Dynamic Preference for Flexibility in a Lucas Tree Economy with Investment

R. Vijay Krishna* Philipp Sadowski†

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1. Introduction

This is a companion note to Krishna and Sadowski [2011] (henceforth KS). KS analyze choice on the recursive domain of Infinite Horizon Consumption Problems (IHCPs) first analyzed by Gul and Pesendorfer [2004]. They provide representations of preference for flexibility that are solutions to Bellman equations. In particular, they consider the case where there is no preference for flexibility with respect to continuation problems, but only with respect to consumption alternatives. Let $x \in Z$ be an IHCP, that is, a menu of lotteries p over present consumption $k \in K$ and continuation problems $z \in Z$. For this case, KS axiomatize a representation of Preference for Flexibility with Stationary Beliefs (PFS),

$$V(x) = \int_{\mathcal{U}_K} \max_{p \in x} \Big[\int_{K \times Z} \big[u(k) + \delta V(z) \big] \, \mathrm{d}p(k, z) \Big] \, \mathrm{d}\mu(u)$$

where $\delta \in (0,1)$ is the discount factor, $\mathcal{U}_K$ is the space of consumption utilities (ie, the space of vN-M utilities over the set of consumption prizes K that are identified up to additive constant), and μ is a probability measure on $\mathcal{U}_K$. The measure μ is unique, up to scaling (see definition 2).

Following KS, we say that one DM has *greater preference for flexibility* than another DM, if the first prefers a nondegenerate menu over a singleton whenever the second does. KS characterize this property in terms of DM's beliefs in the context of the PFS representation: one DM has a greater preference for flexibility than another if, and only if, the corresponding beliefs are dominated in the increasing convex order. If the relevant part of the space of consumption utilities is one dimensional, this condition on beliefs amounts to second order stochastic dominance.

(∗) University of North Carolina, Chapel Hill <rvk@unc.edu>
(†) Duke University <p.sadowski@duke.edu>

KS introduce a version of the Lucas tree economy, where the representative agent behaves as in their model and is uncertain about his degree of future risk aversion. They show that in such an economy, the representative agent's uncertainty about his future risk aversion drives price volatility (in the sense of second order stochastic dominance).

In such an economy, if ownership of the productive asset constrains liquidity, we should expect correlation between underinvestment (a lower average price for the productive asset) and higher price volatility (in the sense of second order stochastic dominance), as both are driven by the representative agent's degree of uncertainty about future risk aversion. In this note we reconsider the Lucas tree economy introduced in KS and add an additional market stage, where the productive asset cannot be traded. We affirm the intuition that a representative agent with a higher degree of uncertainty about his future risk aversion implies less investment and more price volatility in the economy.

2. PFS representation

Let K be a finite set of prizes with typical member k. We follow GP in defining an infinite horizon consumption problems (IHCP) as a collection of lotteries that yield a consumption prize in the present period and a new infinite horizon problem starting in the next period. Let Z be the collection of all IHCPs.[1] GP show that Z is a compact metric space, and that each $z \in Z$ can be identified with a compact set of probability measures over $K \times Z$. For the compact metric space $K \times Z$, let $\mathscr{P}(K \times Z)$ denote the space of probability measures endowed with the topology of weak convergence, so that $\mathscr{P}(K \times Z)$ is compact and metrizable. Let $\mathscr{F}(\mathscr{P}(K \times Z))$ denote the space of closed subsets of $\mathscr{P}(K \times Z)$, endowed with the Hausdorff metric, such that $\mathscr{F}(\mathscr{P}(K \times Z))$ is a compact metric space. It can be shown that Z is linearly homeomorphic to $\mathscr{F}\big(\mathscr{P}(K \times Z)\big)$. We shall denote this linear homeomorphism as $Z \simeq \mathscr{F}\big(\mathscr{P}(K \times Z)\big)$. Typical elements $x, y, z \in Z$ are interpreted as ***menus*** of lotteries over consumption and continuation problems.

