Stairway to Heaven or Highway to Hell: Liquidity, Sweat Equity, and the Uncertain Path to Ownership[*]

R. Vijay Krishna[†] Giuseppe Lopomo[‡] Curtis R. Taylor[§]

December, 2011

Abstract

We study a setting in which a principal contracts with an agent to operate a firm over an infinite time horizon when the agent is liquidity constrained and privately observes the sequence of cost realizations. We formulate the principal's problem as a dynamic program in which the state variable is the agent's continuation utility, which is naturally interpreted as his equity in the firm. We establish a bang-bang property of an optimal contract wherein the agent is incentivized only through adjustments to his future utility until achieving a critical level of equity, after which he may be incentivized through cash payments. Thus the incentive scheme resembles what is commonly regarded as a *sweat equity* contract, with all rents back loaded. The critical level of sweat equity obtains when none of the agent's liquidity constraints bind. At this point, the contract calls for efficient production in all future periods and the agent attains a vested ownership stake in the firm. Finally, properties of the theoretically optimal contract are shown to be similar to features common in real-world work-to-own franchising agreements and venture capital contracts.

Key Words: recursive contracts, dynamic screening, franchising, venture capital.

JEL Classifications: C61, D82, D86, L26

(∗) We thank Rachael Kranton and Philipp Sadowski for helpful comments, and are grateful for the able research assistance of Joe Mazur and Sergiu Ungureanu.

(†) University of North Carolina, Chapel Hill <rvk@unc.edu>

(‡) Duke University <giuseppe.lopomo@duke.edu>

(§) Duke University <crtaylor@duke.edu>

Contents

1. Introduction

Consider the common situation in which two parties form a partnership in order to operate jointly a business enterprise. The equity or cash partner (principal) possesses capital, but is unable, either due to lack of expertise or because her time and energy is best spent elsewhere, to operate the firm. By contrast, the managing partner (agent) possesses technical knowhow, but lacks access to the financial resources necessary to launch the enterprise or keep it afloat.

Real-world examples of this type of situation abound: retail franchising, venture capital, real estate development, newly minted professionals joining established firms. The salient features of these contractual settings are that: (i) the agent is liquidity constrained and cannot purchase or finance the enterprise himself, (ii) the relationship is of a long term nature, (iii) the agent has private access to knowledge regarding certain factors influencing profitability, and (iv) the principal maintains control rights over some aspects of the operation. In this paper, we provide a normative analysis of the optimal dynamic contract for the principal in a general setting possessing these characteristics.

Formally, we study an infinite-horizon discrete-time model in which the marginal cost of production evolves according to an iid process that the agent privately observes. Both principal and agent have quasilinear time-separable von Neumann-Morgenstern preferences and discount the future at the same rate. Since contracting occurs before the agent learns any private information and because allocation of risk is not germane, full efficiency could be achieved by selling the firm to the agent at its first-best expected present value. This solution, however, is assumed infeasible by supposing that the agent does not possess the requisite capital. In particular, the agent is presumed to be severely liquidity constrained and cannot experience negative cash flow in any period.[1]

These assumptions give rise to a dynamic intratemporal screening model in which the principal incentivizes the agent through both instantaneous payments as well as promised future payments. The principal also manages information rents through control of the scale of operations, that is, the output of the firm.

Our findings relate the evolution of firm dynamics to other features of the contractual relationship. In particular, we show that there is a maximal firm size, ie, scale of operations, that is achieved if (and only if) the agent becomes a ***fully vested partner*** in the firm. Moreover, we show:

- ***Backloading of rents:*** The optimal contract incentivizes the agent exclusively via promised future payments before he becomes a fully vested partner, and exclusively via instantaneous payments if he becomes a fully vested partner.
- ***Easing of liquidity:*** Liquidity constraints ameliorate as the firm grows, and vanish completely if the agent becomes a fully vested partner.
- ***Heaven or Hell:*** In the long run, with probability 1, the firm either grows to the point where the agent becomes a fully vested partner or it shrinks to the point where the principal replaces him.

In fact, our main results are best summarized collectively as a theory of ***sweat equity***, wherein the agent works for the principal without receiving rents until the scale of the firm and his equity position grow to the level of ownership or shrink to the point where he is replaced. We summarize evidence below in section 8 showing that these characteristics of the optimal dynamic contract have close parallels in real-world work-to-own franchise programs and venture capital covenants. They also resonate with features of contracts involving newly hired members of professional partnerships: a doctor joining a medical practice, an attorney joining a law firm,

(1) We discuss situations in which the agent possesses initial positive wealth in subsection 7.2.

an economist joining a consulting group, etc.

In the next section we briefly survey the relevant literature. We introduce the model formally in section 3, and describe the recursive approach we employ in section 4, where we also establish basic properties of the principal's value function, prove that the optimal contract backloads all rents, and derive a simplified version of the principal's contract design problem that is more amenable to analysis. In section 5 we derive necessary and sufficient first order conditions characterizing the solution to the principal's problem. We also derive an expression for the critical level of equity at which the agent achieves a vested ownership stake in the firm. In section 6 we describe the short and long-run dynamics induced by the optimal contract. The Lagrange multipliers associated with the liquidity constraints, or more precisely, their sum, can be interpreted as the marginal social cost of illiquidity. This, and other issues, related to various levels of ownership, path dependence of the optimal contract, and the version of the model where the principal can fire the agent, are analyzed in section 7. Section 8 contains the applications of our model mentioned above to work-to-own franchising programs and to venture capital covenants, and some concluding remarks appear in section 9. Formal proofs and some purely technical results are relegated to the appendix.

2. Related Literature

This paper contributes to a growing literature on optimal dynamic incentive schemes spanning a diverse set of research areas including: social insurance (eg, Fernandes and Phelan, 2000), taxation (eg, Albanesi and Sleet, 2006), and executive compensation (eg, Sannikov, 2008). As is common in this body of work, we employ the recursive techniques for analyzing dynamic agency problems pioneered by Green (1987) (who studied social insurance), Spear and Srivastava (1987) (who studied dynamic moral hazard), and especially Thomas and Worrall (1990) (who examined income smoothing under private information), in which shocks are iid over time and the state variable is taken to be the expected present value of the agent's utility under the continuation contract.

Of particular relevance is the recent literature on optimal financial contracting in the face of moral hazard. Specifically, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Biais et al. (2007) study various dynamic incarnations of the celebrated ***cash flow diversion***

(CFD) model.[2] Roughly, DeMarzo and Fishman (2007) explore optimal financial contracting in a general finite-horizon CFD model which DeMarzo and Sannikov (2006) formulate in continuous-time with an infinite horizon, and Biais et al. (2007) provide a model bridging the two environments. Clementi and Hopenhayn (2006) study optimal investment and capital structure in a discrete-time infinite-horizon model and Quadrini (2004) derives the optimal renegotiation-proof contract in a similar environment.

As in our setting, all of these papers assume a risk-neutral but liquidity constrained agent and a risk-neutral wealthy principal. There are, however, several key differences between the environment we study and the one analyzed in the dynamic CFD literature. First and foremost, the underlying problem facing the principal in CFD models involves moral hazard in which the agent must be given incentives either not to expropriate privately observed cash flows for his personal use or to privately exert personally costly effort. (As DeMarzo and Fishman, 2007 demonstrate, these two situations are formally equivalent.) In particular, the information privately observed by the agent in the CFD models is of no operational use to the principal—she always wants him either to not divert funds or to work hard, depending on the context of the model. Hence, her contemporaneous policy decision of how much to invest is not sensitive to the agent's private information about his action (regarding the amount of cash he expropriated or his effort choice).

Our focus, by contrast, is not on optimal investment dynamics or capital structure, but on the day-to-day operation of the firm. The principal in our model wishes to tailor her contemporaneous policy decision of how much to produce to the agent's private information regarding the marginal cost of operation. Thus, ours is a dynamic model of intratemporal screening that cannot properly be viewed as a setting of moral hazard.[3] To see this plainly, note that in the CFD models each value of the state variable gives rise to a distinct level of optimal investment, while in our setting each value of the state variable gives rise to a menu of output levels from which the agent must be given incentives to select the optimal one. While our investigation clearly touches on issues of corporate finance, our focus is rooted in questions of procurement and monopolistic screening more readily identified with industrial organization.[4]

Clearly, some of our results do have parallels in the CFD literature. For

(2) See Bolton and Scharfstein (1990) for a canonical two-period CFD model.
(3) The conditions under which *ex post* hidden information, as in the CFD models, is analogous to moral hazard are articulated in Milgrom (1987).
(4) See, for example, Laffont and Martimort (2002, p 86).

instance, we discover a bang-bang property of an optimal contract common among the CFD papers under which the agent is incentivized only through adjustments in his future utility up to a threshold, after which he is incentivized with cash payments. The CFD papers naturally interpret this as optimal financial structure; eg, debt must be retired before dividends can be paid. We, on the other hand, interpret the bang-bang property of the optimal incentive scheme as a sweat equity contract under which the agent works for the principal until he is fired or earns a permanent ownership stake in the firm. However, in both the CFD models as well as in ours, the backloading of rents is a consequence of the twin assumptions that the agent is risk neutral *and* liquidity constrained.

Questions of interpretation and implementation aside, a number of our results have no counterpart in the CFD literature. For instance, we show that there is an endogenously determined positive level of equity that the principal optimally grants the agent at the beginning of the contract. We also characterize the production mandates used to control information rents including the familiar result from static mechanism design of *no distortion at the top,* which holds in our setting for all values of the state.

Defining dead-weight loss to be the difference between the first-best value of the firm and its value (principal's share plus agent's share) at any state, allows us to relate the social cost of illiquidity to the analytical measure of the *price* of the constraints. Namely, dead-weight loss under the contract is the integral of the sum of the Lagrange multipliers between the current state and the state at which firm value is maximized (where all the multipliers drop to zero and the agent achieves a vested ownership stake in the firm).

In addition to this study, there are several other recent investigations of screening mechanisms in dynamic environments. For instance, Bergemann and Välimäki (2010) introduce and analyze a dynamic version of the *VCG* pivot mechanism. (In a similar vein, see Athey and Segal, 2007 and Covallo, 2008.) In two recent working papers, Pavan, Segal and Toikka (2009); Pavan, Segal and Toikka (2010) study dynamic screening in a setting in which the distribution of types may be non-stationary and agents' payoffs need not be time-separable. These authors derive a generalization of the envelope formula of Mirrlees (1971) for incentive compatible static mechanisms and use this to compute a dynamic representation for virtual surplus in the case of quasi-linear preferences. While their analysis is illuminating, the generality of their model prohibits use of the recursive methods that are the lynchpins of our study. Moreover, Pavan, Segal, and Toikka do not address directly the question of contracting for ownership in the face of liquidity constraints that is the focus of our investigation. Boleslavsky (2009) explores a dynamic selling mechanism in which a consumer possesses both permanent private information about his propensity to have high or low taste shocks and transitory private information about his current (conditionally independent) shock. The optimal contract in Boleslavsky's model exhibits a type of *immiseration*, in the sense that after a sufficiently long time horizon, the supplier will eventually refuse to serve the consumer.

Battaglini (2005) investigates a dynamic selling procedure in a model where a consumer's taste parameter follows a two-state (high or low) Markov process. The consumer has private information about the initial state of the process as well as subsequent states. For an initial string of reported low-demand realizations, the consumer's allocation is distorted down from the efficient level, but the distortions diminish after each report in the string. Moreover, the first time the consumer reports high demand, the contract calls for efficient output for both types from that point forward. These dynamics contrast sharply with our findings in which each *bad* report leaves the agent in a worse position and efficiency obtains only after a sufficiently long string of *good* reports. In analyzing the process of ownership acquisition, Battaglini (2005) emphasizes the role of initial and persistent private information, while we focus on the complementary part played by illiquidity and transitory private information.

