ theorem. The calculation should be similar to the calculation in the proof of the affine Desargues’ theorems in Chapter 1.

PROJECT 8. Let \( S_1 \) and \( S_2 \) be planes in \( \mathbb{R}^3 \) intersecting in a line. Let \( A \) be a point in \( \mathbb{R}^3 \), not on these planes. There is a natural notion of projecting points of \( S_1 \) to points of \( S_2 \) from \( A \) (see Figure ?). We obtain a map \( p_A \) from a subset of \( S_1 \) to \( S_2 \). Observe that this map is not defined for all points of \( S_1 \) and that it is not surjective. Find a natural extension

\[
\tilde{p}_A : \mathbb{P}^2(S_1) \to \mathbb{P}^2(S_2)
\]

where \( \mathbb{P}^2(S_i) \) is the projective plane obtained by extending \( S_i \), and conclude that \( \tilde{p}_A \) is bijective. Interpret the outcome as a visualization of the ideal line of \( S_1 \) in \( S_2 \). Use this idea to prove the projective Desargues’ theorem, based on the fact that we already proved the affine Desargues’ theorem in Chapter 1.
Pappus’ theorem holds in \( \mathbb{P}^2 \).

**Theorem 2.3.2** (Pappus’ theorem—projective version). If \( A, B, C \) are collinear and \( A', B', C' \) are collinear, then \( K = AB' \cap A'B, L = BC' \cap B'C, \) and \( M = CA' \cap C'A \) are also collinear.

**PROJECT 9.** Prove the projective Pappus’ theorem using projective coordinates, c.f. Project 7.

**PROJECT 10.** Prove the projective Pappus’ theorem using the idea of Project 8.

### 2.4 Problems

**48.** Prove Proposition 2.2.1.

The next three problems refer to the following part of the projective Desargues theorem: If \( ABC \triangle \) and \( A'B'C' \triangle \) are perspective w.r.t. a point \( P \), then \( K = AB \cap A'B', \ L = BC \cap B'C', \) and \( M = CA \cap C'A' \) are collinear.

**49.** Phrase the affine geometry theorem that we obtain from this theorem if \( K \) is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

**50.** Phrase the affine geometry theorem that we obtain from this if theorem \( A \) is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

**51.** Phrase the affine geometry theorem that we obtain from this if theorem \( P, A, A' \) is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

The next two problems refer to the projective Pappus’ Theorem 2.3.2.
52. Phrase the affine geometry theorem that we obtain from this theorem if $K$ and $L$ are ideal points (and none of $A, B, C, A', B', C'$ are ideal). You do not need to prove this theorem, just phrase it.

53. Phrase the affine geometry theorem that we obtain from this theorem if $A, B, C$ are ideal points (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.
Chapter 3

Spherical Geometry

3.1 Points, lines, triangles, polarity

In spherical geometry the role of the “plane” will be played by the sphere $S^2 = \{ x \in \mathbb{R}^3 : ||x|| = 1 \}$. When we say point, we mean a point on $S^2$.

The role of straight lines will be played by the “great circles” in $S^2$. A great circle is defined to be the intersection $S^2 \cap P$ where $P$ is a plane in $\mathbb{R}^3$ containing the origin.

Now we have a geometry of “points” and “lines”. The following point-line incidence properties are obvious.

**Proposition 3.1.1.** In spherical geometry

- given two non-antipodal points there is a unique line passing through them;
- the intersection of any two lines is exactly two points.

**Definition 3.1.2.**

- For two (not antipodal) points of $S^2$ there is exactly one great circle connecting them. The two points divide this great circle to two arcs. The shorter of the two will be called a *spherical segment* connecting the two points.

- For three spherical points $A, B, C$ (that are not contained in a spherical line) the union of the three spherical segments $AB$, $BC$, $CA$ is called a *spherical triangle*.
• For a spherical line \( a = S^2 \cap P \) there are two points \( X, Y \) on \( S^2 \) with \( X \perp P, Y \perp P \).

We will call these two points the poles of \( a \).

• Let \( ABC \) be spherical triangle. Let \( C' \) be a pole of \( AB \) contained in the same hemisphere as \( C \). Let \( B' \) be a pole of \( CA \) contained in the same hemisphere as \( B \). Let \( A' \) be a pole of \( BC \) contained in the same hemisphere as \( A \). The spherical triangle \( A'B'C' \) is called the polar triangle of \( ABC \).

**Theorem 3.1.3** (Bipolar Theorem). *The polar triangle of the polar triangle of \( ABC \triangle \) is itself.*

**Proof.** For the purpose of this proof, if \( X \) and \( Y \) are on \( S^2 \) and \( X \perp Y \) then we will call the spherical segment \( XY \) a quadrant.

We have that \( AB' \) is a quadrant because \( B' \) is a pole of \( AC \). The segment \( AC' \) is a quadrant, because \( C' \) is a pole of \( AB \).

Since \( AB' \) and \( AC' \) are quadrant, \( A \) must be a pole of \( B'C' \). What remains to be proved is that \( A \) is on the same side of \( B'C' \) as \( A' \).

The points \( A \) and \( A' \) are on the same side of \( BC \) by definition of \( A' \). Moreover \( A' \) is a pole of \( BC \). Hence the segment \( AA' \) is less than a quadrant.

Since \( AA' \) is less than a quadrant, and \( A \) is a pole of \( B'C' \) (proved above), we must have that \( A \) and \( A' \) is on the same side of \( B'C' \).

\[\Box\]

### 3.2 Length, angle

**Definition 3.2.1.** A vector \( \mathbf{v} \in \mathbb{R}^2 \) is called tangent to \( S^2 \) at \( X \in S^2 \) if \( \mathbf{v} \perp X \). The collection of tangent vectors to \( S^2 \) at \( X \) is called the tangent plane \( T_X S^2 \).