KS provide representations of choice over Z with the understanding that DM will choose from the IHCP he faces in every subsequent period. Take p, q to be typical lotteries in $\mathscr{P}(K \times Z)$. We will also consider the space of menus of consumption lotteries, $\mathscr{F}\big(\mathscr{P}(K)\big)$, with typical members being a, b. By the recursive nature of Z, continuation problems are members of Z. Let A, B denote typical elements of the collection of menus of continuation lotteries, $\mathscr{F}\big(\mathscr{P}(Z)\big)$. When there is no risk of confusion, we identify prizes and continuation problems with degenerate lotteries and lotteries with singleton menus.

As in KS, let $\mathscr{U}_K := \big\{u \in \mathbb{R}^K : \sum u_i = 0\big\}$ be the set of all vN-M utility

(1) See GP for the recursive construction of Z.

functions over instantaneous consumption (ie, over K) that are identified up to a constant. The subjective state space relevant for the PFS representation is $\mathscr{U}_K$. To ensure that expected consumption utility under a measure μ is well defined, the measure μ must be ***nice***, ie, must satisfy $\int_{\mathscr{U}_K} \|u\|_2 \, \mathrm{d}\mu(u) < \infty$, where $\|u\|_2 := \big(\sum_i u_i^2\big)^{1/2}$.

Subjective states $u \in \mathscr{U}_K$ are naturally interpreted as consumption utilities, and the two terms are treated as synonyms. Similarly, in what follows, all probability measures are interpreted as subjective beliefs, and the two terms are used interchangeably. For $p \in \mathscr{P}(K \times Z)$, let p_k and p_z denote the marginal distributions on K and Z respectively.

Definition 1. Let $\mathscr{U}_K$ be defined as above, μ a nice probability measure on (the Borel sigma-algebra of) $\mathscr{U}_K$, and $\delta \in (0,1)$. We say that $\succsim$ has a representation of ***Preference for Flexibility with Stationary Beliefs*** (PFS), (μ, δ), if there exists a continuous function $V : Z \to \mathbb{R}$, linear on Z, that satisfies

$$V(x) = \int_{\mathscr{U}_K} \max_{p \in x} \big[u(p_k) + \delta V(p_z)\big] \, \mathrm{d}\mu(u) \tag{2.1}$$

and represents $\succsim$.

In the presentation above, $u(p_k) = \sum_{k' \in K} p_k(k') u(k')$, and $V(p_z)$ is the extension of V to $\mathscr{P}(Z)$ by linearity (and continuity), ie $V(p_z) = \int_Z V(z') \, \mathrm{d}p_z(z')$. KS provide axioms on a preference relation, $\succsim$, that are equivalent to the existence of a PFS representation, (μ, δ), of $\succsim$. Moreover, they establish that δ is unique and μ is unique up to scaling (Theorem 1 in KS).

Definition 2. Two probability measures μ and μ' on $\mathscr{U}_K$ are ***identical up to scaling,*** if there is $\lambda > 0$ such that $\mu(E) = \mu'(\lambda E)$ for all measurable $E \subset \mathscr{U}_K$, where $\lambda E := \{\lambda u : u \in E\}$.

Preference for flexibility is the preference for non degenerate menus over singletons. Intuitively, one DM has more preference for flexibility than another if she has a stronger preference for menus over singletons. The following definition allows us to formulate this condition in terms of preferences over consumption alternatives:

Definition 3. Fix $A \in \mathscr{F}\big(\mathscr{P}(Z)\big)$. Let $a \succsim_K b$ if, and only if, $(a, A) \succsim (b, A)$.

KS show that, in the context of the PFS representation, $\succsim_K$ is independent of the choice of $A \in \mathscr{F}\big(\mathscr{P}(Z)\big)$.

Definition 4. $\succsim^*$ has a ***greater preference for flexibility*** than $\succsim$ if

$$a \succsim_K \{\beta\} \ \text{ implies } \ a \succsim_K^* \{\beta\}$$

for all $\beta \in \mathscr{P}(K)$ and $a \in \mathscr{F}\big(\mathscr{P}(K)\big)$.