3. The Model

A principal contracts with an agent to produce output in each period $t = 0, 1, 2, \ldots$. Both parties are risk-neutral, have time-separable preferences, and have a common discount factor $\delta \in (0, 1)$. If the agent produces q units in a given period, then a contractually verifiable monetary benefit (revenue) $R(q)$ is generated, where $R : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly concave, and $R(0) = 0$.[5]

The principal is *not* a bank who simply lends the agent capital. Instead, we suppose the firm possesses some market power, which leads naturally to the assumption $R'' < 0$, and which we associate with control of specialized assets such as brand recognition, an exclusive location, a proprietary business formula, or

(5) As long as revenue is contractible, it does not matter whether it accrues directly to the principal (who then compensates the agent for costs) or to the agent (who then delivers profits to the principal). We assume the former case in the text.

physical capital. The principal generally retains ownership of these assets, although they may be transferred to the agent under certain situations as we discuss in section 7.2 below.

The agent's cost of producing q units of output in a given period is θq, where $\theta \in \Theta := \{\theta_1, \ldots, \theta_n\}$, and $0 < \theta_1 < \cdots < \theta_n < \infty$.[6] We will frequently abuse notation and refer to $i, j \in \Theta$ rather than saying $\theta_i, \theta_j \in \Theta$. The cost parameter θ is drawn independently in each period according to the cumulative probability distribution F_i, where $\Pr\{\theta = \theta_i\} := f_i > 0$ for all $i \in \Theta$.

To ensure an interior solution to the contracting problem, we assume

$$[\text{MR}_0] \qquad R'(0) = \infty$$

and

$$\lim_{q \to \infty} R'(q) < \theta_1$$

Then, implicitly define the first-best output levels by $R'(q_i^*) = \theta_i$ for all $i \in \Theta$. For future reference, note that $\infty > q_1^* > q_2^* > \cdots > q_n^* > 0$; ie, first-best output is monotone decreasing in type. As always, the agent can leave at any moment in time, to an outside option worth 0 utiles.[7] There are two crucial sources of friction in the model. First, the agent is liquidity constrained and cannot incur a negative cash flow in any period. Second, the realization of the cost parameter θ in each period is observed only by the agent.[8] If either of these conditions were relaxed, it would be possible to implement the first best outcome. For instance, if θ was observed publicly in each period, the principal could simply write a forcing contract that dictated the efficient level of output q_i^* and compensated the agent for his actual costs $\theta_i q_i^*$. If, on the other hand, the agent possessed sufficient liquid resources, he could purchase the franchise from the principal at the outset for its first-best expected present value,

$$[\text{FB}] \qquad v^{\text{FB}} := \frac{1}{1-\delta} \sum_{i \in \Theta} f_i \Big(R(q_i^*) - \theta_i q_i^* \Big)$$

(6) Consider the seemingly more general specification in which output is $x \geqslant 0$; concave revenue is $B(x)$; and increasing convex cost is $\theta C(x)$. This is equivalent to the specification given in the text under the change of variables $q := C(x)$ and $R(q) := B(C^{-1}(q))$. Moreover, our results also hold under an alternative specification in which revenue is $\theta B(x)$ which is observed only by the agent and cost is $C(x)$ which is contractually verifiable.

(7) In fact, the agent's individual rationality constraint never binds (as we discuss below), so the analysis is unaltered whether we assume he has the option to quit in any period or is committed to work for the principal indefinitely.

(8) Implicit in this assumption is that the agent can divert any excess funds to his own consumption without being observed by the principal.

in which case there would be no residual incentive problem. Hence, it is the combination of illiquidity and private information that links the present with the future, giving rise to a non-trivial dynamic contracting problem.

The timing runs as follows. At the beginning of the game the principal offers the agent an infinite-horizon contract which he may accept or reject. If he rejects, then the game ends and each party receives a reservation payoff of zero. If the agent accepts the principal's offer, the contract is executed.

4. Contract Design

When designing an optimal contract, the Revelation Principle implies that the principal may restrict attention to incentive compatible direct mechanisms. Moreover, it is well known (see, eg, Thomas and Worrall, 1990) that in the setting under study, she also may restrict attention to recursive mechanisms in which the state variable is the agent's lifetime promised expected utility under the contract, denoted by v. For reasons discussed below, we refer to v as the agent's ***equity*** (or ***sweat equity***) in the firm. Hence, if the agent's current equity is v and he reports θ_i, then the contract specifies the amount of output he is to produce $q_i(v)$, the amount he is to be compensated by the principal $m_i(v)$, and his level of equity starting next period $w_i(v)$. (To ease notation, we frequently suppress dependence of the contractual terms on v.)

In fact, it is convenient, both notationally and conceptually, to define the agent's instantaneous rent as $u_i := m_i - \theta_i q_i$ and to consider contracts of the form (u, q, w) rather than (m, q, w). We now present the contractual constraints under this formulation.

Promise Keeping: The ***promise keeping*** constraint that the contract must obey is written

[PK] $$\sum_{i=1}^{n} f_i\left(u_i + \delta w_i\right) = v$$

The agent's lifetime expected payoff, v, is composed of his expected payoff in the current period, u_i, and his expected continuation payoff, δw_i.

Incentives: The set of incentive constraints is

$$[\mathrm{C}_{ij}] \qquad u_i + \delta w_i \geqslant u_j + \delta w_j + (\theta_j - \theta_i) q_j$$

for all $i, j \in \Theta$. Simply put, the agent's payoff from truthfully reporting i must be no less than what he could obtain by reporting any j, namely the truthful payoff from reporting j plus the applicable cost difference.

Liquidity: The agent's liquidity constraints are written

$$[\mathrm{L}'_i] \qquad u_i \geqslant 0$$

for all $i \in \Theta$. That is, when the agent reports truthfully, the monetary transfer he receives from the principal, m_i, must cover his production costs $\theta_i q_i$.[9] As written, the liquidity constraints do not permit wealth accumulation by the agent. In other words, he has no method for saving any positive rents $m_i - \theta_i q_i > 0$ to ease liquidity in the future. While this appears to be a restrictive assumption, it is actually completely innocuous because the principal saves (and dis-saves) on the agent's behalf by adjusting his equity v. Of course, the contract could specify that the agent save any positive rents in a verifiable bank account, but this would be functionally equivalent to using equity adjustments and operationally more cumbersome.[10]

Participation: The continuation utility w_i is the sum of expected future rents, and because instantaneous rents to the agent can never be negative, it follows that we must include feasibility constraints that require $w_i \geqslant 0$ for all i. Thus, promise keeping (PK) implies that the agent's lifetime expected utility v is always nonnegative, and the participation constraint that the contract initially offer him nonnegative lifetime utility may be ignored.

The following proposition shows that the principal's problem can be written as a dynamic program, and establishes that an optimal contract exists by virtue of being the corresponding policy function.

(9) Strictly speaking, the full set of liquidity constraints is $m_j - \theta_i q_j \geqslant 0$ for all $i, j \in \Theta$. That is, the agent cannot spend more money than the principal gives him in any period, whether or not he reports truthfully. However, because we are imposing incentive compatibility (C_{ij}), there is no loss in generality from restricting attention to the subset of liquidity constraints associated with truthful reporting.

(10) See Edmans et al. (2010) for a novel use of 'incentive accounts' in the context of executive compensation.

Theorem 1. *The principal's discounted expected utility under an optimal contract, (u, q, w), is represented by a unique, concave, and continuously differentiable function $P : \mathbb{R}_+ \to \mathbb{R}$ that satisfies*

$$\text{[VF}'\text{]} \qquad P(v) = \max_{(u,q,w)} \sum_i f_i\Big[\Big(R(q_i) - \theta_i q_i\Big) - u_i + \delta P(w_i)\Big]$$

subject to: promise keeping (PK), incentive compatibility (C_{ij}), liquidity (L_i'), and feasibility $q_i \geqslant 0$ and $w_i \geqslant 0$ for all $i \in \Theta$. Moreover, there exists $v^* \in (0, \infty)$ such that $P'(v) > -1$ for $0 \leqslant v < v^*$ and $P'(v) = -1$ for $v \geqslant v^*$, and $P'(0) = \infty$.

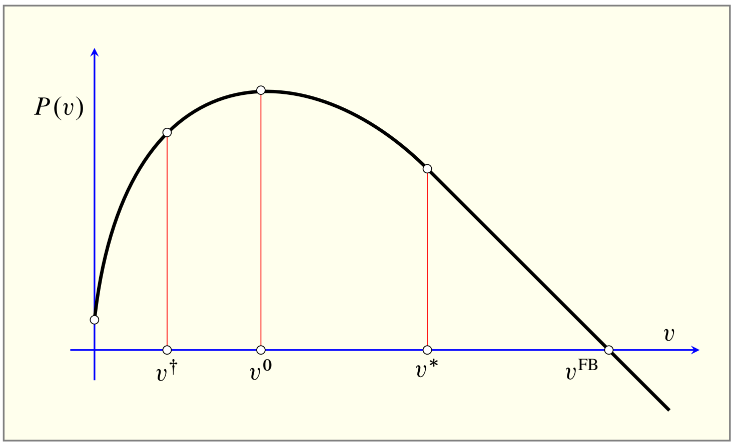

Figure 1: *Principal's Value Function*

Theorem 1 provides some clues to the structure of an optimal contract. In particular (MR_0), namely the assumption that $R'(0) = \infty$, ensures $P'(0) = \infty$. In other words, the principal's payoff is initially increasing in the agent's equity. This, along with the facts that $P'(v) = -1$ for $v \geqslant v^*$ and that $P(v)$ is concave, implies that there exists a level of equity $v^0 \in (0, v^*)$ satisfying $P'(v^0) = 0$ at which the principal's discounted expected payoff is maximized (see figure 1). This is the level of equity at which the principal initially stakes the agent upon signing the contract.

Note, however, that the social surplus (ie, firm value) $P(v) + v$ is maximized at any $v \geqslant v^*$.[11] In other words, the value of the contractual relationship continues to grow until $v = v^*$. The following result shows that any optimal contract must have a *bang-bang* structure in the sense that all rents are backloaded.

(11) This follows since $P(v) + v$ is continuously differentiable, and has derivative $P'(v) + 1$, which is strictly positive for all $v < v^*$, and is 0 for all $v \geqslant v^*$.

Proposition 4.1. For any optimal contract (u, q, w), incentives are provided purely through adjustments in the agent's equity whenever his stake in the firm is sufficiently low – in particular,

$$w_i(v) < v^* \text{ implies } u_i(v) = 0$$

Moreover, there exists a ***maximal rent*** optimal contract in which incentives are provided purely through payment of rents if the agent's stake in the franchise is sufficiently high – specifically, for all v, $w_i(v) \leqslant v^*$ and

$$u_i(v) > 0 \text{ implies } w_i(v) = v^*$$

Proposition 4.1 underpins the interpretation of the optimal incentive scheme as a sweat equity contract. For $v < v^*$, if it is the case that $w_i(v) < v^*$, that is, the agent does not reach $v = v^*$ in the next period, it must be that the agent earns no instantaneous rents, but instead is incentivized purely through adjustments to his equity position. Once $v = v^*$, however, the agent – as we discuss below – achieves a permanent ownership stake in the firm and earns nonnegative instantaneous rents from that point forward.

The intuition behind this result is that in the dynamic setting, the principal can induce truth telling via two instruments: instantaneous rent u_i and continuation utility w_i, the latter being the sum of expected future rents. The problem with providing incentives through current rent, u_i, is that this must be non-negative due to illliquidity; ie, the agent can only be rewarded and never penalized. Moreover, any instantaneous rent awarded to the agent is spent outside the contractual relationship and therefore does not benefit the principal. If, however, the principal chooses to provide the necessary incentives through continuation payoffs w_i, then she can reward the agent by adjusting his equity up or penalize him by adjusting it down. Hence, providing incentives through continuation utility has two advantages: it keeps payments inside the relationship and it permits penalties. Once $v = v^*$, liquidity constraints no longer bind (ie, penalties become irrelevant,) and the principal can provide the requisite incentives purely through instantaneous rents.

To aid with analysis and obtain a sharper characterization of an optimal contract, It is helpful to reformulate the principal's program in a simpler way (with fewer constraints and choice variables). To this end, first consider the following definition.

Definition 4.2 (Monotonicity in Type)**.** Output is said to be ***monotonic in***

type if for all $v \geqslant 0$,

$$[\mathrm{M}_i] \qquad q_i(v) \geqslant q_{i+1}(v)$$

for all $i = 1, \ldots, n-1$. Analogous definitions apply for rent $u_i(v)$ and promised utility $w_i(v)$.