It is natural to imagine arrows representing tangent vectors in such a way that the arrows start at \( X \). Thus \( T_X S^2 \) coincides with the usual notion of a plane tangent to \( S^2 \) at \( X \).
3.2. LENGTH, ANGLE

Given a spherical segment connecting $X$ and $Y$ there is a unique unit vector $v \in T_X S^2$ which is determined by $Y = \alpha X + \beta v$ with $\beta > 0$. The vector $v$ will be called the unit tangent vector at $X$ to the segment from $X$ to $Y$.

**Definition 3.2.2.** The distance $d(X,Y)$ between points $X$ and $Y$ is defined to be

$$d(X,Y) = \arccos(X \cdot Y).$$

Because of the geometric interpretation of dot product, $d(X,Y)$ is in fact the angle between the vectors $X$ and $Y$. Also, since the radius of $S^2$ is 1, this angle is the same as the length of the spherical segment $XY$. Thus the odd looking definition of distance above is natural: it is the length measured on the sphere.

Let $v \in T_X S^2$ be a unit vector and consider the parameterized curve

$$s(t) = (\cos t)X + (\sin t)v. \quad (3.1)$$

First observe that this curve lies in $S^2$. Indeed, for any $t$ we have

$$s(t)s(t) = ((\cos t)X + (\sin t)v)((\cos t)X + (\sin t)v) =$$

$$XX \cos^2 t + 2Xv \cos t \sin t + vv \sin^2 t = \cos^2 t + \sin^2 t = 1.$$ 

The curve is also in the plane spanned by $X$ and $v$. Hence the curve is in the intersection of a plane with the sphere, that is in a spherical line.

It will be important to study the parametrization (3.1) with respect to the distance notion defined above. Let $a$ be in $(0, \pi)$. Then $s([0,a])$ is a spherical segment and $v$ is the unit vector tangent to the spherical segment $Xs(a)$ at $X$.

We claim that the length of $s([0,a])$ is $a$. Indeed,

$$d(X, s(a)) = \arccos(X \cdot ((\cos a)X + (\sin a)v)) = \arccos(\cos a) = a.$$ 

Moreover, for any $t_1, t_2$ with $|t_1 - t_2| < \pi$ we have that the length of the $s(t_1)s(t_2)$ segment is $|t_1 - t_2|$ (Ex. 54 below.) This remarkable property of the parametrization (3.1) is called “arc-length parametrization”, or “natural parametrization”.
Chapter 3. Spherical Geometry

Definition 3.2.3. The $XYZ\angle$ is defined to be the angle of the following two vectors: the unit tangent vector at $Y$ to the segment $YX$, and the unit tangent vector at $Y$ to the segment $YZ$.

3.3 Sides and angles of polar triangles

Theorem 3.3.1. Let $a, b, c$ be the sides and $\alpha, \beta, \gamma$ be the angles of a triangle $ABC\triangle$. Let $a', b', c'$ be the sides and $\alpha', \beta', \gamma'$ be the angles of the polar triangle $A'B'C'\triangle$. Then

\begin{align*}
  a' &= \pi - \alpha, \\
  b' &= \pi - \beta, \\
  c' &= \pi - \gamma, \\
  a &= \pi - \alpha', \\
  b &= \pi - \beta', \\
  c &= \pi - \gamma'.
\end{align*}

Proof. Because of symmetry and the Bipolar Theorem 3.1.3 it is enough to prove one of the six statements. Let $K = AB \cap B'C'$, $L = AC \cap B'C'$. Since $A$ is a pole of $B'C'$ we have $KL = \alpha$.

Then $B'C' = B'L + KC' - KL = \pi/2 + \pi/2 - \alpha$, proving $a' = \pi - \alpha$. \hfill \qed
3.4 Laws of Cosines and Sines

Theorem 3.4.1 (Spherical Law of Cosines). In $ABC\triangle$ on the sphere (with usual notations) we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$  

Proof. Let $u$ be the unit tangent vector of $CA$ at $C$. Let $v$ be the unit tangent vector of $CB$ at $C$. Then $A = \cos b \cdot C + \sin b \cdot u$, $B = \cos a \cdot C + \sin a \cdot v$. Therefore $c = AB = \arccos(A \cdot B) = \arccos(\cos b \cos a + \sin b \sin a \cdot u \cdot v)$, but $u \cdot v = \cos \gamma$. \hfill \Box

Corollary 3.4.2 (Spherical Pythagorean theorem). In a right angled triangle with hypotenuse $c$ on $S^2$ we have $\cos c = \cos a \cos b$.

Proof. This is the Spherical Law of Cosines for $\gamma = \pi/2$. \hfill \Box

Project. In Spherical Law of Cosines and in Spherical Pythagorean do the following. Write $a \cdot t$ for $a$ etc for all side measurements. Then expand in Taylor series with respect to $t$. Study the coefficients of $1, t, t^2$.

Theorem 3.4.3 (Spherical triangle-inequality). In a spherical triangle we have $c < a + b$.