Note that the comparison in the definition requires that $\succsim_K$ and $\succsim_K^*$ rank singleton menus the same; that is, $\alpha \succsim_K \beta$ if and only if $\alpha \succsim_K^* \beta$.

Definition 5 (Increasing convex order)**.** If μ and μ^* are probability measures with supports in $\mathcal{U}_K$, then μ dominates μ^* in the ***increasing convex order*** if for every increasing convex function $\varphi : \mathcal{U}_K \to \mathbb{R}$, $\int \varphi \, \mathrm{d}\mu \leqslant \int \varphi \, \mathrm{d}\mu^*$, and $\int \varphi \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu^*$ if φ is linear.[2]

Theorem 1 (Theorem 3 in KS)**.** *If $\succsim$ and $\succsim^*$ have a PFS representation, then $\succsim^*$ has a greater preference for flexibility than $\succsim$ if, and only if, μ dominates μ^* in the increasing convex order.*

To illustrate this result, consider the example of two decision makers, DM and DM*, both of whom have monotone preferences over $\{0, \frac{1}{2}, 1\}$ and are uncertain about their future risk aversion. Further, suppose their preferences, $\succsim$ and $\succsim^*$ respectively, have a PFS representation and are such that for all u in the supports of μ and μ^*, $u(0) = 0$ and $u(1) = 1$. What is uncertain is $u\left(\frac{1}{2}\right) \in [0, 1]$. Abusing notation, we may summarize the state space as $[0, 1]$. Then DM* has a greater preference for flexibility than agent DM if, and only if, μ second order stochastically dominates μ^* on $[0, 1]$.

3. *A Lucas Tree Economy with Investment*

Intuitively, preference for flexibility is associated with a tendency to invest less in the productive asset, as such investment reduces liquidity. In order to capture this intuition in a simple asset pricing model like a Lucas tree economy, there must be periods where the productive asset, the tree, is not tradeable. Consider the following environment: Odd periods ($t = 1, 3, 5, \dots$) are mornings, and even periods ($t = 2, 4, 6, \dots$) are evenings.

The tree produces a perishable good for every period according to the stationary distribution $F(\omega)$. Importantly, this production now happens with probability 1/2 'just before' and with probability 1/2 'just after' the period (and with it the ownership right) changes. Entering period t, the representative agent is endowed with shares z_t and proportional probabilistic rights to the tree's output, ω_t. In the

(2) The requirement that $\int \varphi \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu^*$ if φ is linear captures the notion that μ and μ^* have the same 'mean'. It is not standard in the general definition of *increasing convex order*. In economics, however, the one dimensional special case of second order stochastic dominance is usually restricted to distributions with the same mean. Our definition retains the conventional definition of second order stochastic dominance as a special case.

morning (t odd) domestic asset markets are open and the probabilistic right, q_t, to current output, ω_t, can be traded for future shares in the tree, z_{t+1}.[3]

The market structure of the economy in the evening is different: Because of the timing of production, in the evening, $t+1$, ownership rights to the tree's output, ω_{t+1}, are determined by q_t with probability 1/2 and by z_{t+1} with probability 1/2. This specification ensures that, in terms of expected received output in period $t+1$, it is as good to own q_t as it is to own z_{t+1}. However, we assume that international goods markets are open only at the beginning of period $t+1$. On those markets, domestic assets cannot be traded, but probabilistic rights to output of $\omega = 1/2$ can be traded at the fixed price $\kappa > 1$ for probabilistic rights to the output of $\omega = 1$. Thus, the tree's output can only be traded on international markets if it is realized before the beginning of period $t+1$. In expectation (from the morning's perspective), therefore, ownership rights to output, q_t, provide the agent with more flexibility or 'liquidity' in the evening than shares of the productive asset, z_{t+1}, do.