In static mechanism design, inequalities analogous to (M_i) are often referred to as ***implementability conditions*** because it is generally not possible to implement non-monotonic allocations.

Next, consider the binding version of the upward adjacent incentive constraints that say the agent must be indifferent between reporting his true marginal cost and one level higher:

$$[\mathrm{C}_i] \qquad u_i + \delta w_i = u_{i+1} + \delta w_{i+1} + \Delta_i q_{i+1}$$

for all $i = 1, \ldots n-1$, where $\Delta_i := \theta_{i+1} - \theta_i$.

The following lemma establishes a result familiar from static mechanism design that the large set of incentive constraints (C_{ij}) may be replaced by a much smaller set, namely (M_i) and (C_i).

Lemma 4.3. If output is monotonic in type (M_i) and the upward adjacent incentive constraints bind (C_i), then all incentive constraints (C_{ij}) are satisfied. Moreover, there exists a maximal rent optimal contract (u, q, w) in which (M_i) and (C_i) hold, and in any such contract, instantaneous rent and promised utility are also monotonic in type.

Next, the following lemma uses (PK) and (C_i) to derive a key expression for the agent's current payoff.

Lemma 4.4. In any optimal contract, the agent's payoff satisfies

$$[\mathrm{U}_i] \qquad u_i + \delta w_i = v - \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1} + \sum_{j=i}^{n-1} \Delta_j q_{j+1}$$

for all $i = 1, \ldots, n$. Moreover, (U_i) implies (PK) and (C_i).

Equation (U_i) says that the current payoff to the agent when he is type i is his promised expected level of equity from the prior period (first term on the right) minus his expected information rent (second term) plus his realized information rent (third term).

The equations (U_i), which imply (PK) and (C_i) can be used to eliminate instantaneous rents, u_i, from the principal's program (VF′). Specifically, the liquidity constraints (L_i'), requiring $u_i \geqslant 0$, can be recast as

$$[\mathrm{L}_i] \qquad \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1} - \sum_{j=i}^{n-1} \Delta_j q_{j+1} + \delta w_i \leqslant v$$

for all $i \in \Theta$. Using this version of the liquidity constraints and substituting (PK) directly into the principal's objective yields the following intuitive reformulation of the contract design program.

Theorem 2. *The principal's value function $P : \mathbb{R}_+ \to \mathbb{R}$ is a solution to the following relaxed program:*

$$[\mathrm{VF}] \qquad P(v) = \max_{(q,w)} \sum_i f_i \Big[\Big(R(q_i) - \theta_i q_i\Big) + \delta\Big(P(w_i) + w_i\Big)\Big] - v,$$

subject to monotonicity in output (M_i), liquidity (L_i), and feasibility $q_n \geqslant 0$ and $w_n \geqslant 0$. Moreover, there is a solution to this program that is a maximal rent contract in which $u_i(v)$ and $w_i(v)$ are monotonic in type. This optimal contract (q, w) is unique and continuous in v.

This version of the principal's program is substantially simpler than the one presented in Theorem 1, involving n^2 fewer constraints and n fewer choice variables. This version of the program also has an intuitive interpretation. The term $\sum_i f_i \Big[R(q_i) - \theta_i q_i\Big]$ is simply expected instantaneous social surplus (current profit), while the term $\sum_i f_i \delta \Big[w_i + P(w_i)\Big]$ is the expected continuation surplus (future profit). Also, v is just the sum of present and future expected rents owed to the agent. Therefore, $P(v)$ is just the dynamic analogue of the objective in the static problem, wherein the principal wants to maximize expected social surplus (ie, the value of the firm) net of any expected information rents.

Note that in the absence of liquidity constraints, the first order condition for q_i would be $R'(q_i) - \theta_i q_i = 0$, implying $q_i = q_i^*$, and the first order condition for w_i would be $P'(w_i) + 1 = 0$, implying $w_i = v^*$. Moreover, the principal would set the agent's initial equity at $v = 0$ to ensure his participation. But then (U_i) and lemma 5.1 in the next section give the agent's first-period rents as

$$u_i = \sum_{j=i}^{n-1} \Delta_j q_{j+1} - v^*,$$

which is evidently negative for high cost realizations. Hence, it is the presence of the binding liquidity constraints that causes the principal to distort output levels away from first-best. We investigate these distortions in the next section.

5. *Optimal Contracts*

In the previous section, we noted that we can formulate the principal's problem as a dynamic program with only liquidity, implementability, and feasibility constraints. For any value of v, the optimal value of $\big(q(v), w(v)\big)$ is the solution to a concave programming problem, hence first order conditions are both necessary and sufficient. Let λ_i be the Lagrange multiplier associated with the liquidity constraint (L_i) and μ_i the Lagrange multiplier of the implementability constraint $q_i \geqslant q_{i+1}$ with $q_{n+1} = 0$ for all i. Since $P'(0) = \infty$, we will ignore the constraint $w_n \geqslant 0$ whenever $v > 0$. Since $P'(v) = -1$ for $v \geqslant v^*$, we can also ignore the constraint $w_i \leqslant v^*$. For the moment, let us ignore the constraint (M_1), that is, the constraint (M_i) for $i = 1$. (proposition 5.2 below shows that this is without loss of generality.)

The first order condition for q_1 is simply $R'(q_1) = \theta_1$, that is $q_1 = q_1^*$. This is the familiar result from static monopolistic screening that there is *no distortion at the top*, which holds here for all $v \geqslant 0$. The first order condition for q_i, for any $i > 1$, is

$$R'(q_i) - \theta_i = \frac{\Delta_{i-1}}{f_i} \sum_{k=1}^{n} \lambda_k \Big[F_{i-1} - \mathbb{I}\{k < i\}\Big] - \frac{1}{f_i}(\mu_i - \mu_{i-1})$$

$$[\mathrm{FO}q_i] \qquad = \frac{\Delta_{i-1}}{f_i} \Big[F_{i-1}\Lambda_n - \Lambda_{i-1}\Big] - \frac{1}{f_i}(\mu_i - \mu_{i-1})$$

where $\Lambda_k = \sum_{j=1}^{k} \lambda_j$ for all k.

By Theorem 1, we know that the value function P is continuously differentiable. Therefore, the first order condition for w_i is

$$[\mathrm{FO}w_i] \qquad P'(w_i) = -1 + \frac{\lambda_i}{f_i}$$

Finally, the envelope condition is

$$[\mathrm{env}] \qquad P'(v) = -1 + \Lambda_n$$

The first order conditions permit calculation of v^* as presented in the following lemma.

Lemma 5.1. The critical level of equity is

$$v^* = \frac{1}{1-\delta}\sum_{j=1}^{n-1} F_j \Delta_j q^*_{j+1}. \qquad \text{[Vest]}$$

Hence, v^* is the present value of receiving expected rents from efficient production (that is, output without distortions) in perpetuity. Moreover, since $P'(v) = -1$ for all $v \geqslant v^*$, it must be that $\lambda_i(v) = 0$ for all i, $v \geqslant v^*$. That is, v^* is the lowest equity level at which none of the agent's liquidity constraints bind and correspondingly the lowest equity level at which no production levels are distorted.

The following proposition establishes a result familiar from static mechanism design; namely that for $v < v^*$, the principal distorts output levels down (and never up) in order to control information rents.

Proposition 5.2. In any maximal rent optimal contract, the agent never produces more than first-best output, that is $q_i(v) \leqslant q_i^*$ for $i = 1\ldots, n$ and $v \in [0, v^*]$.

At $v = 0$, the contract calls for virtual shutdown in the sense that the agent produces positive output only in the lowest-cost state (this is lemma A.1 in the appendix): $q_1(0) = q_1^*$ and $q_i(0) = 0$ for $i = 2, \ldots, n$.[12] As v increases, output restrictions are relaxed until $v = v^*$, at which point the contract calls for efficient production for all cost realizations: $q_i(v^*) = q_i^*$ for $i = 1\ldots, n$. The agent's promised future utility levels also rise in sweat equity. At $v = 0$, he *never* receives any rents, implying $w_i(0) = 0$ for $i = 1, \ldots, n$. Again, as v increases, promised future utility levels rise until $v = v^*$, when the agent becomes a ***vested partner*** achieving a permanent ownership stake, with $w_i(v^*) = v^*$ for $i = 1, \ldots, n$. At low levels of v, the agent's liquidity constraints are tight and the contract imposes stringent output restrictions along with correspondingly low levels of promised future utility. As we prove in the next section, if the agent makes a favorable report at this point, he is rewarded with higher equity. This relaxes his liquidity constraints ultimately leading to less strict output controls and still higher levels of promised future utility.[13]

(12) To be sure, the assumption $R'(0) = \infty$ implies that the limiting case of $v = 0$ and the concomitant virtual shutdown never obtains on any finite sample path; ie $w_i(v) > 0$ for all $v > 0$ and all $i \in \Theta$.

(13) Strictly speaking, this discussion pertains to monotonicity when moving discretely from $v = 0$ to $v = v^*$. The functions $q_i(v)$, $w_i(v)$, and $\lambda_i(v)$ are continuous for $0 \leqslant v \leqslant v^*$, but we have been unable to prove that they are monotone at every point in this range (although we suspect this to be true for suitable specifications of R).

6. Dynamics

We next derive both short- and long-run dynamics of the contractual relationship. Our first observation follows directly from summing the first order conditions for w_i (FOw_i) and substituting from the envelope condition (env).

Lemma 6.1. The optimal contract induces a process P' that is a martingale: ie,

$$P'(v) = \sum_{i=1}^{n} f_i P'(w_i).$$

To see this, consider an increase in v by one unit. This can be achieved by increasing all the w_i's by $1/\delta$. The cost of this to the principal is $\sum_i f_i \Big[1+P'(w_i)\Big]-1$. By the envelope theorem, this is locally optimal, and hence is equal to $P'(v)$.

An important consequence of the martingale property of P' is that a shock of $\theta = \theta_1$ is necessarily *good*, in the sense that the continuation value of sweat equity $w_1 > v$, while a shock of $\theta = \theta_n$ is unambiguously *bad*, $w_n < v$. More generally, we have the following.

Proposition 6.2. In the optimal maximal rent contract, for all $v \in (0, v^*)$, we have $P'(w_n) > P'(v) > P'(w_1)$. Moreover, $w_1(v) > v > w_n(v)$.

This captures the short-run consequences of good and bad shocks. To see the intuition, suppose, for simplicity, that P is strictly concave on $(0, v^*)$. Since P' is a martingale, if the proposition were not true, it would follow that $P'(w_i) = P'(v)$ for all $i \in \Theta$, which implies (if P is strictly concave) that $w_i(v) = v < v^*$ for all $i \in \Theta$. But proposition 4.1 also requires that for such a v, $u_i(v) = 0$, which violates promise keeping (PK), and by incentive compatibility, would require that $q_i = 0$ for all $i > 1$. Therefore, incentive compatibility and promise keeping force the principal to spread out the agent's continuation utilities, rewarding him for favorable (low) cost reports and penalizing him for unfavorable (high) cost ones. While we are unable to establish that P is strictly concave, the proof can be extended to the case where P is merely concave (see the appendix).

We are now in a position to describe the long-run properties of the optimal contract. Recall that the agent becomes a ***vested partner*** if his equity level reaches v^*.

Theorem 3. *The martingale P' converges almost surely to $P'_\infty = -1$. Thus, the agent becomes a vested partner with probability* 1*.*

From the martingale convergence theorem, it follows that P' must converge, almost surely, to an integrable random variable P'_∞. The theorem establishes that along almost all sample paths, this limit must be -1. That P' cannot settle down to a finite limit greater than -1 follows from proposition 6.2 above and the continuity of the contract in v.

Proposition 6.2 says that for $v \in (0, v^*)$, the set of continuation utilities $\{w_1(v), \ldots, w_n(v)\}$ can always be partitioned into two non-empty subsets, one in which the agent is rewarded for reporting low costs and one in which he is penalized for reporting high costs. Because $P'(0) = \infty$, an arbitrarily long string of penalties never pushes the agent's continuation utility into the absorbing state at $v = 0$. An arbitrarily long string of rewards, however, will eventually drive his continuation utility into the absorbing state at $v = v^*$. Theorem 3 says that with probability 1, the agent will eventually experience a sufficiently long sequence of rewards to become a vested partner in the firm.