Proof. We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma; \quad \cos(a + b) = \cos a \cos b - \sin a \sin b.$$  

The first line is the Law of Cosines, the second line is a trig identity. From trig we have $\sin a \sin b \in (0, 1)$, $\cos \gamma \in (-1, 1)$. Therefore the RHS of the first line is larger than the RHS of the second line. Hence the same holds for the LHSs, i.e. $\cos c > \cos(a + b)$. Since $\cos$ is strictly monotone decreasing on $(0, \pi)$ this implies $c < a + b$. \hfill \Box

Theorem 3.4.4 (Dual Law of Cosines). In $ABC\triangle$ on the sphere (with usual notations)

$$-\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c.$$  

Proof. Apply the Law of Cosines to the polar triangle. \hfill \Box

Remark 3.4.5. The dual Law of Cosines calculates $\gamma$ if $\alpha, \beta$ and $c$ are given. In Euclidean geometry one can calculate $\gamma$ as long as $\alpha$ and $\beta$ are given, no need for $c$! Also, this Theorem shows that the sum of the angles of a triangle is not constant, keeping $\alpha$ and $\beta$ the same, but changing $c$ does change $\gamma$. More on this is coming in the next section.
Theorem 3.4.6 (Spherical Law of Sines). In \(ABC\triangle\) on the sphere (with usual notations)

\[
\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.
\]

Proof. From the Law of Cosines we have

\[
\cos \gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.
\]

From this, using \(\sin^2 + \cos^2 = 1\) we have

\[
\sin \gamma = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b}.
\]

Divide by \(\sin c\) and see that the RHS becomes symmetric in \(a, b, c\).

Theorem 3.4.7. The common value of \(\sin \alpha/\sin a = \sin \beta/\sin b = \sin \gamma/\sin c\) is equal

\[
\frac{6 \text{Vol}(OABC\text{-tetrahedron})}{\sin a \sin b \sin c},
\]

in other words the \(\sqrt{-}\)-expression in the proof above is equal 6 times the volume of the parallelepiped.

Proof. We can move the \(ABC\triangle\) into the special position \(A = (1, 0, 0), B = (?, \text{positive}, 0), C = (?, ?, \text{positive})\). Then a good picture and calculation shows that

\[
A = (1, 0, 0), B = (\cos c, \sin c, 0), C = (\cos b, \sin b \cos \alpha, \sin b \sin \alpha).
\]

Using the determinant formula for the volume of the \(OABC\) tetrahedron we obtain the theorem.

3.5 Anglesum, area, Girard formula

If two great circles intersect in angle \(\theta\), then the domain between the two (on both sides) will be called a \(\theta\)-lune.
Lemma 3.5.1. The area of a $\theta$-lune is $4\theta$.

Proof. The area of a $\theta$-lune is a linear function of $\theta$. For $\theta = \pi$ we get the surface area of $S^2$, which is $4\pi$. \hfill $\square$

Theorem 3.5.2 (Girard’s formula). The sum of the angles of a spherical triangle is $\pi$ plus the area of the triangle.

Proof. Consider a spherical triangle $T$ with angles $\alpha, \beta, \gamma$. The $\alpha$-lune, $\beta$-lune, and $\gamma$-lune together cover the sphere 1-sheeted, except they cover $T$ 3 times, as well as a congruent copy $T'$ of $T$ “on the other side” also 3 times.

Hence

$$\text{Area}(\alpha\text{-lune}) + \text{Area}(\beta\text{-lune}) + \text{Area}(\gamma\text{-lune}) = \text{Area}(S^2) + 2\text{Area}(T) + 2\text{Area}(T').$$

And therefore, using Lemma 3.5.1 we have

$$4\alpha + 4\beta + 4\gamma = 4\pi + 4\text{Area}(T),$$
which proves the theorem.

**Corollary 3.5.3.** The sum of the angles of a spherical triangle is always strictly larger than \( \pi \).

**Remark 3.5.4.** In Euclidean geometry the sum of the angles is \( \pi \), so the Girard formula says that in spherical geometry the sum of angles is the Euclidean value plus the area.

A spherical triangle is always contained in a hemisphere. Thus, the area of a spherical triangle is at most half the area of \( S^2 \), that is \( 2\pi \). Hence, from Girard’s formula we obtain an upper bound for the angle sum as well, namely

\[
\pi < \alpha + \beta + \gamma < 3\pi.
\]

**Proposition 3.5.5.** For the perimeter \( p \) of a triangle on the sphere we have \( 0 < p < 2\pi \).

**Proof.** Applying the \( \pi < \alpha + \beta + \gamma < 3\pi \) estimate for the polar triangle we obtain the statement. \( \square \)

**Theorem 3.5.6.** The sum of the angles of a spherical quadrilateral is \( 2\pi \) plus the area.

**Proof.** Cut the quadrilateral by a diagonal to two triangles \( T_1 \) and \( T_2 \). For the two triangles we have Girard’s formula

\[
\alpha + \beta_1 + \delta_1 = \pi + \text{Area}(T_1),
\]

\[
\gamma + \beta_2 + \delta_2 = \pi + \text{Area}(T_2).
\]

Adding together we obtain

\[
\alpha + \beta + \gamma + \delta = 2\pi + \text{Area}(T_1 \cup T_2).
\]

**Theorem 3.5.7.** The sum of the angles of a spherical \( n \)-gon is the Euclidean value \( (n-2)\pi \) plus the area of the \( n \)-gon.

**Proof.** The idea of the preceding proof can be applied iteratively. \( \square \)
3.6 Problems

54. Consider the parametrization (3.1). Prove that for \(|t_1 - t_2| < \pi\) we have that the length of the \(s(t_1)s(t_2)\) segment is \(|t_1 - t_2|\).

55. The coordinates of Chapel Hill NC are 35.93°N, 79.03°W. Look up what these coordinates mean. Look up the coordinates of Paris, France. Find the distance between Chapel Hill and Paris (call the radius of the Earth \(R\)).