First we establish the pricing function $\psi(\omega, u)$ for a representative agent whose value function in odd periods, that is, at the time asset markets are open, takes the form

$$\begin{aligned} v(z,\omega,u) = \max_{q,x} \Big[& q u(\omega) + \tfrac{1}{2}\delta x \int u'(\omega')\ \mathrm{d}F(y')\,\mathrm{d}\mu(u') \\ & + \tfrac{1}{2}\delta q\big(f_{\frac{1}{2}} + \kappa f_1\big) \int \max[u'(\tfrac{1}{2}), \tfrac{1}{\kappa}u'(1)]\ \mathrm{d}\mu(u') \\ & + \delta^2 \int v(x,\omega',u')\ \mathrm{d}F(\omega')\,\mathrm{d}\mu(u') \Big] \end{aligned}$$

subject to $q + p(\omega,u)x \leqslant z + p(\omega,u)z$.

Let us define $u^*(\mu) := \int \max[u(\frac{1}{2}), \frac{1}{\kappa}u(1)]\ \mathrm{d}\mu(s) = \int \max[u\,(1/2)\,, \frac{1}{\kappa}]\ \mathrm{d}\mu(u)$ and $\mathbf{E}[u;F,\mu] := \int u'(y')\ \mathrm{d}F(\omega')\,\mathrm{d}\mu(u')$. Define

$$\phi(\omega,u) := p(\omega,u)\big(u(\omega,u) + \tfrac{1}{2}\delta(f_{\frac{1}{2}} + \kappa f_1)u^*(\mu)\big)$$

and

$$\gamma := \tfrac{1}{2}\delta\,\mathbf{E}[u;F,\mu] + \delta^2 \iint (u'(\omega') + \tfrac{1}{2}\delta(f_{\frac{1}{2}} + \kappa f_1)u^*(\mu))\ \mathrm{d}F(\omega')\,\mathrm{d}\mu(u')$$

Following the standard arguments in Lucas, we find

$$\phi(\omega,u) = \gamma + \delta^2 \iint f(\omega',u')\ \mathrm{d}F(\omega')\,\mathrm{d}\mu(u')$$

(3) Whether or not output is realized just before or just after the beginning of period t will turn out to be irrelevant for odd t, as ownership of the tree is not permitted to change from period $t-1$ to period t.

As in the Lucas model in KS, the unique solution is $\phi(\omega, u) = \gamma/(1-\delta^2)$. Let $\Lambda := \gamma/(1-\delta^2)$ and $\Gamma := \frac{1}{2}\delta(f_{\frac{1}{2}} + \kappa f_1)u^*(\mu)/\Lambda$, so that

$$\psi(\omega, u) = \frac{u(\omega)}{\Lambda} + \Gamma$$

We expect to verify the standard intuition that more preference for flexibility implies more demand for liquidity, or a lower willingness to invest in the productive asset, which will be reflected in a higher price for q_t in terms of z_{t+1}.

Note that Γ increases in the expected utility the agent derives from being able to access international markets.

As in KS, we now confine attention to the example where there are only three levels of output, 0, 1/2 and 1. Consider two exchange economies, A and B, with representative agents A and B respectively. We assume that both agents have period 0 preferences with a PFS representation based on state spaces $\mathcal{U}_i^\dagger$, where $u(0) = 0 \leqslant u(1/2) \leqslant u(1) = 1$ for all $u \in \mathcal{U}_i^\dagger$, $i = A, B$. Thus, agents are uncertain about their risk aversion, which is captured by $u(1/2)$. We also assume that $\succsim^A$ and $\succsim^B$ agree on the intertemporal tradeoff for getting 1 instead of 0, which implies $\delta_A = \delta_B$.

Thus, for $\omega \in \{0, 1/2, 1\}$, prices are given by

$$\begin{aligned}
\psi_i(0, u) &= \Gamma_i \\
\psi_i(1, u) &= \frac{1}{\Lambda_i} + \Gamma_i \\
\psi_i(\tfrac{1}{2}, u) &= \frac{u(1/2)}{\Lambda_i} + \Gamma_i
\end{aligned}$$

Let $H_i(\lambda) := \mathbf{P}(\frac{u(1/2)}{\Lambda_i} < \lambda)$ be the distribution of prices in the two economies for the case where output is $\omega = 1/2$, renormalized such that the two distributions can be compared in terms of second order stochastic dominance. Let $\overline{\psi}_i := \mathbf{E}\left[\psi_i(\frac{1}{2}, u), \mu_i\right]$ be the average domestic price in the case where output is $\omega = 1/2$.

