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Mathematical Social Sciences ▪ (▪▪▪▪) ▪▪▪–▪▪▪

Contents lists available at SciVerse ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

A nonsmooth approach to nonexpected utility theory under risk[☆]

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ARTICLE INFO

Article history:
Received 22 March 2011
Received in revised form
22 July 2011
Accepted 2 August 2011
Available online xxxx

ABSTRACT

We consider concave and Lipschitz continuous preference functionals over monetary lotteries. We show that they possess an envelope representation, as the minimum of a bounded family of continuous vN-M preference functionals. This allows us to use an envelope theorem to show that results from local utility analysis still hold in our setting, without any further differentiability assumptions on the preference functionals. Finally, we provide an axiomatisation of a class of concave preference functionals that are Lipschitz.

1. Introduction

In this paper, we consider concave preference functionals over the space of monetary lotteries that are Lipschitz continuous. Concavity of the preference functional is to be interpreted as representing pessimism on the part of the decision maker about his future (ie, as yet to be determined) preferences. The key property of concave and Lipschitz preference functionals is that they possess an *envelope representation*, that is, they can be written as the minimum of a bounded family of continuous vN-M preference functionals. (Every concave function can be written as the infimum of a family of continuous vN-M preference functionals. Lipschitzness ensures that the infimum is always achieved.) We show that properties of the concave preference functional (say, with regards to risk aversion) can be linked to the properties of the underlying family of vN-M preference functionals that form the envelope. We also provide an axiomatisation of preferences that can be represented by a concave and Lipschitz continuous preference functional. We now describe our results in more detail.

Our first set of results, linking global behaviour of the preference functional to the properties of the family that forms the envelope falls under the rubric of 'local utility analysis'. This stems from the observation by Machina (1982) in his seminal paper, that while expected utility theory itself may be difficult to reconcile with observations, the *theorems* of expected utility analysis are still useful provided the (nonexpected utility) preference functional under consideration is sufficiently smooth. Indeed, the intuition is very straightforward. If the preference functional is sufficiently smooth (and Machina uses the notion of Fréchet differentiability here—see Appendix A.4 for definitions and details), then it can be approximated locally by an expected utility preference functional, the approximation being given by the Fréchet derivative of the functional, and referred to as the *local utility function*. If we consider smooth paths in the space of monetary lotteries, then local and global changes can be linked by integrating the preference functional along these paths. This allows Machina (1982) to obtain elegant characterisations of *all* sufficiently smooth nonexpected utility preference functionals in terms of economic behaviour. Indeed, his analysis is now referred to as *local utility analysis*. (The essential equations of local utility analysis are Eqs. (2)–(4) in what follows.)

We provide a more general intuition for local utility analysis: we show that if a preference functional is a *generalised potential*, that is, if it allows marginal local changes to be aggregated along sufficiently smooth paths to correspond to global changes, then one can do local utility analysis. There are at least two ways a preference functional can be a generalised potential. One kind of generalised potential corresponds to the criterion identified by Machina, namely that the preference functional be Fréchet differentiable. The second is the approach taken here, based on the envelope theorem. We show that a preference functional is a generalised potential if it is concave and Lipschitz, i.e., if it possesses an envelope representation, which allows us to use the envelope theorem to aggregate local changes along suitable paths.

Our results complement the findings of Machina (1982) in the following sense. His analysis works for preference functionals

☆ Chatterjee would like to thank the Human Capital Foundation www.hcfoundation.ru/en/ for its support of the Penn State Department of Economics. We would also like to thank participants at RUD 2009 and other seminars for useful comments. Helpful suggestions were made by Philipp Sadowski, Fabio Maccheroni who also alerted us to his paper, and Edi Karni. Comments from an Associate Editor and referee greatly improved the exposition. Remaining errors are our own.

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doi:10.1016/j.mathsocsci.2011.08.001

that are (L^1) Fréchet differentiable. Unfortunately, there are many useful classes of preference functionals, including Rank Dependent Utility (henceforth RDU; see Section 4.2 for a description) and the Maxmin under Risk model of Maccheroni (2002), that are not Fréchet smooth. We show that if the preference functional is concave (or convex) and Lipschitz, Machina's results can nonetheless be recovered in this nonsmooth setting. Moreover, concavity and Lipschitzness are much easier to verify than Fréchet smoothness.

Our envelope based approach significantly expands the class of preference functionals for which local utility analysis is valid. (It is trivial to note that *convex* Lipschitz preference functionals are also amenable to our analysis since they too possess an envelope representation.) Therefore, our approach also applies to risk averse Lipschitzian RDU preference functionals (which are convex, cf. Section 4.2).

The rest of the paper is structured as follows. We introduce the model and obtain an envelope representation (in Section 3) and show that local utility analysis is valid in our setting in Section 4. An axiomatisation of preferences following Maccheroni (2002) is in Section 5 and Section 2 contains a discussion of the related literature. Omitted proofs are in the Appendix, which also contains a discussion Lipschitz continuity and of various notions of differentiability.

2. Related literature

The standard book on multicriteria utility theory is still Keeney and Raiffa (1993). They start out with a single utility function for each criterion used by the decision-maker and then provide axioms on how these should be combined. The various sets of axioms characterise additive, multiplicative and multilinear representations and require decision makers to make explicit tradeoffs among criteria, for example comparing lotteries where all but one of the criteria are at their lowest levels and one at its highest level. Such trade-offs might be harder for decision-makers to make than using the representation we provide.

The preference functionals above can be viewed as representing a decision maker who aggregates the utility functions of a set of agents in an economy. It is interesting to note that even though each agent in this economy has a vN-M utility function, the decision maker with the aggregated utility function violates the vN-M axioms (in particular the Independence axiom). A similar observation has been made by Schlee (2001), who shows that in an economy with a representative agent, it may be that all agents strictly dislike any improvement in information, but the representative agent is indifferent to more information.