7. Discussion and Extensions

7.1. The Social Cost of Illiquidity

Define firm value, or what is the same in this instance, social surplus, under an optimal contract as $S(v) := P(v) + v$. By Theorem 1, $S(v)$ is an increasing, concave and continuously differentiable function. In particular, we know that $S(v)$ is strictly increasing on $[0, v^*)$, and $S(v) = v^{\mathrm{FB}} = \frac{1}{1-\delta} \sum_i f_i \Big[R(q_i^*) - \theta_i q_i^*\Big]$ for all $v \geqslant v^*$.[14] Moreover, by the envelope condition (env), we see that $S'(v) = P'(v) + 1 = \Lambda_n(v)$. Therefore, Λ_n measures the marginal social cost of illiquidity (which is decreasing in v). Hence, for any $v < v^*$, the ***dead-weight loss*** generated by an optimal

(14) To see this, recall that for all i, $q_i(v^*) = q_i^*$ and $w_i(v^*) = v^*$. Substitution into (VF) then yields $P(v^*) + v^* = \frac{1}{1-\delta} \sum_i f_i \big[R(q_i^*) - \theta_i q_i^*\big] = v^{\mathrm{FB}}$, and hence, $S(v^*) = v^{\mathrm{FB}}$. Moreover, $P(v)$ is continuous and $P'(v) = -1$ for $v \geqslant v^*$, so $S(v) = S(v^*)$ for $v > v^*$. It also follows from this that $P(v^*) > 0$ if, and only if, $v^* > v^{\mathrm{FB}}$, the latter being a condition that depends on the primitives of the model.

contract is

$$v^{\mathrm{FB}} - S(v) \;=\; \int_v^{v^*} \Lambda_n(x)\,\mathrm{d}x.$$

This cost represents the loss in social surplus arising from the output restrictions the principal imposes to control information rents. As the agent's stake in the enterprise grows, his liquidity constraints become less stringent and output restrictions are *relaxed*. At $v = v^*$, all output levels are first-best and dead-weight loss is consequently nil.

7.2. The Path to Ownership

When exploring firm ownership, it is useful to distinguish between two paradigms as discussed by Bolton and Scharfstein (1998). One school of thought, due to Berle and Means (1968), defines ownership as residual claims over the cash flows of the firm. A second school, pioneered by Grossman and Hart (1986) and Hart and Moore (1990), identifies ownership of the firm with control rights over productive assets. In our model, the formal contract between the principal and agent is purely financial, identifying firm ownership with the Berle-Means interpretation. Nevertheless, it is possible to include an option for the principal to transfer control of the productive assets to the agent in certain situations, thereby permitting us to regard firm ownership in the Grossman-Hart-Moore sense as well.

It follows from our assumption on the absence of fixed costs, ie, $R(0) = 0$, that first-best profit is nonnegative in every state.[15] Recall that under a maximal rent optimal contract, the agent's equity is capped at v^* and he is incentivized with cash from that point forward. However, once the agent attains equity of v^*, all output distortions are eliminated, and both the principal and agent are indifferent between providing incentives with cash or further equity adjustments.

Suppose now that $v^* < v^{\mathrm{FB}}$, so that $P(v^*) = v^{\mathrm{FB}} - v^* > 0 = P(v^{\mathrm{FB}})$, and consider a contract under which the agent continues to be incentivized with sweat equity until $v = v^{\mathrm{FB}}$. Indeed, once the agent reaches v^*, he will move monotonically to v^{FB} because (as is easily seen from U_i) $w_n(v) = \dfrac{v - (1-\delta)v^*}{\delta} \geqslant v$ for $v \geqslant v^*$. Once $v = v^{\mathrm{FB}}$, the principal owes the agent cash flows equal to the first-best value of the firm; ie $P(v^{\mathrm{FB}}) = 0$. At this point the principal can simply transfer control of the productive assets to the agent and terminate the contractual relationship.

(15) If this were not the case, then the agents lack of liquidity would prohibit full ownership of the productive assets.

Our model is somewhat less well suited to analyze transfer of asset ownership in the case when $v^* > v^{\mathrm{FB}}$. In this instance, output distortions are not completely eliminated until the principal owes the agent cash flows in excess of the first-best value of the firm. If $v = v^*$, one can imagine the principal paying the agent a termination fee of $v^* - v^{\mathrm{FB}}$ and transferring control of the firm to him. The trouble is that if relinquishing control is an option formally available to the principal, then she should exercise it before $v = v^*$ because $v^* > v^{\mathrm{FB}}$ implies $0 > P(v^{\mathrm{FB}}) > P(v^*)$. The principal could eliminate the negative part of the value function by relinquishing control to the agent at the point when $P(v) = 0$. Of course, this would impact her incentives to distort output at lower equity levels as well as the value function itself, but would not alter the qualitative nature of the optimal contract.

We conclude the discussion of ownership with a few words concerning the situation in which the agent has positive initial wealth. Theorem 1 implies the following result.

Corollary 7.1. Suppose $v^* \leqslant v^{\mathrm{FB}}$ and the agent has initial liquid wealth of $y > 0$.

(a) If $y \leqslant v^0$, then the agent surrenders y to the principal and receives initial equity v^0. Initial welfare is $S(v^0) < v^{\mathrm{FB}}$.

(b) If $v^0 < y < v^*$, then the agent surrenders y to the principal and receives initial equity y. Initial welfare is $S(y) \in (S(v^0), v^{\mathrm{FB}})$.

(c) If $y \geqslant v^*$, then the agent surrenders at least v^* and receives a like amount in initial equity. Initial welfare is v^{FB}.

If the agent possesses initial liquid wealth of $y > 0$, then the principal, who has all the bargaining power, can require the agent to buy his way into the contract. If $y < v^0$, then it is optimal for the principal to demand y from the agent and grant him the starting equity level v^0. If $v^0 < y < v^*$, then the principal receives $S(y)$ by requiring the agent to tender all his wealth. Since $S(y)$ is increasing, higher values of y result in a higher initial payoff for the principal. Finally, if $y \geqslant v^*$, then the agent has enough initial wealth to become a vested partner from the outset; ie, liquidity constraints never bind and the contract is first-best. While it is common wisdom that incentive problems can be eliminated under *ex post* private information by *selling the firm* to the agent, note that if $v^* < v^{\mathrm{FB}}$, then it is not necessary to sell the entire firm because the first-best outcome obtains if the agent's equity position is at least v^*.

7.3. Hiring and Firing

Now, suppose there is an infinite pool of identical agents, but that the principal can only contract with one at a time. The principal may, however, *fire* the current agent and replace him with a new one. If the principal fires an agent, then she must make a severance payment to him equal to the current level of sweat equity.

Proposition 7.2. There exists a critical level of equity $v^\dagger \in (0, v^0)$ such that it is optimal to fire the agent if sweat equity falls below $v^\dagger$ (also see figure 1).

Lemma C.1 in the appendix shows that for any $C > 0$, the process P' is greater than C with strictly positive probability. Hence, there is a strictly positive probability that sweat equity will fall below any positive $v \in (0, v^0)$, and hence, a positive probability that a given agent will get fired. Moreover, Doob's Maximal Inequality (see, for instance, Theorem 2.4 of Steele, 2001) provides a bound for this probability, wherein, the probability that $P'(v) \geqslant C$ is less than $1/(1 + C)$.

To formally incorporate the option to replace an agent it is necessary to introduce a new value function $Q(v)$. For any function $Q : \mathbb{R}_+ \to \mathbb{R}$ bounded above, let $v^0_Q \in \arg\max_x Q(x)$. Now let Q be the unique function that satisfies

$$Q(v) = \max \left[Q(v^0_Q) - v,\ \max_{(q,w)} \mathbf{E} \left[\Big(R(q_i) - \theta_i q_i \Big) + \delta \Big(Q(w_i) + w_i \Big) \right] - v \right]$$
$$\text{s.t. } (\mathrm{M}_i),\ (\mathrm{L}_i),\ q_n \geqslant 0 \text{ and } w_n \geqslant 0$$

At any level of sweat equity v such that it is not optimal to fire the current agent, $Q(v)$ obviously has the same properties as $P(v)$, although it lies above $P(v)$ for $v < v^*$ because the option to replace the agent has positive value since it is exercised with positive probability. Hence, for any $v < v^0_Q$ such that firing is not optimal, $Q(v)$ is increasing. Since $Q(v^0_Q) - v$ is decreasing, there exists a state $v^\dagger$ such that it is optimal to fire the agent if $v < v^\dagger$ and to retain him if $v > v^\dagger$.

In essence, the option to reset the process allows the principal to avoid very low levels of sweat equity and the associated large output restrictions. Rather than waiting for the agent to make the long and erratic climb back to v^0_Q, the principal simply pays him off and begins again with a new agent.

7.4. Path Dependence

The maximal rent optimal contract specifies (q, w) as a function of equity, v. Therefore, the evolution of (q, w) depends on the evolution of v. Typically, the evolution of v along any sample path will depend on the order of shocks, which is true of models of dynamic contracting in general. Nevertheless, there is a very strong form of path dependence that holds in our model. There are two reasons for this: firstly, once $v = v^*$, output is always first-best efficient from then on, and in any optimal contract, v never falls below v^* again, and second, from any initial $v > 0$, v^* can be reached in finitely many periods.[16]

More specifically, for any initial $v_0 \in (0, v^*)$, there exists an integer $\tau < \infty$ such that if the agent repeatedly receives θ_1 shocks over τ periods (which happens with strictly positive probability), he will reach v^*, ie he will have $v_\tau = v^*$, in τ periods. This relies on two observations (see lemma C.1 in the appendix). The first observation is that for any $v \in (0, v^*)$ and γ such that $P'(v) > \gamma > -1$, there is a $\tau < \infty$ such that if state θ_1 is repeated τ times, $P'(v^{(\tau)}) < \gamma$. The second observation is that for $v_\tau < v^*$, the sequence $v_\tau - v_{\tau-1}$ is increasing which implies that v_τ reaches v^* in finitely many steps.

In sum, we have argued that from any initial level of sweat equity, the agent will reach v^* with positive probability in a finite number of periods. Therefore, in an arbitrary sample path, the order of the occurrences of shocks matters greatly. In any sample path where θ_1 occurs sufficiently often, the agent strictly prefers to have all the θ_1 shocks in the beginning, since this will place him at v^* in finitely many periods, giving him a permanent ownership stake in the firm. Notice that this result holds for all revenue functions R that satisfy our assumptions. This is in contrast with a result in Thomas and Worrall (1990), where it is shown that when an agent with a private endowment has CARA utility, the optimal lending contract with a risk neutral principal takes a simple form, where it is only the number of times a particular state (private income shock) has occurred that matters, and the order in which the shocks occur is irrelevant.

(16) This second property is what distinguishes our strong form of path dependence from the results in, for instance, Thomas and Worrall (1990). In that paper, immiseration occurs (with probability 1,) and the agent's lifetime utility goes to $-\infty$, but takes infinitely long to do so.

8. Applications

In order to focus on the underlying fundamental economic forces, the model analyzed above is necessarily stylized. Nevertheless, the environment we investigate, involving a liquidity-constrained entrepreneur who must contract for initial rounds of operating capital, has obvious real-world counterparts. In this section we briefly discuss the two examples mentioned in the introduction, work-to-own franchise programs and venture capital contracts. In each of these settings, numerous features of the agreements closely parallel aspects of the theoretically optimal contract.

Work-to-Own Franchising Programs

Franchising is a ubiquitous organizational form, especially in retailing. According to Blair and Lafontaine (2005, pages 8–13), 34% of US retail sales in 1986 (almost 13% of GDP) derived from franchised outlets. Estimates on the number of US franchisers vary widely, but listings in directories suggest a figure between 2,500 and 3,000. The basic reasons for the prevalence of the franchise relationship accord well with our model. The franchiser wishes to expand into a specific market but lacks idiosyncratic knowledge about local factors influencing profitability such as demand and cost fluctuations. The franchisee observes local conditions but lacks brand recognition and an established business formula. Often, the franchisee also lacks sufficient seed capital for getting the business off the ground. For instance, Blair and Lafontaine (2005, page 97) suggest that franchisee capital constraints partially explain the wide discrepancy between the franchise fee of $125,000 charged by McDonald's in 1982 and the estimated present value of restaurant profits of between $300,000 and $450,000 over the duration of the contract.