56. Let \(L\) and \(W\) be points on the Earth (radius \(R\)), with coordinates 38.8°N, 9.15°W and 38.8°N, 77°W (why \(L\) and \(W\)?). What is their spherical distance?

57. The points \(L\) and \(W\) of the exercise above are on the same latitude (38.8°N). What is their distance measured on that latitude?

58. Prove Theorem 3.5.7 in detail.

59. We saw above that \(\pi < \text{angle sum of a spherical triangle} < 3\pi\). Prove that this estimate is sharp.

60. Let \(P\) be a plane not containing the origin. Prove that if \(P \cap S^2\) is not empty, then it is a circle in spherical geometry. That is, prove that there exist a point \(C \in S^2\) and a nonnegative number \(r\) such that \(P \cap S^2 = \{A \in S^2 : d(A, C) = r\}\).

61. (a) Let the sides of a triangle in spherical geometry be .3, .4, and .5. Find its angles.
   (b) Let the sides of a triangle in spherical geometry be .03, .04, and .05. Find its angles.
   (c) Let the sides of a triangle in spherical geometry be .003, .004, and .005. Find its angles.

62. Consider Chapel Hill, Chicago, and Seattle. Find the distances between any two of these cities online. Using these distances find the angle between the Chapel Hill-Chicago and the Chapel Hill-Seattle segments.

63. Consider Chapel Hill, Chicago, and Seattle. Find the distances between any two of these cities online. Calculate the area of the triangle with these vertices on the surface of the Earth.

64. Let \(a, b, c\) and \(\alpha, \beta, \gamma\) be the side lengths and angles of a spherical triangle. Assume \(\gamma = \pi/2\) (that is, we have a right triangle). Express \(\sin \alpha\) in terms of the sides. Verify that for small sides your result is close to the Euclidean value of \(a/c\).
65. Give a proof of the spherical Law of Sines based on the result of Exercise 64 along the lines of Proof-2 of the Euclidean Law of Sines (Theorem 1.16.4).

66. Let $a, b, c$ and $\alpha, \beta, \gamma$ be the side lengths and angles of a spherical triangle. Assume $\gamma = \pi/2$ (that is, we have a right triangle). Express $\cos \alpha$ in terms of the sides. Verify that for small sides your result is close to the Euclidean value of $b/c$.

67. Let $a, b, c$ and $\alpha, \beta, \gamma$ be the side lengths and angles of a spherical triangle. Assume $\gamma = \pi/2$ (that is, we have a right triangle). Express $\tan \alpha$ in terms of the sides. Verify that for small sides your result is close to the Euclidean value of $a/b$. 
Chapter 4

Hyperbolic Geometry

4.1 The Minkowski space

Definition 4.1.1 (Minkowski Inner Product). For any vectors $x$ and $y$ in 3-space the Minkowski inner product is

$$\Phi((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1 y_1 + x_2 y_2 - x_3 y_3.$$ 

The space $\mathbb{R}^3$ equipped with the Minkowski inner product is called the Minkowski space, the Minkowski space-time, or the space of special relativity.

We will call a vector $x$ in Minkowski space

- time-like if $\Phi(x, x) < 0$,
- light-like if $\Phi(x, x) = 0$, and
- space-like if $\Phi(x, x) > 0$.

Observe that light-like vectors $x = (x_1, x_2, x_3)$ satisfy

$$x_3^2 = x_1^2 + x_2^2$$

and thus form a double cone, called the “light-cone.” Vectors $\overrightarrow{OA}$ with $A$ inside the double cone are time-like, and with $A$ outside the light-cone are space-like, see Figure 4.1.

Definition 4.1.2 (Hyperbolic plane, hyperboloid model). We define $\mathbb{H}^2$ to be the top component of the 2-sheeted hyperboloid $x_1^2 + x_2^2 - x_3^2 = -1$:

$$\mathbb{H}^2 = \{ x \in \mathbb{R}^3 : \Phi(x, x) = -1, x_3 > 0 \}.$$ 

---

\footnotetext[1]{Hermann Minkowski, 1864-1909}
Figure 4.1: The light-cone formed by all light-like vectors. On the outside is the set of all space-like vectors $\mathbf{x}$ satisfying $\Phi(\mathbf{x}, \mathbf{x}) = 5$. On the inside is the set of all time-like vectors $\mathbf{x}$ satisfying $\Phi(\mathbf{x}, \mathbf{x}) = -1$. The top component of this 2-sheeted hyperboloid is the hyperbolic plane $\mathbb{H}^2$. 
4.2. DIGRESSION: HYPERBOLIC TRIG FUNCTIONS

Definition 4.1.3. Lines of $\mathbb{H}^2$ are defined to be the non-empty intersection of $\mathbb{H}^2$ with any plane $P$ containing the origin.

Now we have another geometry of points and lines, so we are interested in the incidence properties, analogous to Proposition 3.1.1. To find those incidence properties a new “model” of $\mathbb{H}^2$ will be useful.

**Klein model.** Let $\mathbb{K}^2 = \{(x_1,x_2,1) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$. Consider the following map $f$ from $\mathbb{H}^2$ to $\mathbb{K}^2$: for $A \in \mathbb{H}^2$ let $f(A) = OA \cap \mathbb{K}^2$.

**Proposition 4.1.4.** The map $f$ is a one-to-one correspondence (a.k.a. bijection) between $\mathbb{H}^2$ and $\mathbb{K}^2$.

**Proof.** Observe that for $x = (x_1,x_2,x_3) \in \mathbb{H}^2$ we have $f(x) = (x_1/x_3, x_2/x_3, 1)$. Injectivity and surjectivity for this map is straightforward calculation. □

Observe that the $f$-image of a line in $\mathbb{H}^2$ is an open chord of $\mathbb{K}^2$ (that is, an open segment connecting two points on the boundary circle of $\mathbb{K}^2$).