A class of concave preference functionals has been axiomatised by Maccheroni (2002) in a very general setting, who shows that such preferences also have envelope representations. By considering an environment with more structure which, in turn, allows us to make stronger assumptions, we are able to sharpen the characterisation obtained. In particular, we augment his axioms with an axiom christened *Uniform Sensitivity* by Rader (1972), which ensures that the functional representing the preference is Lipschitz. We thus axiomatise a class of preferences that possess an envelope representation in our setting. This demonstrates how a set of axioms on preferences that results in a concave preference functional can be enlarged to result in a Lipschitz concave preference functional. We discuss Maccheroni (2002) in greater detail in Section 5.

Another paper that is very relevant is Cerreia-Vioglio (2009) who considers various domains (lotteries over a finite space, lotteries over a compact metric space, menus of lotteries) and studies (suitably) continuous preferences that satisfy Weak Convexity or Mixing (so the upper contour sets are convex). He interprets Weak Convexity as we do, in that it reveals the agent's pessimism about his future tastes and reveals a hedging motive. In particular, if the domain is lotteries over a compact metric space, any continuous preference that satisfies Weak Convexity can be represented by the functional $\inf_{v\in\mathcal{V}} U(\mathbf{E}_v(p), v)$ where $\mathcal{V}$ is a closed, convex family of vN-M utility functions, and U is an aggregator of sorts. Thus, our setting is a special case of his, although our extra assumptions allow us to draw stronger conclusions. The main contribution of Cerreia-Vioglio (2009) is then to show that any model of choice that satisfies Weak Convexity can be viewed as being subject to a form of pessimism and being desirous of instruments that allow the agent to hedge against future subjective uncertainty.

Local utility analysis, as discussed above, was introduced by Machina (1982). Machina (1984) considers smooth and convex preference functionals on the space of monetary lotteries. Convexity of the preference functional is motivated by the observation that in settings where the temporal resolution of uncertainty matters, economic agents may prefer this uncertainty to be resolved sooner rather than later. While our analysis is also valid for convex preference functionals, we drop the smoothness assumption and instead only require that the preference functional is Lipschitz continuous.

The technical result that allows us to obtain an envelope representation is from Ergin and Sarver (2010a,b). They are interested in a model where an agent has preferences over menus of lotteries, and obtain a representation wherein the agent behaves as if there are costs to contemplating about the consequences of choice from a menu.

3. The model and envelope representation

Let $D[0, 1]$ be the space of all probability distribution functions over the unit interval. Distribution functions will be denoted by $F(\cdot)$, $G(\cdot)$ etc. We may think of each distribution function $F(\cdot)$ as a member of $L^1[0, 1]$, the Banach space of all integrable functions on the unit interval, and therefore endow $D[0, 1]$ with the induced metric, i.e., $d(F, G) := \|F - G\|_1$, where $\|\cdot\|_1$ is the L^1 norm and $\|F - G\|_1 = \int |F(x) - G(x)|\ \mathrm{d}x$. As noted by Machina (1982), this metric induces the topology of weak convergence on $D[0, 1]$. In what follows, we regard $D[0, 1]$ as a compact, convex subset of $L^1[0, 1]$.

A preference functional is a function $V : D[0, 1] \to \mathbb{R}$. A preference functional is *concave* if for all $F, G \in [0, 1]$ and $\alpha \in [0, 1]$, $V\big(\alpha F+(1-\alpha)G\big) \geqslant \alpha V(F)+(1-\alpha)V(G)$. It is Lipschitzian of rank $M > 0$ if $|V(F) - V(G)| \leqslant M \|F - G\|_1$ for all $F, G \in D[0, 1]$. In this paper, we shall focus primarily on concave preference[1] functionals, although it will be clear that all our results also hold for convex preference functionals.

Let $C[0, 1]$ denote the space of uniformly continuous functions on the unit interval. A function $u \in C[0, 1]$ is *Lipschitzian of rank* $M > 0$ if $|u(x) - u(y)| \leqslant M |x - y|$ for all $x, y \in [0, 1]$. A subset $\mathcal{U} \subset C[0, 1]$ is *equi-Lipschitzian*[2] if there exists $M > 0$ such that every $u \in \mathcal{U}$ is Lipschitzian of rank M. In what follows, we shall frequently suppress the variable of integration and denote integrals of the form $\int u(x)\, \mathrm{d}F(x)$ and $\int F(t)\varphi(t)\, \mathrm{d}t$ by $\int u\, \mathrm{d}F$ and $\int F\varphi$ respectively.

[1] Clearly, concavity of the preference functional indicates a preference for randomisation in the sense that $V(\frac{1}{2}F + \frac{1}{2}G) \geqslant \frac{1}{2}V(F) + \frac{1}{2}V(G)$. As we will see below, this says nothing about the risk aversion of the agent.

[2] The term is chosen to be in consonance with the notion of *equi-differentiability* used by Milgrom and Segal (2002). It is easily shown that if the family of functions $\mathcal{U}$ is equi-differentiable in the sense of Milgrom and Segal (2002), then it is also equi-Lipschitzian.

Please cite this article in press as: Chatterjee, K., Vijay Krishna, R., A nonsmooth approach to nonexpected utility theory under risk. Mathematical Social Sciences (2011), doi:10.1016/j.mathsocsci.2011.08.001

We begin by obtaining an intermediate representation. It says that every concave preference functional can be written as the pointwise minimum of a family of affine functionals. It is well known that every concave function can be written as the infimum of a family of affine functionals. The proposition below shows that the infimum is actually achieved, and puts some structure on the family of affine functionals.

Theorem 3.1. *For any preference functional $V : D[0,1] \to \mathbb{R}$, the following are equivalent:*

(i) *V is concave and Lipschitzian of rank $M > 0$.*
(ii) *There exists a set $\mathcal{U} \subset C[0,1]$ that is convex, closed, (norm) bounded, and equi-Lipschitzian of rank $M > 0$ (and hence compact), such that*

$$V(F) = \min\left[\int u \, \mathrm{d}F : u \in \mathcal{U}\right] \tag{1}$$

Moreover, for any $a > 0$ and $b \in \mathbb{R}$, $aV + b = \min[\int u \, \mathrm{d}F : u \in a\mathcal{U} + b]$. Finally, there is a $\mathcal{U}$ that is smallest (in the sense of set inclusion) amongst sets satisfying Eq. (1) above.