In fact, many franchisers have explicit work-to-own or sweat equity programs designed to allow liquidity constrained managers to become owners of their own franchises. These arrangements span a wide variety of retail businesses and industries including: *7-11* convenience stores, Big-O-Tires, Charley's Steakery, Fastframe, Fleet Feet Sports, Lawn Doctor, Petland, Outback Steakhouse, and Quiznos sandwiches, to name but a few. While details of sweat equity arrangements vary across franchisers, Quiznos' *Operating Partner Program* is broadly representative, enabling experienced managers to receive financing from the parent company for all but $5,000 of the up-front investment. A recent interview with Quiznos' executive John Fitchett highlights the similarities between the restaurant chain's sweat equity program and the theoretically optimal contract discussed above.[17]

Private information and liquidity constraints: 'The Operating Partner Program was developed in response to a successful pool of qualified, interested entrepreneurs with restaurant experience who would make great franchise owners, but lack access to the necessary financing ... '

Sweat Equity and ownership: 'Operating partners earn a salary and benefits as they work toward full ownership of the restaurant, with 80 percent of profits paying down Quiznos' contribution on a monthly basis. ... we believe an operating partner that successfully operates the restaurant can reach the point of being able to acquire full ownership in two to five years ...'.

Path dependence and replacement: 'For the first year, Quiznos will cover any losses, and the amount will be added to the loan value. After 12 months, if the restaurant has not reached profitability, Quiznos and the operator will determine whether the operator is running his or her restaurant in the most effective way, or if there are other circumstances that may influence the profitability of the restaurant. [We will then] evaluate whether to put a new operator in the restaurant.'

Venture Capital Contracts

Another contractual setting that accords neatly with our model is the venture capital market. Founders often wish to launch a business based on their personal expertise but do not possess sufficient financial resources. Venture capitalists (VCs) provide liquidity to startups staging subsequent investments and founder compensation based on various performance criteria. Indeed, HBS (2000), a case study by Harvard Business School, reports 'A central concept used by VCs in structuring their investments is "earn in", in which the entrepreneur earns his equity through succeeding at value creation ... VCs also insist on vesting schedules for options or stock grants, whereby managers earn their stakes over a period of years'.

In a pioneering article, Kaplan and Stromberg (2003) investigate 213 VC investments in 119 portfolio companies by 14 VC firms. Their findings also corroborate features of our optimal dynamic mechanism.

(17) See Liddle (2010).

> In general, board rights, voting rights, and liquidation rights are allocated such that if the firm performs poorly, the VCs obtain full control. As performance improves, the entrepreneur retains/obtains more control rights. If the firm performs very well, the VCs retain their cash flow rights, but relinquish most of their control and liquidation rights. Ventures in which the VCs have voting and board majorities are also more likely to make the entrepreneur's equity claim and the release of committed funds contingent on performance milestones.

While our stylized model does not directly address the plethora of contingencies and control rights found in typical VC contracts, Kaplan and Stromberg's findings are consistent with the main features of the theoretically optimal mechanism. Specifically, v, or sweat equity, is a summary statistic of past performance, and greater sweat equity leads to reductions in output distortions, less stringent liquidity constraints, and eventually to agent ownership, while lower sweat equity results in higher distortions, more stringent liquidity constraints, and ultimately even to replacement of the agent. In fact, the founders of poorly performing ventures are frequently ousted by the VCs who either take direct control of the company themselves or hire new management. According to White, D'Souza and McIlwraith (2007), VC's replace the founder with a new CEO in up to 50% of all venture-backed startups.

9. Conclusion

In this paper we explore the question of how a principal optimally contracts with an agent to operate a business enterprise over an infinite time horizon when the agent is liquidity constrained and has access to private information about the sequence of cost realizations. We formulate the mechanism design problem as a recursive dynamic program in which promised utility to the agent constitutes the relevant state variable.

We establish a bang-bang property of an optimal contract, wherein the agent is incentivized only through adjustments to his equity until achieving a critical level, after which he may be incentivized through cash payments. We can, therefore, interpret the incentive scheme as a sweat equity contract, where all rent payments are back loaded. The critical level of sweat equity occurs when none of the agent's liquidity constraints bind. At this point, the contract calls for efficient production in all future periods and the agent earns a permanent ownership stake in the enterprise, ie, he becomes a vested partner.

We demonstrate that the derivative of the principal's value function is a martingale, yielding several implications. First, for a given level of sweat equity, the set of cost reports can be partitioned into two subsets, *good* reports leading to higher levels of sweat equity and *bad* reports leading to lower levels. Second, if the principal cannot fire the agent, the Martingale Convergence Theorem implies that he will eventually become an owner with probability 1; ie, the contract provides a *Stairway to Heaven*. On the other hand, if the principal has the option to replace the current agent with a new one, then she will do so after the agent's equity level in the firm becomes sufficiently low, an event that occurs with positive probability. Hence, the contract also embodies a *Highway to Hell*.

Finally, we show that the properties of the theoretically optimal contract square well with features common in real-world work-to-own franchising agreements and venture capital contracts. In both of these settings, managers are incentivized primarily through equity adjustments. Moreover, good outcomes lead to less stringent controls by the franchiser/VC and increased autonomy by the manager, while bad outcomes have the reverse effects.

In essence, this paper can be viewed as addressing the basic question of how an equity partner should optimally contract with a managing partner who possesses no wealth or access to outside sources of capital. The answer we obtain is intuitive. The equity partner should use a sweat-equity contract to incentivize the manager, adjusting his ownership stake up when the firm performs well and down when it performs poorly. We show that the (potentially) long and winding road induced by such a contract must ultimately lead to ownership or to dismissal.

Appendices

A. Proofs from Section 4

We begin with a proof of Theorem 1.

Proof of Theorem 1. The proof is standard, which allows us to make frequent reference to Stokey, Lucas and Prescott (1989). Recall that the state variable, sweat equity or promised utility v, lies in the set $[0, \infty)$. The principal can always just give the agent v utiles without requiring any production. This would give the agent v utiles and cost the principal $-v$ utiles, thus forming a lower bound for her utility. An upper bound for the principal's value function obtains if we consider the case where there is full information, in which case, the principal's utility is

$$\frac{1}{1-\delta}\sum_{i=1}^{n} f_i\Big[R(q_i^*) - \theta_i q_i^*\Big] - v$$

This entails giving the agent exactly v utiles (net of production costs), but getting efficient output in every state, ie there are no output distortions. Therefore, the value function $P(v)$ must lie within these bounds, ie must satisfy

$$0 \;\leqslant\; P(v) + v \;\leqslant\; \frac{1}{1-\delta}\sum_{i=1}^{n} f_i\Big[R(q_i^*) - \theta_i q_i^*\Big]$$

Let $C[0, \infty)$ be the space of continuous functions on $[0, \infty)$, and let

$$\mathscr{F} := \left\{ Q \in C[0,\infty) : 0 \leqslant Q(v) + v \leqslant \frac{1}{1-\delta}\sum_{i=1}^{n} f_i\Big[R(q_i^*) - \theta_i q_i^*\Big] \right\}$$

be endowed with the sup metric, which makes it a complete metric space. Let $\Gamma_0(v)$ be the set of (u, q, w) that satisfy the constraints $q_i \geqslant 0$, (L_i'), (PK), (C_{ij}), and $w_i \geqslant 0$ for all i, j. Define the operator $\mathsf{T} : \mathscr{F} \to \mathscr{F}$ as

$$\begin{aligned}(\mathsf{T}Q)(v) = \max_{(u_i, q_i, w_i)} & \sum_i f_i\Big[\Big(R(q_i) - \theta_i q_i\Big) - u_i + \delta Q(w_i)\Big] \\ \text{s.t.} \;\; & (u, q, w) \in \Gamma_0(v)\end{aligned}$$

for each $Q \in \mathscr{F}$. Since $\Gamma_0(v)$ is compact for each v, the maximum is achieved for each v. Moreover, by the bounds established earlier, it is easily seen that $\mathsf{T}Q \in \mathscr{F}$.

Next, define

$$v^\flat := \frac{1}{1-\delta}\sum_{j=1}^{n-1} F_j(\theta_{j+1}-\theta_j)q^*_{j+1}$$

and let us assume that $Q \in \mathscr{F}$ is such that $Q'(v) = -1$ for all $v \geqslant v^\flat$. Consider the relaxed problem

$$\begin{aligned}&\max_{(u_i,q_i,w_i)} \sum_i f_i\Big[\Big(R(q_i)-\theta_i q_i\Big)+\delta w_i+\delta Q(w_i)\Big]-v\\&\quad\text{s.t. (PK)}\end{aligned}$$

It is easy to see that every solution to this problem must have $q_i = q_i^*$. Moreover, *a* solution (but certainly not the unique solution) to this problem has, in addition, $w_i = v^\flat$. By letting

$$u_i(v) := v - \delta v^\flat - \sum_{j=1}^{n-1} F_j(\theta_{j+1}-\theta_j)q^*_{j+1} + \sum_{j=1}^{n-1}(\theta_{j+1}-\theta_j)q^*_{j+1}$$

we see from (U_i) above that (PK) and ($\mathrm{C}_{i,i+1}$) hold with equality, so that all the constraints, including liquidity, are satisfied. Therefore, the contract $\Big(u(v), q_i^*, w_i = v^\flat\Big) \in \Gamma_0(v)$, and is feasible, and is therefore a solution to the original constrained problem. In particular, for any $Q \in \mathscr{F}$ that is linear, with slope -1 for $v \geqslant v^\flat$,

$$\mathsf{T}Q(v) = \sum_i f_i\Big[\Big(R(q_i^*)-\theta_i q_i^*\Big)+\delta v^\flat(b)+\delta Q\Big(v^\flat\Big)\Big]-v$$

for such $v \geqslant v^\flat$. Indeed, with the contract $\Big(u(v), q_i^*, w_i = v^\flat\Big) \in \Gamma_0(v)$, for any $v, v' \geqslant v^\flat$,

$$\mathsf{T}Q(v) - \mathsf{T}Q(v') = -(v-v')$$

that is, $(\mathsf{T}Q)'(v) = -1$ for all $v \geqslant v^\flat$.

By a variation of Theorem 4.6 of Stokey, Lucas and Prescott (1989), we see that the operator T has a unique fixed point in $\mathscr{F}$, that we shall call P. Moreover, if $Q \in \mathscr{F}$ is concave and is linear with slope -1 beyond $v^\flat$, $\mathsf{T}Q$ has this property too. Therefore, the fixed point of the operator T must also have this property, that is, the value function P is concave and has the property that $P'(v) = -1$ for all $v \geqslant v^\flat$.

We first establish a lower bound on $P'(0)$. By lemma A.1 below, the optimal contract associated with $v = 0$ is $q_1 = q_1^*$, $q_i = 0$ for $i > 1$ and $u_i = w_i = 0$ for all $i \in \Theta$, and we have

$$P(0) = f_1\Big(R(q_1^*)-\theta_1 q_1^*\Big)+\delta P(0).$$

Since P is concave, we know $P'(0) \geqslant \Big[P(\varepsilon) - P(0)\Big]/\varepsilon$ for all $\varepsilon > 0$.

Now, consider a contract such that in the first period $q_1 = q_1^*$, $q_i = x$, for $i > 1$, $w_i = 0$ for all $i \in \Theta$, and $u_i = (\theta_n - \theta_i)x$ for all $i \in \Theta$. From the second period on, the contract reverts to $v = 0$. Define x by

$$\begin{aligned}\varepsilon &= \sum_{i\in\Theta} f_i u_i \\ &= (\theta_n - \mathbf{E}[\theta])\, x.\end{aligned}$$

to satisfy (PK). Note that this contract satisfies all constraints.