**Proposition 4.1.5 (Incidence properties of $\mathbb{H}^2$).** In $\mathbb{H}^2$

1. for and two points there is exactly one line connecting them.

2. The intersection of any two lines is either empty or 1 point.

3. For a line $l$ and point $P \notin l$ there are infinitely many lines $m$ passing through $P$ parallel to $l$ (i.e. not intersecting $l$).

Recall that the first two of these incidence properties hold in Euclidean geometry too. The Euclidean counterpart of the third one is

3’. For a line $l$ and point $P \notin l$ there is exactly one line $m$ passing through $P$ parallel to $l$ (i.e. not intersecting $l$).

Intellectuals of the 19th century argued whether we can decide empirically/theoretically whether (3) or (3’) holds in reality. More on this and other philosophical remarks at the end of the course.
4.2 Digression: Hyperbolic trig functions

**Definition 4.2.1.** Define $\sinh x = \frac{(e^x - e^{-x})}{2}$, $\cosh x = \frac{(e^x + e^{-x})}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$.

To get familiar with these hyperbolic trigonometric functions do the following exercises.

- Draw their graphs. Prove that $\sinh x$ is odd, $\cosh x$ is even.
- Prove that $\cosh^2 x - \sinh^2 x = 1$ for all $x$.
- Prove that $\cosh' x = \sinh x$, $\sinh' x = \cosh x$.
- Define $\text{arccosh}$ as the inverse function of $\cosh|_{[0,\infty)}$. Draw its graph. Determine its domain and range.

4.3 Length, angle

**Definition 4.3.1.** The *distance* between two points $X, Y \in \mathbb{H}^2$ is defined by

$$d(X,Y) = \text{arccosh}(-\Phi(X,Y)).$$

We need to verify that this definition makes sense, that is, $-\Phi(X,Y)$ is in the domain of $\text{arccosh}$.

**Theorem 4.3.2.** For any $X, Y \in \mathbb{H}^2$, the Minkowski inner product satisfies $\Phi(X,Y) \leq -1$. 
4.3. **LENGTH, ANGLE**

**Proof.** By Cauchy-Schwartz, we observe

\[
\Phi(X, Y) = x_1 y_1 + x_2 y_2 - x_3 y_3 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2} - x_3 y_3 = \sqrt{x_3^2 - 1} \sqrt{y_3^2 - 1} - x_3 y_3.
\]

Now we only need to show that \((x_3^2 - 1)(y_3^2 - 1) \leq (x_3 y_3 - 1)^2\). By rearranging terms, we see that this condition is equivalent to \((x_3 - y_3)^2 \geq 0\).

**Proposition 4.3.3.** We have \(d(X, Y) \geq 0\), and \(d(X, Y) = 0\) if and only if \(X = Y\).

**Proof.** The first statement is obvious from the definition of \(\text{arccosh}\), and the second one follows from careful examination of the estimates in the proof above.

**Definition 4.3.4.** For \(X \in \mathbb{H}^2\) define \(X^\perp = \{v \in \mathbb{R}^3 : \Phi(X, v) = 0\}\).

Observe that \(X^\perp\) is a plane in \(\mathbb{R}^3\).

**Theorem 4.3.5.** We have \(X^\perp = T_X \mathbb{H}^2\), that is the tangent plane to \(\mathbb{H}^2\) at \(X\) in \(\mathbb{R}^3\).

**Proof.** Tangent vectors to \(\mathbb{H}^2\) at \(X\) are derivatives of curves in \(\mathbb{H}^2\) passing through \(X\). Let \(\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))\) be such a curve with \(\phi(0) = X\). Then \(\phi(t) \subset \mathbb{H}^2\) for all \(t\) implies \(\phi_1(t)^2 + \phi_2(t)^2 - \phi_3(t)^2 = -1\). Differentiating (and dividing by 2) gives \(\phi_1(t)\phi_1'(t) + \phi_2(t)\phi_2'(t) - \phi_3(t)\phi_3'(t) = 0\). Substituting \(t = 0\) gives \(\Phi(X, \phi'(0)) = 0\). So we obtained that \(T_X \mathbb{H}^2\) is part of \(X^\perp\), but both are planes, so they must be equal.

**Theorem 4.3.6.** All non-zero elements in \(T_X \mathbb{H}^2\) are spacelike.

The proof of this theorem is left as an exercise (Problem 68).

Let \(X \in \mathbb{H}^2\), \(v \in T_X \mathbb{H}^2\) with \(\Phi(v, v) = 1\). Consider the curve

\[
s(t) = \cosh t \cdot X + \sinh t \cdot v.
\]

**Proposition 4.3.7.** The curve \(s\) is in \(\mathbb{H}^2\). We have \(s(0) = x\). The curve \(s(t)\) is contained in a line in \(\mathbb{H}^2\). We have \(d(X, s(t)) = t\).