The representation of V as in (1) with a minimal $\mathcal{U}$ is referred to as a *canonical envelope representation*.

The concave functional V has a very natural interpretation for decision makers. Each vN-M utility function on [0, 1] can be interpreted as a decision making criterion. The decision maker behaves as if she considers many decision making criteria possible, but given the coarseness in her perception, is unsure about the correct criterion to use. The criterion she chooses is the maximum of the minimum values – i.e., the largest amount of utility she can guarantee herself – in the face of this subjective 'ambiguity'.

It is worth mentioning that if $V(F) := \int x \, \mathrm{d}F$, i.e., if V is risk neutral and an expected utility functional, then (the minimal) $\mathcal{U} := \{u \in \mathbb{R}^{[0,1]} : u(x) = x, \ x \in [0,1]\}$, that is, $\mathcal{U}$ consists solely of the linear function $u(x) = x$.

To see why uniqueness of the set $\mathcal{U}$ cannot be guaranteed, consider a set $\mathcal{U}$ as in the theorem. Since [0, 1] is compact, and $\mathcal{U}$ is norm bounded in $C[0,1]$, it follows that there exists an $M < \infty$ such that $\max_{x \in [0,1]} u(x) < M$ for all $u \in \mathcal{U}$. Now, define $v \in \mathbb{R}^{[0,1]}$ as $v(x) = M$ for all $x \in [0,1]$. Also, let $\hat{\mathcal{U}} := \mathrm{conv}(\mathcal{U} \cup \{v\})$. Clearly, for all $F \in D[0,1]$, we still have $\min[\int u \, \mathrm{d}F : u \in \mathcal{U}] = \min[\int w \, \mathrm{d}F : w \in \hat{\mathcal{U}}]$. Thus, it is not possible to pin down the set $\mathcal{U}$ uniquely. (Recall that $\mathcal{U}$ is identified up to the same positive affine transformation as V.) The best we can do, as stated in the theorem, is pin down a smallest set $\mathcal{U}$ that induces (1) above. As mentioned in Section 2, Theorem 3.1 can also be viewed as a refinement of Theorem 20 in Cerreia-Vioglio (2009).

4. Local utility analysis

Machina (1982) observes that the *theorems* of expected utility analysis are useful even for nonexpected utility preference functionals. We now describe how this works in general, nonsmooth settings.

A *path* in $D[0,1]$, denoted by $F(\cdot, t)$, is a continuous function $t \mapsto F(\cdot, t)$ where $t \in [0,1]$. For any $F, G \in D[0,1]$, a path *connects* F and G if there exists a continuous path $F(\cdot, t)$ such that $F(\cdot, 0) = F$ and $F(\cdot, 1) = G$. A path is Lipschitzian if the map $t \mapsto F(\cdot, t)$ is Lipschitz. It is *uniformly Lipschitzian* if there exists an $M > 0$ such that for all $x \in [0,1]$, $F(x, \cdot)$ is Lipschitzian of rank M. It is *uniformly semismooth*[3] if there exists a set $N \subset [0,1]$ of Lebesgue measure 0 such that for all $\varphi \in L_+^\infty[0,1]$ (so that $\varphi \geqslant 0$) with $\|\varphi\|_\infty = 1$ and for all $t^* \in [0,1] \setminus N$, the integral $\int F(\cdot, t)\varphi(\cdot)$ is differentiable (in t) at t^*, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int F(\cdot, t)\varphi(\cdot)\right]\Big|_{t^*} = \int \left[\frac{\partial}{\partial t} F(\cdot, t)\varphi(\cdot)\right]\Big|_{t^*}.$$

Intuitively, a uniformly semismooth path is one where for a set of points of full measure, the order of integration and differentiation can be interchanged. (This will allow us to evaluate an integral over a path, as in Eqs. (3) and (4).)

Let $V : D[0,1] \to \mathbb{R}$ be a preference functional. It is a *generalised potential* if (i) for each $F \in D[0,1]$, there exists a continuous function $v(\cdot, F) : [0,1] \to \mathbb{R}$ such that $V(F) = \int v(\cdot, F) \, \mathrm{d}F$, and (ii) for any path $F(\cdot, t)$ through F that is uniformly Lipschitz and semismooth, $F(\cdot, t^*) = F$ implies

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(F(\cdot, t)\big)\Big|_{t^*} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int v\big(x, F(x, t^*)\big) \, \mathrm{d}F(x, t)\right)\Big|_{t^*}. \tag{2}$$

As we shall see below, for the uniformly Lipschitz and semismooth paths under consideration, this allows us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int v\big(x, F(x, t^*)\big) \, \mathrm{d}F(x, t)\right)\Big|_{t^*} = \int v\big(x, F(x, t^*)\big) \, \mathrm{d}F_t(x, t^*) \tag{3}$$

and we can integrate Eq. (2) and use Eq. (3) to obtain

$$V\big(F(\cdot, 1)\big) - V\big(F(\cdot, 0)\big) = \int \left(\int v\big(x, F(x, t^*)\big) \, \mathrm{d}F_t(x, t^*)\right) \mathrm{d}t^*. \tag{4}$$

The function $v(\cdot, F)$ will be called the *weak local utility function* at F.[4] It is easy to see that weak local utility functions contain information about what happens near a lottery, and for preference functionals that are generalised potentials, it is possible to link local and global by integrating along suitable paths.

Every linear, expected utility preference functional is obviously a generalised potential. It the preference functional is not linear but is Fréchet smooth, then the weak local utility function can be taken to be the Fréchet derivative. It is easy to check that in this case, the preference functional is a generalised potential, and the fundamental theorem of calculus, which allows integration along sufficiently smooth paths, ensures that we can integrate along paths.

The main observation of this paper is that concave (or convex) preference functionals that are Lipschitz continuous are also generalised potentials. Machina (1982, p 279) states that even though the Independence axiom does not hold, i.e., even though the preference functional V is not linear, 'the implications and predictions of theoretical studies which use expected utility analysis *will* (emphasis in original) be valid, *provided preferences are smooth* (emphasis added)'. This is because some of the properties of the local utility functionals correspond to global properties of the preference functional V, and vice versa. Our goal in this section

[3] We note that our definition of uniformly semismooth is nonstandard, and different from that in the literature, for instance Correa and Jofre (1989), which is equivalent to the existence and continuity of directional derivatives in every direction. We do not know how the two definitions are related.