The principal's payoff under the proposed contract is

$$\begin{aligned}Q(\varepsilon) &= f_1 \left(R(q_1^*) - \theta_1 q_1^*\right) + \sum_{j=2}^{n} f_j \Big[R(x) - \theta_j x\Big] - \varepsilon + \delta P(0) \\ &= P(0) + \sum_{j=2}^{n} f_j \Big[R(x) - \theta_j x\Big] - \varepsilon.\end{aligned}$$

Note that $P(\varepsilon) \geqslant Q(\varepsilon)$ and $\lim_{\varepsilon\to 0} Q(\varepsilon) = P(0)$. Moreover,

$$Q(\varepsilon) - P(0) = \frac{\sum_{j=2}^{n} f_j \Big[R(x) - \theta_j x\Big] - \varepsilon}{\varepsilon}$$

so that

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{Q(\varepsilon) - P(0)}{\varepsilon} &= \lim_{\varepsilon\to 0} \frac{\sum_{j=2}^{n} f_j \Big[R(x) - \theta_j x\Big] - \varepsilon}{\varepsilon} \\ &= \sum_{j=2}^{n} f_j \lim_{x\to 0} \left[\frac{R(x) - \theta_j x}{\Big(\theta_n - \mathbf{E}[\theta]\Big) x}\right] - 1 \\ &= \frac{(1 - f_1)R'(0) + f_1\theta_1 - \mathbf{E}[\theta]}{\theta_n - \mathbf{E}[\theta]} - 1 \\ &= \frac{(1 - f_1)R'(0) + f_1\theta_1 - \theta_n}{\theta_n - \mathbf{E}[\theta]},\end{aligned}$$

where we have used $\varepsilon = x\Big(\theta_n - \mathbf{E}[\theta]\Big)$ in the second equality. This gives us the bound

$$\begin{aligned}P'(0) &= \lim_{\varepsilon\to 0} \frac{P(\varepsilon) - P(0)}{\varepsilon} \\ &\geqslant \lim_{\varepsilon\to 0} \frac{Q(\varepsilon) - P(0)}{\varepsilon} \\ &= \frac{(1 - f_1)R'(0) + f_1\theta_1 - \theta_n}{\theta_n - \mathbf{E}[\theta]}\end{aligned}$$

as required. Notice now that if (MR_0) holds, that is if $R'(0) = \infty$, it follows immediately that $P'(0) = \infty$.

Since the optimal contract lies in the interior of the feasible set (in an appropriate sense), the continuous differentiability of P follows from standard results as, for instance, in Theorem 4.11 on p 85 of Stokey, Lucas and Prescott (1989). Since P is concave and $P'(v) = -1$ for all $v \geqslant v^\flat$, there is a smallest v such that $P'(v) = -1$; let $v^* := \min\{v : P'(v) = -1\}$. In sum, $P'(v) = -1$ for all $v \geqslant v^*$ and $P'(v) > -1$ for all $v < v^*$. Moreover, by construction, $v^* \leqslant v^\flat$. (Of course, it is shown in section 5 that in fact $v^* = v^\flat$.) □

While we do not (yet) know much about the optimal contract, the following lemma tells us what any optimal contract must look like at $v = 0$.

Lemma A.1. If $v = 0$, any optimal contract entails $u_i = w_i = 0$ for all i, $q_1 = q_1^*$, and $q_i = 0$ for all $i > 1$.

Proof. To see this, recall that feasibility implies $u_i, w_i \geqslant 0$ for all i. Promise keeping (PK) requires $\sum_i f_i \big[u_i + \delta w_i\big] = 0$, which implies $u_i = w_i = 0$ for all i. This observation and $C_{i,i+1}$ in turn imply that $q_i = 0$ for all $i > 1$. The intuition is simply that if there is any output for $i > 1$, then there must be some rent paid to $i = 1$ which, due to the liquidity and feasibility constraints, would violate (PK). □

Proof of proposition 4.1. Notice that from (PK) the value function can be written as

$$P(v) = \max_{(u,q,w)} \sum_i f_i \Big[\Big(R(q_i) - \theta_i q_i\Big) + \delta w_i + \delta P(w_i)\Big] - v$$

subject to all the constraints. So suppose $w_i(v) < v^*$ for some v, and by way of contradiction, $u_i(v) > 0$. Notice that in all the constraints (C_{ij}) and (PK), u_i and w_i appear in the form $u_i + \delta w_i$. Since $u_i > 0$, we can reduce it by an appropriately chosen $\varepsilon > 0$ and increase w_i by ε/δ. This leaves all the (PK) and (C_{ij}) constraints unchanged. Moreover, liquidity constraint (L_i) is also unaffected. Lastly, the q_i's are left unchanged. Therefore, this new contract is feasible, and is also a strict improvement, since $w_i + P(w_i)$ is strictly increasing for $w_i < v^*$ (by Theorem 1), which contradicts the optimality of the original contract. Therefore, it must be that for any optimal contract, $w_i(v) < v^*$ implies $u_i(v) = 0$.

Suppose now that the contract is a maximal rent contract, where $w_i(v) \leqslant v^*$ for all v. We have proved above that $u_i(v) \neq 0$ implies $w_i(v) \geqslant v^*$. But since u_i is non-negative, and the contract is maximal rent, this is equivalent to saying that $u_i(v) > 0$ implies $w_i(v) = v^*$, as required. □

The following lemma breaks down the proof of Lemma 4.3 into easily digestible parts.

Lemma A.2. (a) For all v, q is monotone in type, that is $q_i(v) \geqslant q_{i+1}(v)$.

(b) The constraints $C_{i,i+1}$ and $C_{i,i-1}$ imply all other constraints C_{ij}.

(c) If the constraints $C_{i,i+1}$ hold with equality and $q_i(v) \geqslant q_{i+1}(v)$ for all $i < n$, the constraints $C_{i,i-1}$ holds for all $i > 1$.

(d) We may assume that the constraints $C_{i,i+1}$ hold with equality; that is, if any of the constraints $C_{i,i+1}$ are slack, there is another contract that gives the principal the same utility, but where $C_{i,i+1}$ holds with equality.

(e) In any maximal rent contract, ($C_{i,i+1}$) and (L_i) imply that u and w are monotone in type.

Proof. (a) That $q_i \geqslant q_{i+1}$ follows by adding $C_{i,i+1}$ and $C_{i+1,i}$.

(b) Suppose all the constraints $C_{i,i+1}$ and $C_{i,i-1}$ hold. Define $z_i = u_i + \delta w_i$, fix some i, and suppose $j > i$. Then,

$$\begin{aligned} z_i &\geqslant z_{i+1} + (\theta_{i+1} - \theta_i)q_{i+1} \\ &\geqslant z_{i+2} + (\theta_{i+2} - \theta_i)q_{i+2} \\ &\geqslant z_j + (\theta_j - \theta_i)q_j \end{aligned}$$

where we have used the facts that $z_i \geqslant z_{i+1}$ and $q_i \geqslant q_{i+1}$. The proof that any constraint C_{ij} also holds for $j < i$ is similar, and therefore omitted.

(c) That monotonicity of q_i and the equality of $C_{i,i+1}$ implies $C_{i,i-1}$ is standard, and therefore omitted.

(d) We want to show that $C_{i,i+1}$ holds with equality for all $i < n$. By the results above, we may restrict attention to upward incentive constraints and assume that q is monotone in type. Suppose that some constraint $C_{i,i+1}$ is slack, so that $u_i + \delta w_i > u_{i+1} + \delta w_{i+1} + \Delta_i q_{i+1}$. There are two cases to consider. The first case is when $u_i > u_{i+1}$. We can increase u_{i+1} by ε and reduce u_i by $(f_i/f_{i+1})\varepsilon$, so that (PK) still holds, none of the upward incentive constraints are upset, and the objective is unchanged. We may choose ε so that $C_{i,i+1}$ holds with equality, which proves this case.

The second case is where $u_i \leqslant u_{i+1}$, which implies $w_i > w_{i+1}$. Replace w_i with $w_i' := w_i - \varepsilon$, and replace w_{i+1} with $w_{i+1}' := w_{i+1} + (f_i/f_{i+1})\varepsilon$, where $\varepsilon > 0$ is chosen so that $C_{i,i+1}$ holds with equality. Notice that since $q_{i+1} \geqslant 0$, it must be that $w_i' \geqslant w_{i+1}'$. We want to show this change does not leave the principal any worse off.

To see this, notice that by construction, $f_i w_i + f_{i+1} w_{i+1} = f_i w_i' + f_{i+1} w_{i+1}'$. Therefore, it only remains to show that $f_i P(w_i') + f_{i+1} P(w_{i+1}') \geqslant f_i P(w_i) + f_{i+1} P(w_{i+1})$, which holds if, and only if, $f_{i+1}\Big[P(w_{i+1}') - P(w_{i+1})\Big] \geqslant f_i\Big[P(w_i) - P(w_i')\Big]$. Recall that P is continuously differentiable, so that if $w_{i+1}' \leqslant w_i'$, the concavity of P implies $P'(w_{i+1}') \geqslant P'(w_i')$. We then observe

$$
\begin{aligned}
f_{i+1}\Big[P(w_{i+1}') - P(w_{i+1})\Big] &\geqslant f_{i+1} P'(w_{i+1}')(w_{i+1}' - w_{i+1}) \\
&= f_i P'(w_{i+1}')\varepsilon \\
&\geqslant f_i P'(w_i')(w_i - w_i') \\
&\geqslant f_i\Big[P(w_i) - P(w_i')\Big]
\end{aligned}
$$

where we have used the fact that $f_i(w_i - w_i') = f_i\varepsilon = f_{i+1}(w_{i+1}' - w_{i+1})$, and the first and last inequality follow from the definition of the subdifferential, and the second follows from the concavity of P. This proves our claim.

(e) We shall show that u and w are monotone in type in any maximal rent contract. Suppose first that $u_i < u_{i+1}$ for some i. Then, by the liquidity constraint (L_i'), it must be that $u_{i+1} > 0$. But by proposition 4.1, this implies $w_{i+1} = v^*$. Now, $(C_{i,i+1})$ with implies $\delta w_i \geqslant (u_{i+1} - u_i) + \delta w_i + \Delta_i q_{i+1} > \delta w_i$, which implies $w_{i+1} > v^*$, which is impossible in a maximal rent contract. Therefore, it must be that u is monotone in type.

Next, let us assume that $w_{i+1} > w_i$ for some i. Once again, $(C_{i,i+1})$ implies $u_i - u_{i+1} \geqslant \delta(w_{i+1} - w_i) + \delta_i q_{i+1} > 0$, which implies, by (L_{i+1}), that $u_i > 0$. But proposition 4.1 says we must have $w_i = v^*$, which in turn implies $w_{i+1} > v^*$, which is impossible in a maximal rent contract. Therefore, w must also be monotone in type. □

Proof of Lemma 4.4. Note that (C_i) can be rewritten, for each i from 1 to $n-1$, as

$$
u_i + \delta w_i = u_n + \delta w_n + \sum_{j=i}^{n-1} \Delta_j q_{j+1}
$$

Taking the expectation of both sides and – as usual – reversing the order of the double summation on the right yields

$$\sum_{j=1}^{n} f_j \left(u_j + \delta w_j\right) = u_n + \delta w_n + \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1}$$

Notice from (PK), that the left side of this expression is simply v. Hence, we have

$$u_n + \delta w_n = v - \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1}$$

Substituting this back into the original equation above yields (U_i). □

Proof of Theorem 2. The only part that remains to be proved is the uniqueness of the maximal rent contract. The first claim is that for any $v \geqslant 0$, there is a unique $q_i(v)$ for each i for all maximizers (q, w). To see this, suppose (q, w) and (q', w') are optimal at some v, but $q \neq q'$. Then, since the feasible set is convex, $\left(\frac{1}{2}(q + q'), \frac{1}{2}(w + w')\right)$ is also feasible, and moreover, is a strict improvement over (q, w) and (q', w'), since $R(q)$ is strictly concave. Therefore, it must be that $q = q'$ across all optimal contracts.

By proposition 4.1 and lemma 4.3, we know that for each v, there exists a type i such that for $u_i = 0$ implies $u_j = 0$ for all $j > i$. Suppose v is such that $u_i(v) = 0$ for some i. We have already established that $q_j(v)$ is unique for all j. This implies that there is a unique $w_i(v)$ such that (L_i) holds with equality. On the other hand, if $u_i > 0$ for some v, then it must be that $w_i = v^*$, since we have a maximal rent contract. In either case, $w_i(v)$ is uniquely determined in a maximal rent contract.