**Proof.** The calculation

\[
\Phi(s(t), s(t)) = \Phi(\cosh t \cdot X + \sinh t \cdot v, \cosh t \cdot X + \sinh t \cdot v) = \cosh^2 t \cdot \Phi(X, X) + 0 + 0 + \sinh^2 t \cdot \Phi(v, v) = \sinh^2 t - \cosh^2 t = -1
\]

shows that \(s(t) \subset \mathbb{H}^2\). Since \(s(t)\) is also in the plane spanned by the vectors \(X, v\), we have that \(s(t)\) is in a hyperbolic line.
The distance formula is proved by

\[ d(X, s(t)) = \text{arccosh}(\Phi(X, \cosh t \cdot X + \sinh t \cdot v)) = \]

\[ \text{arccosh}(-\cosh t \cdot \Phi(X, X) + 0) = \text{arccosh}(\cosh t) = t. \]

\[ \square \]

Remark 4.3.8. In Calculus the length of a curve \( \gamma \) is defined as \( \int_a^b ||\gamma'(t)|| dt \). When defining distance on \( \mathbb{H}^2 \) we could have used the obvious analogue \( \int_a^b \sqrt{\Phi(\gamma'(t), \gamma'(t))} dt \) of this definition (note that all \( \gamma'(t) \) vectors are spacelike, so the square root makes sense!). Let us calculate the length of \( s([0,t_0]) \) above, using this Calculus motivated definition too: \( \int_0^{t_0} \sqrt{\Phi(s'(t), s'(t))} dt \). Since \( s'(t) = \sinh t \cdot X + \cosh t \cdot v \), we have

\[ \Phi(s'(t), s'(t)) = \sinh^2 t \cdot \Phi(X, X) + \cosh^2 t \cdot \Phi(v, v) = -\sinh^2 t + \cosh^2 t = 1. \]

Hence the Calculus motivated definition gives \( \int_0^{t_0} \sqrt{t} dt = t_0 \). This is the same expression we obtained from Definition 4.3.1. Our strange-looking \( d(X, Y) = \text{arccosh}(-\Phi(X, Y)) \) definition is in fact consistent with the Calculus motivated formula.

Angle.

We also want to define the notion of angle. Let \( X \in \mathbb{H}^2 \), \( u, v \in T_X \mathbb{H}^2 \) (\( u, v \) non zero), and consider the rays starting at \( X \) and going in the \( u \), respectively \( v \) direction. It is natural to define the angle of these two rays to be the angle of \( u \) and \( v \) in \( T_X \mathbb{H}^2 \), see Figure 4.2. Since the vectors of \( T_X \mathbb{H}^2 \) are space-like, the angle of \( u \) and \( v \) is naturally defined using the usual calculus notion.

Figure 4.2: Angle in \( \mathbb{H}^2 \)
Definition 4.3.9. For non-zero $u, v \in T_X \mathbb{H}^2$, define

$$\angle(u, v) = \arccos \left( \frac{\Phi(u, v)}{\sqrt{\Phi(u, u)\Phi(v, v)}} \right).$$

4.4 Laws of Cosines and Sines

Theorem 4.4.1 (Hyperbolic Law of Cosines). In $ABC\triangle$ on the $\mathbb{H}^2$ (with usual notations)

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$ 

Proof. Let $u$ be the unit tangent vector of $CA$ at $C$. Let $v$ be the unit tangent vector of $CB$ at $C$. Then $A = \cosh b \cdot C + \sinh b \cdot u$, $B = \cosh a \cdot C + \sinh a \cdot v$. Therefore $c = AB = \arccosh(\Phi(A, B)) = \arccosh(\cosh b \cosh a + \sinh b \sinh a \Phi(u, v))$, but $\Phi(u, v) = \cos \gamma$. \qed

Corollary 4.4.2 (Hyperbolic Pythagorean theorem). If $\gamma = \pi/2$ then $\cosh c = \cosh a \cosh b$.

PROJECT 11. In the Hyperbolic Law of Cosines and in the Hyperbolic Pythagorean theorem do the following. Write $a \cdot t$ for $a$ etc for all side measurements. Then expand in Taylor series with respect to $t$. Study the coefficients of $1, t, t^2$.

Theorem 4.4.3 (Hyperbolic triangle-inequality). In a triangle on $\mathbb{H}^2$ we have $c < a + b$.

Proof. We have

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

$$\cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b$$

The first line is the Law of Cosines, the second line is a hyperbolic trig identity (prove it). We have $\sinh a \sinh b > 0$, $\cos \gamma \in (-1, 1)$. Therefore the RHS of the first line smaller than the RHS of the second line. Hence the same holds for the LHSs, i.e. $\cosh c < \cosh(a + b)$. However, $\cosh$ is strictly monotone increasing on $(0, \infty)$. \qed

Theorem 4.4.4 (Hyperbolic Law of Sines). In $ABC\triangle$ in $\mathbb{H}^2$ (with usual notations)

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$
Proof. From Law of Cosines we have

$$\cos \gamma = \frac{-\cosh c + \cosh a \cosh b}{\sinh a \sinh b}.$$  

From this, using $1 + \sinh^2 = \cosh^2$ we have

$$\sin \gamma = \frac{\sqrt{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2 \cosh a \cosh b \cosh c}}{\sinh a \sinh b}.$$  

Divide by $\sinh c$ and see that the RHS becomes symmetric in $a, b, c$. \hfill \Box  

**Proposition 4.4.5.** In $ABC\triangle$ let $\gamma = \pi/2$, and let us use the standard notations. Then we have

$$\sin \alpha = \frac{\sinh a}{\sinh c}.$$  

The proof follows from the Law of Sines.

