STATISTICS 133 MIDTERM: MARCH 3 1999

ANSWERS

1. (a) The conditions required for the filter to preserve quadratic trends are (i) $\sum_j a_j = 1$, (ii) $\sum_j j a_j = 0$, (iii) $\sum_j j^2 a_j = 0$. Here (ii) is automatic by symmetry. (iii) requires $4A + 4 = 0$, hence $A = -1$. Then (i) leads to $B = 14$.

(b) $\mathrm{Var}(Y_j) = \left\{ \left(-\frac{1}{14}\right)^2 + \left(\frac{4}{14}\right)^2 + \left(\frac{8}{14}\right)^2 + \left(\frac{4}{14}\right)^2 + \left(-\frac{1}{14}\right)^2 \right\} \sigma^2 = \frac{\sigma^2}{2}$.

2 (a). $\phi(z) = 1 - (\alpha + \beta) z + \alpha\beta z^2 - (1 - \alpha z)(1 - \beta z)$ which has roots at $z = 1/\alpha,\ 1/\beta$. For causality, both roots must lie outside the unit circle, so we have $|\alpha| < 1$, $|\beta| < 1$.

(b) $\mathrm{Cov}\{X_t - (\alpha+\beta)X_{t-1} + \alpha\beta X_{t-2}, X_{t-k}\}$ $=\mathrm{Cov}\{Z_t, X_{t-k}\}$ and the latter expression is 0 if $k > 0$, σ^2 if $k = 0$ (using the expansion $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, in which $\psi_0 = 1$). Thus

$$\gamma_k - (\alpha+\beta)\gamma_{k-1} + \alpha\beta\gamma_{k-2} = \begin{cases} 0, & \text{if } k > 0, \\ \sigma^2, & \text{if } k = 0. \end{cases} \tag{1}$$

(c) Since $\alpha \neq \beta$ we can always find A and B so that the equation

$$\gamma_k = A\alpha^k + B\beta^k \tag{2}$$

holds for $k = 0$ and 1. However if $K \geq 2$ and (2) holds for $k = K - 2$ and $k = K - 1$, then (2) holds for $k = K$, as is easily seen by substituting in the equation $\gamma_K = (\alpha+\beta)\gamma_{K-1} - \alpha\beta\gamma_{K-2}$. However it then follows by induction that (2) holds for all $k \geq 0$. (*Note*: An argument based on the general theory of linear difference equations would also be acceptable for this part.)

(d) The given equations are the special cases of (1) for $k = 1$ and $k = 0$, after writing $\gamma_{-k} = \gamma_k$ and substituting the general form of solution (2) for $k \geq 0$.

3(a). From the general theory for forecasting an ARMA process, we deduce $\widehat{X}_{T,1} = .8X_T + \widehat{Z}_{T+1} + .4\widehat{Z}_T$ where $\widehat{Z}_{T+1}$ and $\widehat{Z}_T$ are optimal forecasts of Z_{T+1} and Z_T based on data up to time T. But in this notation, $\widehat{Z}_{T+1} = 0$ and we are assuming Z_T is known, so $\widehat{X}_{T,1} = .8X_T + .4Z_T$. To forecast X_{T+k} for $k > 1$, we iterate this procedure, but assuming $\widehat{Z}_t = 0$ for all $t > T$. Thus $\widehat{X}_{T,2} = .8\widehat{X}_{T,1} + \widehat{Z}_{T+2} + .4\widehat{Z}_{T+1} = .8\widehat{X}_{T,1}$, $\widehat{X}_{T,3} = .8\widehat{X}_{T,2} + \widehat{Z}_{T+3} + .4\widehat{Z}_{T+2} = .8\widehat{X}_{T,2} = (.8)^2\widehat{X}_{T,1}$, and by induction $\widehat{X}_{T,k} = (.8)^{k-1}\widehat{X}_{T,1}$ for all $k \geq 1$.

(b) Applying the formula in (a) to $T = 98$, we deduce $\widehat{X}_{98,1} = (.8)(1.25) + (.4)(-.5) = .8$, $Z_{99} = X_{99} - \widehat{X}_{98,1} = .9 - .8 = .1$. Repeating this process for $T = 99$, $\widehat{X}_{99,1} = (.8)(.9) + (.4)(.1) = .76$, $Z_{100} = X_{100} - \widehat{X}_{99,1} = .92 - .76 = .16$. Then $\widehat{X}_{100,1} = (.8)(.92) + (.4)(.16) = .8$, and hence $\widehat{X}_{100,2} = (.8)^2 = .64$, $\widehat{X}_{100,3} = (.8)^3 = .512$.