[4] Notice that the weak local utility function defined here is distinct from Machina's local utility function, since the weak local utility function does not provide a uniform local approximation of the preference functional, while Machina's local utility function does. Put differently, Machina's local utility function $v(\cdot, F)$ is the Fréchet derivative of V at F and hence provides an approximation of the preference functional in an open neighbourhood of F, while the weak local utility function is merely a subgradient of V at F. (See Appendix A.4 for a definition of Fréchet differentiability.) Clearly, every local utility function is a weak local utility function, but the converse is not true.

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is to show that Machina's statement above holds for all Lipschitz concave preference functionals, even the ones that are not smooth.

The Eqs. (2)–(4) are the essential tools underlying our analysis. Using an appropriate combination of these equations, it is possible to link local and global valuations of the preference functional. We shall now establish that concave and Lipschitz preference functionals are generalised potentials, so that Eqs. (2)–(4) hold. We can now prove our first main theorem.

Theorem 4.1. *Let V be a preference functional that is concave and Lipschitz. Then, it is a generalised potential.*

Theorem 4.1 is closely related to work by Krishna and Maenner (2001), who are motivated by problems in multidimensional mechanism design. Their Theorem 1 is, modulo notational differences, a special case of the theorem above. We improve on their Theorem 1 by allowing for a compact, convex domain in a Banach space (their domain is a convex set with interior in a finite dimensional Euclidean space) and by dispensing with their requirement that the path under consideration be smooth (since we only require that the path be uniformly semismooth).[5] The infinite dimensional setting is needed since our domain $D[0, 1]$ lives in $L^1[0, 1]$, an infinite dimensional Banach space, and in (the proof of) Theorem 4.5, we will consider uniformly semismooth paths.

We are now ready to prove non-smooth analogues of Machina's local utility theorems. In order to use Eqs. (2)–(4) above, all that needs to be established in the proofs is that the paths under consideration are uniformly Lipschitzian and uniformly semismooth.

Theorem 4.2. *Let V have a canonical envelope representation. Then, the following are equivalent:*

(i) *$V(F^*) \geqslant V(F)$ for all $F, F^* \in D[0, 1]$ where F^* first order stochastically dominates F, and*
(ii) *$u(x)$ is increasing in x for each $u \in \mathcal{U}$.*

The proof of the preceding theorem considers paths of the form $F(\cdot, \alpha) := F + \alpha(F^* - F)$. It is clear that paths of this kind are equi-Lipschitzian and uniformly semismooth. (Indeed, they are straight lines.) As in Machina (1982), to ensure the strict preference for stochastic dominance, we shall assume from now on the $u(x)$ is strictly increasing in x for all $u \in \mathcal{U}$. We note that the implication '(ii) implies (i)' does not require local utility analysis. The next proposition relates risk aversion to the concavity of the local utility functions. It follows from Theorem 4.5.

Proposition 4.3. *Let V have a canonical envelope representation. Then, the following are equivalent:*

(i) *$V(F) \geqslant V(F^*)$ for all $F, F^* \in D[0, 1]$ such that F^* differs from F by a mean preserving increase in risk, and*
(ii) *$u(x)$ is a concave function of x for each $u \in \mathcal{U}$.*

As in Machina (1982), more can be said about the local utility functions, and their relation to the preference functional V. Towards this end, we begin with a definition from Machina (1982).

Definition 4.4. If F and F^* are two cumulative distribution functions over wealth in $[0, 1]$, then F^* is said to differ from F by a *simple compensated spread* if the agent is indifferent between F and F^*, and if $[0, 1]$ may be partitioned into disjoint intervals I_L and I_R (with I_L to the left of I_R) such that $F^*(x) \geqslant F(x)$ for all $x \in I_L$ and $F^*(x) \leqslant F(x)$ for all $x \in I_R$.

For an expected utility maximiser, sequences of simple compensated spreads are equivalent to mean utility preserving increases in risk. As in Machina (1982), comparison of increased risk aversion can be characterised in terms of the local utility functions. In what follows, we shall assume that all the local utility functions are strictly increasing. For any $x^* \in [0, 1]$, $G_{x^*}(\cdot)$ is the distribution concentrated at x^*.

Following Machina (1982), it is useful to define the conditional demand for a risky asset as the value of α that yields the most preferred distribution in the set $\left\{(1-p)F^{**} + pF_{(1-\alpha)r+\alpha\tilde{z}}\right\}$, where $r > 0$, $\tilde{z}$ is a nonnegative random variable with mean greater than r and $F_{(1-\alpha)r+\alpha\tilde{z}}$ is the distribution function of $(1-\alpha)r + \alpha\tilde{z}$. A risk averse individual is said to be a *diversifier* if, for all distributions $F^{**}, p \in (0, 1), r > 0$, and nondegenerate $\tilde{z}$, the individual's preferences over the set of distributions $\left\{(1-p)F^{**} + pF_{(1-\alpha)r+\alpha\tilde{z}}\right\}$ are strictly quasiconcave in α.

The following is the analogue of Theorem 4 in Machina (1982).