**Proposition 4.4.6.** In $ABC\triangle$ let $\gamma = \pi/2$, and let us use the standard notations. We have

$$\cos \alpha = \frac{\tanh b}{\tanh c}.$$  

Proof. We have

$$\cos \alpha = \frac{\sqrt{\sinh^2 c - \sinh^2 a}}{\sinh c} = \frac{\sqrt{\cosh^2 a - \cosh^2 c}}{\sinh c}.$$  

The first equality follows from the Proposition above, and the second equality is obtained by applying the identity $\sinh^2 + 1 = \cosh^2$ in the numerator for both terms. Using the Pythagorean theorem and the $\sinh^2 + 1 = \cosh^2$ identity again we obtain

$$\cos \alpha = \frac{\sqrt{(\cosh^2 c/ \cosh^2 b) - \cosh^2 c}}{\sinh c} = \frac{\sqrt{\frac{1}{\cosh^2 b} - 1}}{\sinh \frac{c}{\cosh c}} = \frac{\sqrt{\frac{\sinh^2 b}{\cosh^2 b}}}{\tanh c}.$$  

Proof. We have

$$\cos \alpha = \frac{\tanh b}{\tanh c}.$$  

**Proposition 4.4.7.** In $ABC\triangle$ let $\gamma = \pi/2$, and let us use the standard notations. We have

$$\tan \alpha = \frac{\tanh a}{\sinh b}.$$  

The proof is left as exercise, see Problem 74.
4.5. ANGLE OF PARALLELISM

**Theorem 4.4.8** (Hyperbolic Dual Law of Cosines). In $ABC$ triangle we have

$$\cos \gamma = - \cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c.$$ 

*Proof.* Drop a perpendicular from $C$ to $AB$ and let the resulting segments be $c_1$ and $c_2$, and the altitude $h$. We have $\cosh a = \cosh h \cosh c_2$, $\cosh b = \cosh h \cosh c_1$ from Pythagoras. Multiply these two equations. Then multiply the equality by $\cosh(c_1 + c_2)$ and rewrite $\cosh h^2$ as $\sinh^2 h + 1$. We obtain

$$\cosh a \cosh b (\cosh c_1 \cosh c_2 + \sinh c_1 \sinh c_2) = (1 + \sinh^2 h) \cosh c_1 \cosh c_2 \cosh c.$$ 

Divide by $\cosh c_1 \cosh c_2$ and rearrange, we obtain

$$\cosh a \cosh b - \cosh c = \sinh^2 h \cosh c - \cosh a \cosh b \tanh c_1 \tanh c_2.$$ 

Apply the Law of Cosines on the LHS, and sin and cos formulas for right angled triangles above on the RHS and we obtain the formula of the theorem. 

*Remark 4.4.9.* The Hyperbolic Dual Law of Cosines calculates $\gamma$ if $\alpha$, $\beta$ and $c$ are given. In Euclidean geometry one can calculate $\gamma$ as long as $\alpha$ and $\beta$ are given, no need for $c$! Also, this theorem shows that the sum of the angles of a triangle is not constant, keeping $\alpha$ and $\beta$ the same, but changing $c$ does change $\gamma$. More on this in Section 4.6.

4.5 Angle of parallelism

**Definition 4.5.1.** Two lines $a$ and $b$ in $\mathbb{H}^2$ are called *limiting parallel* if in the Klein model $\mathbb{K}^2$ the corresponding chords meet on the unit circle, see Picture 4.3.

Recall that the points of the circle are *not* part of the Klein model (points of $\mathbb{K}^2$ are in the *interior* of the unit circle), so limiting parallel lines are disjoint, that is, parallel.

Consider a point $P$ not on a line $\ell$. If we connect $P$ with points $A_1, A_2, \ldots$ of the line $\ell$, and let $A_i$ go to infinity in one direction of the line $\ell$, then the limit position of the intersecting lines will be limiting parallel to $\ell$, see Figure 4.4.

**Definition 4.5.2.** Consider a line $\ell$ and $A \in \ell$ (see Figure 4.5). Let $m$ be perpendicular to $\ell$ at $A$. Measure distance $d$ on $m$ from $A$, and let the resulting point be $B$. Consider the limiting parallel $x$ to $\ell$ passing through $B$. Denote the angle between $x$ and $m$ by $\Pi(d)$. Define $\Pi(d)$ the *angle of parallelism* corresponding to the distance $d$. 
Theorem 4.5.3. We have $\Pi(d) = \arcsin(1 / \cosh d)$.

Proof. Intuitively, use the dual law of cosines to the “ideal triangle” $ABC$ where $C$ is the “ideal point” where $\ell$ and $x$ meet in angle 0.

More precisely, consider the triangles in the second picture in Figure 4.6, with $C_1, C_2, \ldots \to \infty$. The dual law of cosines gives

$$\cos \gamma_i = \sin \beta_i \cosh d.$$  

As $i \to \infty$, we have $\gamma_i \to 0$ due to Proposition 4.4.7, and $\beta_i \to \Pi(d)$. Hence in the limit we obtain

$$1 = \sin(\Pi(d)) \cosh d,$$

what proves the Theorem. \qed
Remark 4.5.4. Using Calculus we can deduce from Theorem 4.5.3 that $\Pi(d)$ is strictly monotone decreasing, its limit in 0 is $\pi/2$ and its limit in $\infty$ is 0. Think over what these properties mean geometrically.

4.6 Area and angle-sum of a triangle

- Define 1x, 2x, and 3x ideal triangles by replacing 1, 2, and 3 vertexes by pairs of limiting parallel sides. A 3x ideal triangles are often called “ideal triangles”.
- Theorem: The area of an ideal triangle is $\pi$. No proof in this course. Remarks: this is an integral problem, first we need to figure out what an “area form” is on $\mathbb{H}^2$ (based on that we know what length and angle are), then integrate that form on an ideal triangle, and find that the integral is $\pi$. A shortcut is if we show first that any two ideal triangles can be mapped into each other by a congruence (length preserving transformation) of $\mathbb{H}^2$. Then we just need to calculate one integral. Still somewhat messy. More on congruences of $\mathbb{H}^2$ later.
- Consider a 2x ideal triangle with angle $\theta$. Let $f(\theta)$ be its area. Lemma1: $f(\theta_1 + \theta_2) = f(\theta_1) + f(\theta_2) - \pi$. Proof: draw a picture. Lemma2: $f(\pi) = 0$ (in fact $\lim_{\theta \to \pi} f(\theta) = 0$). Proof: intuitive statement, the 2x ideal triangle with angle $\pi$ has area 0. Lemma3: $f$ is
continuous. Proof: intuitive statement: small change of θ causes small change in the 2x ideal triangle with angle θ, hence its area changes by a small amount.