Finally, Theorem 4.6 of Stokey, Lucas and Prescott (1989) shows that the optimal maximal rent contract must be continuous in v. □

B. Proofs from Section 5

First we present the derivation of v^* given in (Vest).

Proof of Lemma 5.1. Since $P'(v^*) = -1$, we have $\Lambda_n(v^*) = 0$, and since $\lambda_i \geqslant 0$ for all i, it must be that $\lambda_i(v^*) = 0$ for all i.

By the definitions of P and v^*, we also have $\Lambda_n(v) > 0$ for all $v < v^*$. Lemma 4.3, which says that rents are monotone in type, now implies $\lambda_n(v) > 0$ for all $v < v^*$. But by complementary slackness, $u_n(v) = 0$ for all $v < v^*$. Since the optimal contract is continuous in v (see Theorem 2), it follows that $u_n(v^*) = 0$.

From the first order conditions, if $v = v^*$, then $P'(w_i) = -1$, which implies $w_i(v^*) = v^*$ for all i (in a maximal rent contract). Therefore, (Vest) holds by (U_i) for $i = n$. $\square$

Next, for ease of exposition, we shall provide some lemmas that are of independent interest and present results in an order somewhat different from the text, which allows this material to be relatively self contained. We begin with an observation about the implications of local linearity of the value function.

Lemma B.1. Let $0 \leqslant v_\circ < v^\circ$. If P is linear on $[v_\circ, v^\circ]$, any optimal contract must have q constant on $[v_\circ, v^\circ]$, that is $q(v) = q(v')$ for all $v, v' \in [v_\circ, v^\circ]$.

Proof. It is easy to see that at each v, if (u, q, w) and (u', q', w') are part of optimal contracts (maximal rent or not), it must necessarily be that $q = q'$. This follows from the convexity of the set of maximizers, and the strict concavity of R. It is easily seen that we may consider, without loss of generality, maximal rent contracts.

We will prove the contrapositive of the assertion. Let $v, v' \in [v_\circ, v^\circ]$, and suppose q, q' are optimal at v and v' respectively, with $q \neq q'$. For any $\alpha \in (0, 1)$, let $(q^\alpha, w^\alpha) = \alpha(q, w) + (1 - \alpha)(q', w')$. Notice that the constraint (L_i) can be written as $\langle a_i, q\rangle + \delta w_i \leqslant v$, where $a_i \in \mathbb{R}^{n-1}$. Therefore, (q^α, w^α) is certainly feasible at $v^\alpha := \alpha v + (1 - \alpha)v'$, that is (q^α, w^α) satisfies (L_i) and (M_i) for all i. Then,

$$\begin{aligned}
&P\Big(\alpha v + (1-\alpha)v'\Big) \\
&\qquad \geqslant \sum_i f_i\Big[\Big(R(q_i^\alpha) - \theta_i q_i^\alpha\Big) + \delta w_i^\alpha + \delta P(w_i^\alpha)\Big] - v^\alpha \\
&\qquad > \alpha \sum_i f_i\Big[\Big(R(q_i) - \theta_i q_i\Big) + \delta w_i + \delta P(w_i)\Big] - \alpha v \\
&\qquad\quad + (1-\alpha)\sum_i f_i\Big[\Big(R(q_i') - \theta_i q_i'\Big) + \delta w_i' + \delta P(w_i')\Big] - (1-\alpha)v' \\
&\qquad = \alpha P(v) + (1-\alpha)P(v')
\end{aligned}$$

where the strict inequality follows from the strict concavity of R. This proves the strict concavity of P, as required. $\square$

The following lemma provides some useful bounds on the Lagrange multipliers. As in the text, we shall assume, unless otherwise mentioned, that the contracts under question are maximal rent contracts.

Proposition B.2. The Lagrange multipliers satisfy the following inequalities:

(a) $\dfrac{\lambda_i}{f_i} \geqslant \dfrac{\lambda_j}{f_j}$ for $i \geqslant j$

(b) $\dfrac{\lambda_i}{f_i} \geqslant \dfrac{\Lambda_{i-1}}{F_{i-1}}$ for $i > 1$

(c) $\dfrac{\Lambda_{i+1}}{F_{i+1}} \geqslant \dfrac{\Lambda_i}{F_i}$ for $i < n$

(d) $\Lambda_n \geqslant \dfrac{\Lambda_i}{F_i}$ for all $i \leqslant n$

Proof. (a) By Lemma 4.3, we know that in a maximal rent contract, for each v, $w_i(v) \geqslant w_{i+1}(v)$. The concavity of P then implies that $P'(w_i) \leqslant P'(w_{i+1})$. By the first order condition for w_i, namely (FOw_i), we see that $-1 + \lambda_i/f_i = P'(w_i) \geqslant P'(w_{i-1}) = -1 + \lambda_{i-1}/f_{i-1}$. This allows us to conclude that $\lambda_i/f_i \geqslant \lambda_j/f_j$ for all $i \geqslant j$.

(b) The previous part tells that for all $i \geqslant j$, $\lambda_i f_j \geqslant \lambda_j f_i$. Summing over $j \leqslant i-1$, we see that $\lambda_i \sum_{j=1}^{i-1} f_j \geqslant f_i \sum_{j=1}^{i-1} \lambda_j$, which can be rewritten as $\lambda_i F_{i-1} \geqslant f_i \Lambda_{i-1}$, ie $\lambda_i/f_i \geqslant \Lambda_{i-1}/F_{i-1}$, as required.

(c) From the previous part, we know that $F_i \lambda_{i+1} \geqslant f_{i+1}\Lambda_i$. Adding $F_i\Lambda_i$ to both sides of the inequality, we get $F_i(\Lambda_i + \lambda_{i+1}) \geqslant (F_i + f_{i+1})\Lambda_i$, which can be rewritten as $F_i\Lambda_{i+1} \geqslant F_{i+1}\Lambda_i$, as required.

(d) The previous part tells us that $\Lambda_n \geqslant \Lambda_{n-1}/F_{n-1} \geqslant \ldots \geqslant \Lambda_1/F_1$, as required. □

The following is an easy corollary of the proposition above.

Corollary B.3. If $\lambda_i/f_i = \lambda_j/f_j$ for all i, j, $\Lambda_n = \Lambda_k/F_k$ for all $k < n$.

Proof. Again, $\frac{\lambda_i}{f_i} = \frac{\lambda_i}{f_i}$ can be rewritten as $f_j\left(\frac{\lambda_i}{f_i}\right) = \lambda_j$. Summing over $j = 1, \ldots, k$, we get $F_j\left(\frac{\lambda_i}{f_i}\right) = \Lambda_k$. This gives us the equalities $\dfrac{\lambda_i}{f_i} = \dfrac{\Lambda_i}{F_i} = \Lambda_n$, as required. □

An obvious question is whether the optimal contract can ever have greater than optimal production, which was stated as proposition 5.2 in the main text. We are now in a position to establish this.

Proof of proposition5.2. Recall the first order condition for q_i, (FOq_i), is

$$R'(q_i) - \theta_i = \frac{\Delta_{i-1}}{f_i}\Big[F_{i-1}\Lambda_n - \Lambda_{i-1}\Big] + \frac{1}{f_i}(\mu_{i-1} - \mu_i)$$

where μ_i is the Lagrange multiplier of the constraint $q_i \geqslant q_{i+1}$, and q_{n+1} is taken to be 0. Also recall that $R(q_i) - \theta_i q_i$ is concave in q_i, achieving its maximum at q_i^*. Therefore, for any $q_i > q_i^*$, $R'(q_i) - \theta_i < 0$. We shall prove the proposition via the following claims.

Claim 1. If, for some $v \geqslant 0$, there is an i such that the optimal $q_i > q_i^*$, then $\mu_i > 0$.

Proof of Claim 1. This follows from inspection of the first order condition, which requires that $\frac{\Delta_{i-1}}{f_i}\Big[F_{i-1}\Lambda_n - \Lambda_{i-1}\Big] + \frac{1}{f_i}(\mu_{i-1} - \mu_i) < 0$. But proposition B.2(d) tells us that $F_{i-1}\Lambda_n - \Lambda_{i-1} \geqslant 0$. Moreover, by virtue of being Lagrange multipliers, $\mu_{i-1}, \mu_i \geqslant 0$. Therefore, it must be that $\mu_i > 0$. ▲

Claim 2. If, for some $v \geqslant 0$, there is an i such that the optimal $q_i > q_i^*$, then $q_{i+1} = q_i$.

Proof of Claim 2. In claim 1 above, we established that $q_i > q_i^*$ implies $\mu_i > 0$. But the KKT complementary slackness condition requires that $\mu_i(q_i - q_{i+1}) = 0$, which implies $q_i = q_{i+1}$. ▲

Returning to the proof at hand, suppose for some $v \geqslant 0$, there is an i such that $q_i > q_i^*$. Then, by claims 1 and 2, it must be that $q_i = q_{i+1} > q_{i+1}^*$. This, in turn, implies that $\mu_{i+1} > 0$. Therefore, by induction, we see that $q_n > q_n^* > 0$ (where the second inequality is by assumption) and $\mu_n > 0$. But this is impossible, since the KKT complementary slackness condition requires that $\mu_n(q_n - 0) = 0$, which proves the claim. □

C. *Proofs from Section 6*

Proof of proposition 6.2. Recall that w is monotone in type, that is $w_1 \geqslant \ldots \geqslant w_n$, which implies $P'(w_n) \geqslant \ldots \geqslant P'(w_1)$. The claim is that for all $v \in (0, v^*)$, $P'(w_n) > P'(v) > P'(w_1)$. So suppose the claim is not true. Since P' is a martingale, the only possibility then is that $P'(w_1) = \cdots = P'(w_n) = P'(v)$. (Notice that this does not imply $w_1 = \cdots = w_n = v$, since we haven't established that P is strictly concave.)

Since $P'(0) = \infty$, we know that $w_n > 0$. The first order condition (FOw_i) then implies $\lambda_i(v)/f_i = \lambda_j(v)/f_j$ for all i, j. Corollary B.3 then implies $F_{i-1}\Lambda_n = \Lambda_{i-1}$, and (FO$q_i$) then implies $q_i(v) = q_i^*$ for all i.

In this case, proposition 4.1 and lemma 4.4 give $\delta w_1 = v + \sum_{j=1}^{n-1}(1-F_j)\Delta_j q_{j+1}^*$, and $\delta w_n = v - \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1}^*$. The first equation establishes $w_1(v) > v$ directly. From the second equation, $w_n(v) < v$ if and only if $\left[v - \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1}^*\right]/\delta < v$ if, and only if, $v < \frac{1}{1-\delta} \sum_{j=1}^{n-1} F_j \Delta_j q_{j+1}^* = v^*$, which is true by assumption. Therefore, we have $w_1(v) > v > w_n(v)$. Moreover $\delta w_1 - \delta w_n = \sum_{j=1}^{n-1} \Delta_j q_{j+1}^* =: K$.

Let $v_0 := v$, so that $w_n(v_0) < v_0 < w_1(v_0)$, $w_1(v_0) - w_n(v_0) = K/\delta$, and $P'(v) > -1$ and $q_i(v) = q_i^*$ for all $v \in [w_n(v_0), w_1(v_0)]$. Consider the sequence $v_k := w_1(v_{k-1})$, and suppose, as the induction hypothesis, that $P'(\cdot)$ is constant (and strictly greater than -1) on the interval $[w_n(v_{k-1}), v_k]$, with $v_{k-1} \in (w_n(v_{k-1}), v_k)$.

Since $q_i(v) \leqslant q_i^*$ for all v, it follows that $v_k = w_1(v_{k-1}) > w_n(v_k) > w_n(v_{k-1})$, which implies $P'(v_k) = P'(w_n(v_k))$, which in turn implies that $P'(v_k) = P'(v_{k+1})$. Therefore, $P'(\cdot)$ is constant (and strictly greater than -1) on the interval $[w_n(v_0), v_{k+1}]$. Since (v_k) is a strictly increasing sequence that diverges to infinity, we see that $P'(\cdot)$ must then be constant and strictly greater than -1 on the interval $[w_n(v_0), \infty)$, which is impossible because $P'(v) = -1$ for $v \geq v^*$. This completes the proof. □

We now prove another useful lemma that shows that with positive probability, the martingale P' can take all values in $(-1, \infty)$.