Theorem 4.5. *Let V and V^* have canonical envelope representations, with weak local utility functions $\upsilon(\cdot, F)$ and $\upsilon^*(\cdot, F)$ respectively. Then, the following are equivalent:*

(i) *For arbitrary distributions $F(\cdot)$ and $F^{**}(\cdot)$ and $p \in (0, 1)$, if c and c^* respectively solve $V\big((1-p)F^{**} + pF\big) = V\big((1-p)F^{**} + pG_c\big)$ and $V^*\big((1-p)F^{**} + pF\big) = V^*\big((1-p)F^{**} + pG_{c^*}\big)$, then $c \leqslant c^*$.*
(ii) *For any cdf $F(\cdot)$, and any weak local utility functions $\upsilon(\cdot, F)$ and $\upsilon^*(\cdot, F)$, $\upsilon(\cdot, F)$ is at least as concave as $\upsilon^*(\cdot, F)$.*
(iii) *If the distribution $F^*(\cdot)$ differs from $F(\cdot)$ by a simple compensated spread from the point of view of $V^*(\cdot)$ so that $V^*(F^*) = V^*(F)$, then $V(F^*) \leqslant V(F)$.*

If both individuals are diversifiers and have differentiable weak local utility functions, then the above conditions are equivalent to:

(iv) *For an distribution $F^{**} \in D[0, 1], p \in (0, 1), r > 0$, $\tilde{z}$ a nonnegative random variable with $\mathbf{E}[\tilde{z}] > r$, if α and α^* yield the most preferred distributions of the form $(1-p)F^{**} + pF_{(1-\alpha)r+\alpha\tilde{z}}$ for $V(\cdot)$ and $V^*(\cdot)$ respectively, then $\alpha \leqslant \alpha^*$.*

As in Machina (1982), the proof establishes that (i) implies (ii) implies (iii) which implies (i). That (iii) (or (ii)) implies (i) is straightforward, and does not require local utility analysis. Details of this are provided in the Appendix A.2. The equivalence of (ii) and (iv) follows from the observations in Wang (1993).

4.1. A useful family of preferences

We now introduce a useful family of preference functionals that admit a simple envelope representation, and correspond to ε-contamination in the literature on ambiguity.[6] As Yaari (1987) notes, 'In expected utility theory, the agent's attitude towards risk and the agent's attitude towards wealth are forever bonded together'. Our objective is to find a one-parameter family of utility functions that disentangle risk aversion and marginal utility of wealth.[7] Specifically, we would like to vary risk aversion while keeping the utility over certain amounts of wealth (and hence marginal utility) fixed. In a problem with intertemporal choice with separable utility, this would have the happy consequence of keeping the marginal rate of substitution of certain consumption fixed between consecutive periods, while the parameter allows us to vary risk aversion.

5 Actually, more can be shown. In the finite dimensional setting of Krishna and Maenner (2001), it can be shown that every Lipschitz path is uniformly semismooth, hence their Theorem 1 actually holds for all Lipschitz paths.

6 In the working paper version, we explore the consequences of increased risk aversion in a portfolio problem with two risky assets and in an asset pricing model, based on the Lucas tree.

7 Another nonexpected utility theory with this feature is the model of disappointment aversion due to Gul (1991).

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Let $u : [0, 1] \to \mathbb{R}$ be a utility function that is concave, Lipschitz and increasing, and suppose for simplicity that u is also differentiable everywhere.[8] Consider now, for each $\alpha \in [0, 1]$, the function $V(\cdot, \alpha)$ over lotteries, given by

$$V(F;\alpha) = \alpha u(\mu_F) + (1-\alpha)\int u\,\mathrm{d}F. \tag{5}$$

Notice that for $1 \geqslant \alpha > \beta \geqslant 0$, $V(\cdot;\beta)$ is more risk averse than $V(\cdot;\alpha)$ in the following two (typically distinct) senses: (i) For any lottery F, $c_\alpha \geqslant c_\beta$, where c_α and c_β are, respectively, the certainty equivalents for the utility functions $V(\cdot;\alpha)$ and $V(\cdot;\beta)$, and (ii) If lottery F is a simple compensated mean preserving spread of the lottery G (see definition above) such that $V(F;\alpha) = V(G;\alpha)$, then $V(F;\beta) \leqslant V(G;\beta)$.

The function $V(\cdot, 1)$ is risk neutral, since $V(F, 1) = u(\mu_F)$. For any $x \in [0, 1]$, G_x is the distribution function concentrated at x. Then, $V(G_x, \alpha) = \alpha u(x) + (1-\alpha)u(x) = u(x)$ for all $\alpha \in [0, 1]$, that is the 'marginal utility' for wealth is independent of α, while risk aversion varies with α.

Thus, in the one-parameter family $V(\cdot;\alpha)$ with $\alpha \in [0, 1]$, α can be thought of as measuring risk aversion (relative to risk neutrality and the Arrow–Pratt measure of the function u), while utility for money remains unchanged. Proposition A.3 in the Appendix shows that the family $V(\cdot, \alpha)$ has an envelope representation, and provides an explicit construction of the envelope. The construction also demonstrates that the family $(V\cdot, \alpha)$ is naturally related to the notion of ε-contaminated beliefs in the literature on ambiguity.

4.2. Rank dependence

The class of Rank Dependent Utility (RDU) functionals was introduced by Quiggin (1982) and Yaari (1987). Chew et al. (1987) first show that RDU functionals are not Fréchet differentiable, and then show that local utility analysis is possible for Gâteaux differentiable RDU functionals. An RDU functional is a functional of the form

$$V(F) = \int v(x)\ \mathrm{d}(g \circ F)(x)$$

where $F \in D[0, 1]$, $v : [0, 1] \to \mathbb{R}$ is continuous and strictly increasing, and $g : [0, 1] \to [0, 1]$ is continuous, strictly increasing and onto. Chew et al. (1987, Corollary 2) and Wakker (1993, Theorems 20 and 25) show that an RDU preference functional is risk averse if, and only if, both g and v are concave, in which case the preference functional is quasiconvex.

We shall now show that risk averse RDU functionals are, in fact, convex. Indeed, we shall show that an RDU preference functional is convex as long as g is concave.

