STATISTICS 133 MIDTERM: MARCH 3 1999

Time allowed: 75 minutes, or as extended by instructor

ANSWER ALL QUESTIONS. The exam is closed book and notes, but calculators are allowed. You should show all working, but detailed proofs and derivations are not required (i.e. if you can remember the formula, quote it).

1. A filter

$$Y_t = \sum_j a_j X_{t-j}$$

has weights

$$a_2 = a_{-2} = \frac{A}{B}, \qquad a_1 = a_{-1} = \frac{4}{B}, \qquad a_0 = \frac{8}{B}, \qquad a_j = 0 \text{ for all } |j| > 2,$$

where A and B are constants.

(a) If the filter is to have the property of preserving quadratic trends (i.e. $Y_t = X_t$ whenever $X_t = C_0 + C_1 t + C_2 t^2$ for constants C_0, C_1, C_2), what should A and B be?

(b) Suppose the filter is in fact applied to a series in which $\{X_t\}$ are uncorrelated random variables with common variance σ^2. What is the variance of Y_t?

2. Consider the AR(2) process

$$X_t = (\alpha + \beta) X_{t-1} - \alpha\beta X_{t-2} + Z_t,$$

where α and β are constants and $\{Z_t\}$ are uncorrelated random variables with mean 0 and variance σ^2.

(a) Show that the process is stationary (causal) if and only if $|\alpha| < 1$ and $|\beta| < 1$.

(b) If γ_k denotes the k'th order autocovariance, show that

$$\gamma_k - (\alpha + \beta)\gamma_{k-1} + \alpha\beta\gamma_{k-2} = 0, \quad k \geq 1,$$

and write down the corresponding equation when $k = 0$.

(c) Show that the general solution to this set of equations is

$$\gamma_k = A\alpha^k + B\beta^k, \quad k \geq 0,$$

where A and B are constants. Assume $\alpha \neq \beta$.

(d) In the case (c), show that A and B may be found by solving the equations

$$(1 + \alpha\beta)(A\alpha + B\beta) - (\alpha + \beta)(A + B) = 0,$$
$$A + B - (\alpha + \beta)(A\alpha + B\beta) + \alpha\beta(A\alpha^2 + B\beta^2) = \sigma^2.$$

Note: It is not necessary to find actual values for A and B.

3. We wish to obtain forecasts for the process

$$X_t - .8X_{t-1} = Z_t + .4Z_{t-1},$$

where $\{Z_t\}$ are uncorrelated random variables of mean 0 and variance σ^2, based on observations $\{X_t,\ t = 1, 2, ..., 100\}$. The last few values are $X_{98} = 1.25$, $X_{99} = 0.9$, $X_{100} = 0.92$. Let $\widehat{X}_{T,k}$ denote the optimal forecast of X_{T+k} given observations $\{X_t,\ t \leq T\}$.

(a) Assume Z_t is known for $t \leq 100$. Write down a general expression for $\widehat{X}_{100,k},\ k \geq 1$.

(b) In fact we are told $Z_{98} = -0.5$. Use the innovations formula

$$Z_t = X_t - \widehat{X}_{t-1,1}$$

to obtain numerical values for Z_{99} and Z_{100}, and hence write down the numerical values of $\widehat{X}_{100,k}$ for $k = 1, 2, 3$.