Lemma C.1. For any $v \in (0, v^*)$, and $\gamma > P'(v)$, if state θ_n is repeated τ times consecutively, then $P'(v_\tau) > \gamma$ for τ large enough. Similarly, for $-1 < \gamma < P'(v)$, if state θ_1 is repeated τ times consecutively, then $P'(v_\tau) < \gamma$ for τ large enough. Moreover, there exists $\tau < \infty$ such that if state θ_1 is repeated τ times consecutively, then $P'(v_\tau) = -1$; ie, $v_\tau = v^*$.

Proof. Suppose state θ_n occurs repeatedly. This gives us a sequence $v_0 = v$, $v_1 = w_n(v_0) < v_0$, and $v_\tau = w_n(v_{\tau-1}) < v_{\tau-1}$. Since (v_τ) is a decreasing sequence that is bounded below by 0, it has a limit. The first part is proved if we can show that this limit is 0, since $P'(0) = \infty$.

Therefore, suppose the claim is not true. This implies there is some $y > 0$ such that $\lim_{\tau\to\infty} v_\tau = y$. In other words, $\lim_{\tau\to\infty} w_n(v_\tau) = y$. Since the optimal contract is continuous in v, $w_n(\cdot)$ is continuous in v. Therefore, $w_n(y) = y$, which contradicts proposition 6.2 which requires that $w_n(y) < y$. This gives us the desired contradiction. The proof of the second part is similar and therefore omitted.

To prove the third part of the claim, consider the sequence $v_\tau = w_1(v_{\tau-1})$. We know that $\{v_\tau\}$ is a strictly increasing sequence and that $\lim_{\tau\to\infty} v_\tau = v^*$. By proposition 4.1 and lemma 4.4 we know that for $v_{\tau-1} < v^*$,

$$\delta v_\tau = v_{\tau-1} + \sum_{i=1}^{n-1}(1 - F_i)\Delta_i Q_{i+1}(v_{\tau-1}),$$

and hence

$$v_\tau - v_{\tau-1} \geq \frac{(1-\delta)v_{\tau-1}}{\delta}.$$

Thus $v_\tau - v_{\tau-1}$ is a positive increasing sequence which implies that v_τ achieves the limit v^* in a finite number of steps. □

We now move to the proof of proposition 3. Once again, we follow Thomas and Worrall (1990).

Proof of proposition 3. Since P' is a martingale that is bounded below by -1, it follows that $P' + 1$ is a nonnegative martingale. The (positive) Martingale Convergence Theorem (see, for instance, Theorem 22 of Pollard, 2002), says that $P' + 1$ converges almost surely to a nonnegative, integrable limit, $P'_\infty + 1$. Therefore, P' converges almost surely to P'_∞, and the limit is integrable (which implies that $P'_\infty = \infty$ with zero probability). We want to show that $P'_\infty = -1$ almost surely.

Consider a sample path with the properties that (i) $\lim_{t\to\infty} P'(v^t) = C \notin \{-1, \infty\}$, and (ii) state θ_n occurs infinitely often, and define $C =: P'(y)$, so that $\lim_{t\to\infty} v^t = y$. Consider a subsequence $(\sigma(t))$ such that $\theta^{\sigma(t)} = \theta_n$ for all t, ie this is the subsequence consisting of all the θ_n shocks in the original sequence. Since $(v^{\sigma(t)})$ is a subsequence of (v^t), it also converges to y.

Recall that the evolution of promised utility along any sample path can be written as $\varphi(v^t, \theta_i) = v^{t+1}$, where $\varphi(v, \theta_i)$ is continuous in v. This induces the function $\varphi^\sigma(v, \theta_n)$ where $\varphi^\sigma(v^{\sigma(t)}, \theta_n) = v^{\sigma(t+1)}$. Since $\varphi(v, \theta_i)$ is continuous in v, it follows that $\varphi^\sigma(v, \theta_n)$ is also continuous in v. Therefore, the sequence $\varphi^\sigma(v^{\sigma(t)}, \theta_n)$ converges to $\varphi^\sigma(y, \theta_n)$. Moreover, $\varphi^\sigma(y, \theta_n) = \varphi(y, \theta_n) = y$, since $\varphi^\sigma(v^{\sigma(t)}, \theta_n) = v^{\sigma(t+1)}$, and $\lim_{t\to\infty} v^{\sigma(t)} = \lim_{t\to\infty} v^t = y$.

But $\lim_{t\to\infty} P'(v^{\sigma(t)}) = C$ and $\lim_{t\to\infty} P'(v^{\sigma(t+1)}) = C$, so by the continuity of P' we have $P'(y) = P'\Big(\varphi^\sigma(y, \theta_n)\Big) = P'\Big(\varphi(y, \theta_n)\Big) = C$, contradicting proposition 6.2 which states that $P'(y) < P'\Big(\varphi(y, \theta_n)\Big)$. But paths where state θ_n does not occur infinitely often are of probability zero, which proves the proposition. □

Proof of proposition 7.2. By way of contradiction, suppose it is never optimal to fire an agent, then for all $v \in [0, v^0]$ the principal's payoff is $P(v)$. If she fires the current agent when his sweat equity is v (and never fires another), then her payoff is $P(v^0) - v$. Hence, a contradiction will obtain if there exists $v \in (0, v^0)$ such that $P(v^0) - v > P(v)$ or $P(v) + v < P(v^0)$. Note that $P(0) < P(v^0)$ by definition of v^0. Since P is continuous, there exists $v > 0$ such that $P(v) + v < P(v^0)$. □

References

Albanesi, Stefania and Christopher Sleet (2006). 'Dynamic Optimal Taxation with Private Information'. In: *Review of Economic Studies* 73.1, pp. 1–30 (cit. on p. 4).

Athey, Susan and Ilya Segal (2007). *An Efficient Dynamic Mechanism*. Tech. rep. Stanford University (cit. on p. 6).

Battaglini, Marco (2005). 'Long-Term Contracting with Markovian Consumers'. In: *American Economic Review* 95.3, pp. 637–658 (cit. on p. 7).

Bergemann, Dirk and Juuso Välimäki (2010). 'The Dynamic Pivot Mechanism'. In: *Econometrica* 78.2, pp. 771 –789 (cit. on p. 6).

Berle, Adolf A and Gardiner C Means (1968). *The Modern Corporation and Private Property*. revised. Harcourt, Brace and World (cit. on p. 19).

Biais, Bruno et al. (2007). 'Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications'. In: *Review of Economic Studies* 74.2, pp. 345–390 (cit. on pp. 4, 5).

Blair, Roger D and Francine Lafontaine (2005). *The Economics of Franchising*. New York, NY: Cambridge University Press (cit. on p. 23).

Boleslavsky, Raphael (2009). *Dynamic Screening in a Long Term Relationship*. Tech. rep. University of Miami (cit. on p. 7).

Bolton, Patrick and David S Scharfstein (1990). 'A Theory of Predation Based on Agency Problems in Financial Contracting'. In: *American Economic Review* 80.1, pp. 93–106 (cit. on p. 5).

– (1998). 'Corporate Finance, the Theory of the Firm, and Organizations'. In: *Journal of Economic Perspectives* 12.4, pp. 95–114 (cit. on p. 19).

Clementi, Gian Luca and Hugo A Hopenhayn (2006). 'A Theory of Financing Constraints and Firm Dynamics'. In: *Quarterly Journal of Economics* 121, pp. 229–265 (cit. on pp. 4, 5).

Covallo, Ruggiero (2008). 'Efficiency and Redistribution in Dynamic Mechanism Design'. In: *Proceedings of the 9th ACM Conference on Electronic Commerce*. ACM (cit. on p. 6).

DeMarzo, Peter and Michael J Fishman (2007). 'Agency and Optimal Investment Dynamics'. In: *Review of Financial Studies* 20.1, pp. 151–188 (cit. on pp. 4, 5).

DeMarzo, Peter and Yuliy Sannikov (2006). 'Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model'. In: *Journal of Finance* 61.6, pp. 2681–2724 (cit. on pp. 4, 5).

Edmans, Alex et al. (2010). *Dynamic Incentive Accounts*. Tech. rep. NYU (cit. on p. 10).

Fernandes, Ana and Christopher Phelan (2000). ‘A Recursive Formulation for Repeated Agency with History Dependence’. In: *Journal of Economic Theory* 91, pp. 223–247 (cit. on p. 4).

Green, Edward J (1987). ‘Lending and the Smoothing of Uninsurable Income’. In: *Contractual Agreements for Intertemporal Trade*. Ed. by Edward C Prescott and Neil Wallace. University of Minnesota Press (cit. on p. 4).

Grossman, Sanford J and Oliver D Hart (1986). ‘The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration’. In: *Journal of Political Economy* 94.4, pp. 691–719 (cit. on p. 19).

HBS Case Study, 9-800-170 (2000). *Venture Capital Negotiations: VC versus Entrepreneur*. Tech. rep. Harvard Business School (cit. on p. 24).

Hart, Oliver D and John Moore (1990). ‘Property Rights and the Nature of the Firm’. In: *Journal of Political Economy* 98.6, pp. 1119–1158 (cit. on p. 19).

Kaplan, Steven N and Per Stromberg (2003). ‘Financial Contracting Theory Meets the Real World: An Empirical Analysis of Venture Capital Contracts’. In: *Review of Economic Studies* 70.2, pp. 281–315 (cit. on p. 24).

Laffont, Jean-Jacques and David Martimort (2002). *The Theory of Incentives*. Princeton, NJ: Princeton University Press (cit. on p. 5).

Liddle, Alan (2010). ‘Investing in Sweat Equity’. Nation’s Restaurant News. URL: http://www.nrn.com/article/investing-sweat-equity (cit. on p. 24).

Milgrom, Paul (1987). *Adverse Selection without Hidden Information*. Tech. rep. 8742. University of California, Berkeley (cit. on p. 5).

Mirrlees, James A (1971). ‘An Exploration in the Theory of Optimum Income Taxation’. In: *Review of Economic Studies* 38.114, pp. 175 –208 (cit. on p. 6).

Pavan, Alessandro, Ilya Segal and Juuso Toikka (2009). *Dynamic Mechanism Design: Incentive Compatibility, Profit Maximization and Information Disclosure*. Tech. rep. Stanford University (cit. on p. 6).

– (2010). *Infinite-Horizon Mechanism Design: The Independent-Shock Approach*. Tech. rep. Stanford University (cit. on p. 6).

Pollard, David (2002). *A User’s Guide to Measure Theoretic Probability*. New York, NY: Cambridge University Press (cit. on p. 38).

Quadrini, Vincenzo (2004). ‘Investment and liquidation in renegotiation-proof contracts with moral hazard’. In: *Journal of Monetary Economics* 51.4, pp. 713–751 (cit. on pp. 4, 5).

Sannikov, Yuliy (2008). ‘A Continuous-Time Version of the Principal–Agent Problem’. In: *Review of Economic Studies* 75.3, pp. 957–984 (cit. on p. 4).

Spear, Stephen E and Sanjay Srivastava (1987). ‘On Repeated Moral Hazard with Discounting’. In: *Review of Economic Studies* 54.4, pp. 599–617 (cit. on p. 4).

Steele, J Michael (2001). *Stochastic Calculus and Financial Applications*. New York, NY: Springer (cit. on p. 21).

Stokey, Nancy L, Robert E Lucas Jr and Edward C Prescott (1989). *Recursive Methods in Economic Dynamics*. Cambridge, Ma: Harvard University Press (cit. on pp. 27, 28, 30, 33).

Thomas, Jonathan and Tim Worrall (1990). 'Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem'. In: *Journal of Economic Theory* 51, pp. 367–390 (cit. on pp. 4, 9, 22, 38).

White, Rebecca J, Rodney R D'Souza and John C McIlwraith (2007). 'Leadership in venture backed companies: going the distance'. In: *Journal of Leadership and Organizational Studies* (cit. on p. 25).