Proposition 4.6. *Suppose V is a RDU preference functional V as defined above, and g is concave. Then, V is convex.*

Proof. Let $F, G \in D[0, 1]$ and $\alpha \in [0, 1]$. Then, by the concavity of g, for every $x \in [0, 1]$, $g\big(\alpha F(x) + (1-\alpha)G(x)\big) \geqslant \alpha g\big(F(x)\big) + (1-\alpha)g\big(G(x)\big)$. Put differently, $\alpha(g\circ F) + (1-\alpha)(g \circ G)$ first order stochastically dominates $g \circ (\alpha F + (1-\alpha)G)$. This implies,

$$\begin{aligned} V\big(\alpha F + (1-\alpha)G\big) &= \int v(x)\ \mathrm{d}\big(g \circ (\alpha F + (1-\alpha)G)\big)(x) \\ &\leqslant \int v(x)\ \mathrm{d}\big(\alpha(g\circ F) + (1-\alpha)(g\circ G)\big)(x) \\ &= \alpha V(F) + (1-\alpha)V(G) \end{aligned}$$

where the inequality is due to the first order stochastic dominance established above. □

It is easy to see that if v and g are Lipschitz, then V is also Lipschitz. Therefore, for risk averse (and hence convex) Lipschitz RDU functionals, our theorems apply, without any additional differentiability hypotheses. In the setting of RDU preference functionals, Chateauneuf and Cohen (1994) show that it is possible to disentangle, to an extent, risk aversion and marginal utility for wealth. This result can be seen as a variation of the representation of Yaari (1987), who axiomatises an instance of RDU preference functionals, where the function v is linear, so that his preference functional is risk averse even though the marginal utility of wealth is constant.

5. Axioms

In this section, we will augment the representation of Maccheroni (2002) when specialised to our setting. Maccheroni (2002) provides very general (axiomatic) conditions under which a concave preference functional obtains and provides a representation of such preference functionals analogous to the representation obtained in Theorem 3.1. With the additional structure of our domain, we can make stronger assumptions about preferences and reach stronger conclusions. We provide below a more careful comparison between his theorem and our representation. A preference $\succsim \subset D[0, 1] \times D[0, 1]$ is a complete and transitive binary relation on $D[0, 1]$. Our first axiom stipulates that preferences are continuous.

Axiom (*Continuity*)**.** The sets $\{G : G \succsim F\}$ and $\{G : F \succsim G\}$ are closed for all $F \in D[0, 1]$.

For our next axiom, we follow Rader (1972). Essentially, we require that the indifference sets do not become too close to each other too quickly. Rader (1972) calls this property *uniform sensitivity*. Continuous preferences that are uniformly sensitive can be represented by a Lipschitz continuous utility function, which will be crucial in what follows. We emphasise that uniform sensitivity, like continuity, is a structural axiom.

To state the axiom, we need some additional notation. For $A, B \subset D[0, 1]$ disjoint, define

$$\mathfrak{H}(A, B) := \inf\{d(F, G) : F \in A, G \in B\}.$$

If $A = \{F\}$ is a singleton, we shall write $\mathfrak{H}(F, B)$, dropping the set notation for F. Notice that $\mathfrak{H}$ is a semimetric (or pseudometric), but not a metric.

For any $F \in D[0, 1]$, let the indifference class of F be denoted by $[F]$, the upper contour set be denoted by $L^+_{\succsim}(F) := \{G \in D[0, 1] : G \succsim F\}$ and the lower contour set by $L^-_{\succsim}(F) := \{G \in D[0, 1] : F \succsim G\}$. We are now ready to state our axiom.

Axiom (*Uniform Sensitivity*)**.** Preference $\succsim$ is *uniformly sensitive* if there exists $t > 0$ such that for all $F, G \in D[0, 1]$,

$$d(F^*, G^*) \geqslant t\mathfrak{H}(F, [G])$$

for all $F^* \in [F]$ and $G^* \in [G]$.

To see the intuition behind the requirement, consider some F and G such that $F \nsim G$. $\mathfrak{H}(F, [G])$ represents the smallest distance from F and the indifference class of G. Uniform sensitivity says that the distance from any other F^* indifferent to F to any other G^* indifferent to G is at least some fraction of the distance $\mathfrak{H}(F, [G])$, and moreover, this fraction is independent of F and G. Another axiomatisation for Lipschitzian utility is provided by Mas-Colell (1977), who describes the assumption as '…concept …is devised to capture the idea of indifference curves fitting together not too wildly'.

[8] We note that the requirement that u be Lipschitz rules out some natural utility functions such as $u(x) = \sqrt{x}$. But such a function can be approximated by a Lipschitz function, since Lipschitz functions are dense in the space of continuous functions.

Rader (1972, Chapter 6, Theorem 23) shows that a preference is continuous and uniformly sensitive if and only if it admits a Lipschitz continuous utility representation. Lipschitz continuity of a *concave* utility representation will be useful to us in two ways. Firstly, Lipschitz continuity of the representation says that the utility function has a superdifferential (in the sense of convex analysis) at each point on the domain. Second, Lipschitz continuity ensures that we can integrate marginal utility along Lipschitzian paths in $D[0, 1]$, thus enabling us to perform local utility analysis, even though the objects of our analysis are not smooth.

Our main behavioural axioms are a weakening of the classical Independence axiom. The first axiom states that the agent's preferences are convex.

Axiom (*Weak Convexity—WC*)**.** If $F \sim G$, then $\frac{1}{2}F + \frac{1}{2}G \succsim F$.

Clearly, this represents a pessimism in the agent's approach to valuing lotteries. (It is also a weakening of Independence.) Moreover, it represents the preference of the parent in the example of Machina's Mom. Weak Convexity, also referred to as Mixing, is the central axiom in Cerreia-Vioglio (2009), who provides a representation for preferences that satisfy Weak Convexity and some variations of Continuity.

Our next axiom is another weakening of Independence, to which we shall refer as *Zero Independence* and was introduced by Safra and Segal (1998).

Axiom (*Z-Independence*)**.** $F \sim F^*$ and $\alpha \in [0, 1]$ implies $\alpha F + (1-\alpha)G_0 \sim \alpha F^* + (1-\alpha)G_0$.

Roughly put, the idea behind the axiom is that the decision maker can easily visualise mixtures with G_0, the probability measure concentrated at 0. (But also see Safra and Segal, 1998 for a more thorough discussion of this axiom.) The next axiom is an extremely weak monotonicity condition.

Axiom (*Weak Monotonicity—WMon*)**.** For all $F \neq G_0$, $F \succ G_0$.

Weak Monotonicity is very mild, merely requiring that every lottery be strictly preferred to the lottery that gives 0 with probability one. We now turn to our representation theorems.