• Thm: \( f(\theta) = \pi - \theta \). Proof: Let \( g(\theta) = \pi - f(\theta) \). The three lemmas above claim that \( g(\theta_1 + \theta_2) = g(\theta_1) + g(\theta_2) \), \( g(\pi) = \pi \), and \( g \) is continuous. Calculus arguments show that only the identity function satisfies these three properties (first prove for rational multiples of \( \pi \) then extend by continuity).

• Thm: Area(\( ABC \triangle \)) = \( \pi - (\alpha + \beta + \gamma) \). Proof: draw a picture where \( ABC \triangle \) is extended by three 2x ideal triangles (with angles \( \pi - \alpha, \pi - \beta, \pi - \gamma \) respectively) obtaining a 3x ideal triangle. Hence: Area(\( ABC \triangle \)) = \( \pi - (\pi - (\pi - \alpha)) - (\pi - (\pi - \beta)) - (\pi - (\pi - \gamma)) \).

• Cor: In a triangle in \( \mathbb{H}^2 \) we have \( \alpha + \beta + \gamma = \pi - \text{Area} \).

• Cor: Sum of angles in a triangle in \( \mathbb{H}^2 \) is always less than \( \pi \). For small triangles it is close to \( \pi \). For large triangles it is close to 0.

4.7 Problems

68. Let \( X \in \mathbb{H}^2 \). Let \( X^\perp = \{v \in \mathbb{R}^3 : \Phi(X,v) = 0\} \). Prove that \( X^\perp \) is a plane (in the usual Euclidean sense) and all vectors in it are space-like.

69. Find and prove a formula for \( \sinh(x + y) \) in terms of \( \sinh x, \cosh x, \sinh y, \cosh y \).

70. Prove that \( \Phi \) is non-degenerate. That is, suppose that \( v \in \mathbb{R}^3 \) is such that \( \Phi(v,u) = 0 \) for all \( u \in \mathbb{R}^3 \). Prove that \( v = 0 \).

71. Prove the following version of Desargues’ theorem in \( \mathbb{H}^2 \). The triangles \( ABC \) and \( A'B'C' \) are perspective w.r.t. to the point \( P \). Suppose that \( K = AB \cap A'B', L = BC \cap B'C', M = CA \cap C'A' \) exist. Prove that \( K, L, M \) are collinear. [Hint: consider \( \mathbb{K}^2 \) instead of \( \mathbb{H}^2 \). The problem should reduce the a version we already know.]

72. Consider a horizontal plane in \( \mathbb{R}^3 \) whose intersection with \( \mathbb{H}^2 \) is not empty. (a) Prove that the intersection is a circle in the Euclidean geometry of \( \mathbb{R}^3 \). (b) Prove that the intersection is a circle in the hyperbolic geometry \( \mathbb{H}^2 \).

73. Consider the curve \( \phi(t) = (t, 0, \sqrt{t^2 + 1}) \) in \( \mathbb{R}^3 \).

(a) Prove that this curve is in \( \mathbb{H}^2 \).

(b) Prove that the image of this curve in \( \mathbb{H}^2 \) is a line.

(c) What is the speed of this curve at \( t = -2, -1, 0, 1, 2 \). [Hint: speed=\text{length (using } \Phi \text{) of the derivative.]}
4.7. PROBLEMS

74. Consider a right angled triangle in $\mathbb{H}^2$, with right angle $\gamma$. Find a formula for $\tan \alpha$ in terms of $a$ and $b$.

75. (a) In $\mathbb{H}^2$ find the angles of the triangle with sides $3, 4, 5$.
(b) In $\mathbb{H}^2$ find the angles of the triangle with sides $0.03, 0.04, 0.05$.
(c) Find the angles of the triangle with sides $3, 4, 5$ in Euclidean geometry.
(d) Find the angles of the triangle with sides $0.03, 0.04, 0.05$ in Euclidean geometry.

76. Let $ABC\triangle$ and $A'B'C'\triangle$ have the same corresponding angles in $\mathbb{H}^2$ (that is, they are similar). Prove that they have the same corresponding sides as well (that is, they are congruent).

77. Let $ABC\triangle$ and $A'B'C'\triangle$ be perspective w.r.t. a line $\ell$ in $\mathbb{H}^2$, and suppose that $AA'$ and $BB'$ are limiting parallel. Prove that $AA'$ and $CC'$ are also limiting parallel.

78. Prove that the function $\Pi(d)$ is strictly monotone decreasing on $[0, \infty)$, and find its limit at $d = 0$ and $d = \infty$.

79. Let $X = (2, 2, 3)$, $u = (1, 5, 4)$, $v = (-5, -4, -6)$. Verify that $X \in \mathbb{H}^2$, $u, v \in T_X \mathbb{H}^2$. Let $a$ be the ray starting in $X$ in the direction of $u$. Let $b$ be the ray starting at $X$ in the direction of $v$. Give parametrizations of $a$ and $b$ (warning: $u$ and $v$ are not unit vectors w.r.t. $\Phi$). Find the angle between $a$ and $b$. 