Proposition 6. In the two economies above, (a) and (b) below are equivalent and imply (c).

(a) Agent B has greater preference for flexibility than agent A.

(b) H_A second order stochastically dominates H_B.

(c) $\overline{\psi}_B \geqslant \overline{\psi}_A$ and $\psi_B(1) - \psi_A(1) = \psi_B(0) - \psi_A(0)$

Furthermore, if (c) holds for all prices $\kappa > 1$, then (a) and (b) are also implied.

The condition $\psi_B(1) - \psi_A(1) = \psi_B(0) - \psi_A(0)$ is the manifestation of the fact that singletons are ranked identically by both agents in terms of prices. Proposition **??** tells us to expect correlation between the price fluctuations and the degree of underinvestment in small economies that have open goods markets but closed asset markets: the average price for the productive asset is higher in economy A, or conversely the average price for current dividends is lower, $\overline{\psi}_B \geqslant \overline{\psi}_A$. The model suggests that the determinant of these two effects might be the level of uncertainty about future risk aversion in the economy.

Proof of Proposition 6. To investigate the effect of the liquidity provided by holding the right to output rather than shares, consider

$$\Gamma^{-1} = \frac{1}{1-\delta^2}\left(\frac{(1+2\delta)\,\mathbf{E}[u;F,\mu]}{(f_{\frac{1}{2}}+\kappa f_1)u^*(\mu)} + \delta^2\right)$$

Observe that $\frac{\partial \Gamma^{-1}}{\partial u^*(\mu)} < 0$ and therefore $\frac{\partial \Gamma}{\partial u^*(\mu)} > 0$. The proof rests on the following claim.

Claim: If μ_A SOSD μ_B, then $u^*(\mu_B) > u^*(\mu_A)$. Furthermore, if $u^*(\mu_B) > u^*(\mu_A)$ for all $\kappa > 1$, then μ_A SOSD μ_B.

Proof of Claim: Notice that $u^*(\mu) = \int \max[u(\frac{1}{2}), \frac{1}{\kappa}]\, \mathrm{d}\mu\big(u(\frac{1}{2})\big)$. By definition μ_A SOSD μ_B if, and only if, $\int \max[u(\frac{1}{2}), \frac{1}{\kappa}]\, \mathrm{d}\mu_A\big(u(\frac{1}{2})\big) > \int \max[u(\frac{1}{2}), \frac{1}{\kappa}]\, \mathrm{d}\mu_B\big(u(\frac{1}{2})\big)$ for all $\kappa > 1$ (eg, page 33 in Laffont [1989]).

We can now establish the proposition. Equivalence of (i) and (ii) follows as in the case without investment. To establish that (ii) implies (iii), note $\psi_B(0) - \psi_A(0) = \psi_B(1) - \psi_A(1)$ if, and only if, $\Lambda_A = \Lambda_B = \Lambda$, if, and only if, $\mathbf{E}[u(\frac{1}{2}); F, \mu_A] = \mathbf{E}[u(\frac{1}{2}); F, \mu_B] = \overline{u}$. By the claim, $u^*(\mu_B) > u^*(\mu_A)$ for all $\kappa > 1$ if, and only if, μ_A SOSD μ_B, which is obviously the case if, and only if, H_A SOSD H_B. By the observation above $\Gamma_B > \Gamma_A$, if, and only if, $u^*(\mu_B) > u^*(\mu_A)$. Hence, $\overline{\psi}_B = \overline{u}/\Lambda + \Gamma_B \geqslant \overline{u}/\Lambda + \Gamma_A$.

Conversely, if (iii) holds for all $\kappa > 1$, then $\overline{u}_A = \overline{u}_B$, and $\Lambda_A = \Lambda_B$, and hence $\Gamma_B \geqslant \Gamma_A$, which implies $u^*(\mu_B) \geqslant u^*(\mu_A)$, and hence by the claim, μ_A SOSD μ_B. □

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