We note that the axioms Continuity and Weak Convexity appear in Maccheroni (2002), while our Z-Independence is analogous to his Best Outcome Independence. He assumes that there is a best outcome, which is analogous to our requirement of Weak Monotonicity. His abstract setting does not permit an analogue of our axiom Uniform Sensitivity unless more assumptions are made on the set of outcomes, which is what we have done.

5.1. Representation theorems

We are now able to state our main representation theorem.

Theorem 5.1. *Let $\succsim$ be a preference that satisfies Weak Monotonicity. Then, the following are equivalent:*

(i) *$\succsim$ satisfies Continuity, Uniform Sensitivity, weak Convexity, Z-Independence.*
(ii) *There is an equi-Lipschitzian set $\mathcal{U}$ of utility functions on $[0, 1]$ such that*

$$V(F) = \min\left[\int u \, \mathrm{d}F : u \in \mathcal{U}\right] \tag{6}$$

is Lipschitz continuous and represents $\succsim$, and where $\mathcal{U}$ is norm bounded in $C[0, 1]$, weakly closed and convex, and satisfies $\int u \, \mathrm{d}G_0 = \int v \, \mathrm{d}G_0$ for all $u, v \in \mathcal{U}$.

Given such a set $\mathcal{U}$, any other set of the form $a\mathcal{U} + b$ where $a > 0, b \in \mathbb{R}$ also induces a utility function (as in Eq. (6)) that represents $\succsim$. Finally, there is a $\mathcal{U}$ that is smallest (in the sense of set inclusion) amongst sets satisfying Eq. (6) above.

Once again, we note that a version of the representation (6) above has been obtained by Maccheroni (2002). The difference between his analysis and ours stems from the choice of domain. His choice of domain is lotteries with finite support over an arbitrary set Z. The set Z need not have any topological structure, and hence the vN-M utilities obtained are just functions from Z to $\mathbb{R}$. Since Z has no topology, we cannot say anything about their continuity. In contrast, we let $Z := [0, 1]$, and assume Uniform Sensitivity. This lets us obtain a set of utility functions $\mathcal{U}$ in the representation, where every function in $\mathcal{U}$ is Lipschitz continuous. Finally, in spite of the similarity between the representations obtained here (Eq. (6)) and in Maccheroni (2002), the proofs are quite different, and we cannot appeal directly to the analysis in Maccheroni (2002).

We end with the observation that there are many preference functionals that have an envelope representation but do not satisfy Z-Independence (Axiom in Section 5). (Note that any Lipschitz continuous, concave preference functional immediately satisfies Continuity, Uniform Sensitivity, and Weak Convexity.) To see this, consider two vN-M utility functions $u_1, u_2 \in C[0, 1]$ that are Lipschitz continuous such that $u_1(0) \neq u_2(0)$, and consider the set $\mathcal{U} := \mathrm{conv}\{u_1, u_2\}$. Then, the preference functional $V(F) := \min[\int v \, \mathrm{d}F : v \in \mathcal{U}]$ has a canonical envelope representation as in (1), but clearly does not satisfy Z-Independence.

Appendix. Proofs

For expositional convenience, we repeat some of the definitions and ideas in the text. Appendix A.1 proves the existence of an envelope representation (1). Finally, Theorems 4.1 and 4.2 and Proposition 4.3 representing the local utility analysis are proved in Appendix A.2.

A.1. Envelope representation

In this subsection, we complete the proof of Theorem 3.1. We first recall a (version of a) result of Ergin and Sarver (2010a) (a generalisation of which is in Ergin and Sarver (2010b)), which in turn is an extension of the Mazur density theorem.

Theorem A.1. *Let X be a separable Banach space, and $C \subset X$ a convex set whose affine hull is dense in X. Let $f : C \to \mathbb{R}$ be a concave function that is Lipschitz. Then, there exists a weak* compact, convex set $\mathcal{M} \subset X^*$ such that*

$$f(x) = \min_{x^* \in \mathcal{M}} \left[\langle x, x^* \rangle - f^*(x^*)\right]$$

where $f^*(x^*) := \inf_{x \in C}\left[\langle x, x^* \rangle - f(x)\right]$ is the Fenchel conjugate of f. Moreover, there exists a smallest set $\mathcal{M}$ that satisfies the relation above.

The last part of the theorem can be interpreted as saying that there exists a greatest concave function $g : X \to \mathbb{R}$ that dominates f, that is $g \geqslant f$, that is finite everywhere, and is equal to f on C. Since X is separable, it follows that $\mathcal{M}$, since it is weak* compact, is also metrisable (Theorem 6.34 of Aliprantis and Border, 1999). If C is compact, it follows immediately from Berge's Theorem of the Maximum – see, for instance, Ok (2007) – that f^* is (weak*) continuous on $\mathcal{M}$. We now move to the proof of Theorem 3.1.

Proof. For ease of exposition, we reproduce some of the definitions from the text. Recall that a functional $\varphi_0 \in L^\infty[0, 1]$ is a *subgradient* of V at $F_0 \in D[0, 1]$ if for all $F \in D[0, 1]$, $V(F) - V(F_0) \leqslant \langle F - F_0, \varphi_0 \rangle$ (where $\langle F, \varphi \rangle = \int F\varphi$). The set of all subgradients of V at F_0 is denoted by $\partial V(F_0)$. Since V is Lipschitz continuous, it follows from a theorem of Gale (1967) that V has a subgradient at each point in the domain. Moreover, the subgradient has is bounded uniformly by M.

The conjugate functional $V^*(\varphi_0) = \inf_{F \in D[0,1]} \left[\langle F, \varphi_0 \rangle - V(F) \right]$ is (weak*) continuous, and $\varphi \in \partial V(F_0)$ if, and only if, $V(F_0) + V^*(\varphi_0) = \langle F_0, \varphi_0 \rangle$. By the theorem above, V can be written in the form $V(F) = \min\left[\int F\varphi - V^*(\varphi) : \varphi \in \mathcal{F} \right]$ where $\mathcal{F}$ is a weak* compact, convex subset of $L^\infty[0, 1]$, i.e., V is the minimum of a family of affine functionals, and the family is indexed by a weak* compact convex set. Also, for each $\varphi \in \mathcal{F}$, $\|\varphi\|_\infty \leqslant M$.

For any $\varphi \in \mathcal{F}$, define $u(x) := -\int_0^x \varphi + \int \varphi - V^*(\varphi)$ which is clearly Lipschitz continuous of rank M. This gives us the set $\mathcal{U}$ and integration by parts gives us the representation (1).

It is easily seen that we can replace $\mathcal{U}$ by its closed, convex hull, therefore we will take $\mathcal{U}$ to be closed and convex. Since each $u \in \mathcal{U}$ is Lipschitz of rank $M > 0$, it follows that $\mathcal{U}$ is norm bounded, and also compact. □

A.2. Proofs from Section 4

We begin by stating a variation of Theorem 1 of Milgrom and Segal (2002). Let X be a choice set, $t \in [0, 1]$ and $g : X \times [0, 1] \to \mathbb{R}$. Then, the value function and choice correspondence are

$$\Phi(t) := \min\left[g(x, t) : x \in X \right]$$

$$X^*(t) := \left\{ x \in X : g(x, t) = \Phi(t) \right\}.$$

We shall say that g is uniformly semismooth if, there exists a set $E \subset [0, 1]$ of full measure, i.e., $\text{Leb}(E) = 1$, such that for all $x \in X$, $g_t(x, t^*)$ exists if $t^* \in E$.

Theorem A.2. *Let $t^* \in [0, 1]$ and $x^* \in X^*(t^*)$, and suppose $g_t(x^*, t^*)$ exists. Then, if Φ is differentiable at t^*, $\Phi'(t^*) = g_t(x^*, t^*)$. If Φ is differentiable almost everywhere and if g is uniformly semismooth, there exists a set $E_0 \subset [0, 1]$ with $\text{Leb}(E_0) = 1$ such that for all $t^* \in E_0$, $\Phi'(t^*) = g_t(x^*, t^*)$, that is, for each $t^* \in E_0$, $\Phi'(t^*)$ and $g_t(x^*, t)$ exist and are equal.*

Proof. The first part of the theorem is Theorem 1 of Milgrom and Segal (2002), so we do not provide a proof. For the second part, let Φ be differentiable on E_1 where $\text{Leb}(E_1) = 1$. Since g is uniformly semismooth, there is a set E of full measure such that for all $x \in X$, $g_t(x, t^*)$ exists if $t^* \in E$. Let $E_0 := E_1 \cap E$. Notice that $E_0^c = E_1^c \cup E^c$, so $\text{Leb}(E_0^c) = 0$, which implies $\text{Leb}(E_0) = 1$. By construction, Φ is differentiable on E_0 and for all $x \in X$, $g_t(x, t^*)$ exists if $t^* \in E_0$. The conclusion then follows from the first part of the theorem. □

We turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. Again, we recall some notation from the text. Let $F(\cdot, t)$ be a path that is equi-Lipschitzian and uniformly semismooth, we see that the function $\Phi(t) := V\big(F(\cdot, t)\big)$ is also Lipschitz (since it is the composition of two Lipschitz functions), hence differentiable almost everywhere, and also the integral of its derivative, i.e., Eq. (4) holds. Therefore, all that remains to be shown is that Eqs. (2) and (3) hold.

Notice that for any path $F(\cdot, t)$, we may write

$$\Phi(t) = \min\left[g(\varphi, t) : \varphi \in \mathcal{F} \right]$$

where $g(\varphi, t) = \int F(\cdot, t)\varphi(\cdot) - V^*(\varphi)$ and $\mathcal{F} \subset L^\infty[0, 1]$ is weak* compact and convex (as in Theorem 3.1 above), with $V^*(\varphi) := \sup_{F \in D[0,1]} \left[\int F\varphi - V^*(\varphi) \right]$ being the Fenchel conjugate of V.

In other words, Φ is the pointwise minimum of a family of functions, and Eq. (2) corresponds to an envelope condition and (3) corresponds to interchanging the order of differentiation and integration. If Eq. (2) can be established, Eq. (3) follows by assumption.

But the path $F(\cdot, t)$ is uniformly semismooth, so the function $g(\varphi, t) = \int F(\cdot, t)\varphi(\cdot) - V^*(\varphi)$ is uniformly semismooth in the sense of Theorem A.2 above. Therefore, by Theorem A.2, Eq. (2) holds. Moreover, by the assumption that $F(\cdot, t)$ is uniformly semismooth, Eq. (3) also holds, which completes the proof. □

We now present a proof of Theorem 4.2. Since the proofs follow from the local utility analysis of Machina (1982), we shall only provide a sketch of the main argument.

Proof of Theorem 4.2. (ii) implies (i). Suppose F^* first order stochastically dominates F. Then, the path $F(\cdot, \alpha) := F + \alpha(F^* - F)$ is uniformly Lipschitz and semismooth. By Theorem 4.1, we may use Eq. (4) so that $V(F^*) - V(F) = \int_0^1 \left[\int_0^1 v\big(x, F(x, \alpha^*)\big) \, \mathrm{d}F^*(x) - \mathrm{d}F(x) \right] \mathrm{d}\alpha^* \geqslant 0$ since $v\big(x, F(x, \alpha^*)\big)$ is nondecreasing in x for all $F(\cdot, \alpha^*)$.

(i) implies (ii). Suppose that for the cdf $\tilde{F}(\cdot)$, there are $0 \leqslant x^* < x^{**} \leqslant 1$ such that for some local utility function of $\tilde{F}$, say $v(\cdot, \tilde{F})$, we have $v(x^*, \tilde{F}) > v(x^{**}, \tilde{F})$. Let $F^*(\cdot, \alpha) := \alpha G_{x^*} + (1 - \alpha)\tilde{F}$ and $F^{**}(\cdot, \alpha) := \alpha G_{x^{**}} + (1 - \alpha)\tilde{F}$. Clearly, these paths are uniformly Lipschitz and semismooth. Then, by Eq. (3),

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \left(\int_0^1 v\big(x, F(x, \alpha^*)\big) \, \mathrm{d}F(x, \alpha) \right) \bigg|_{\alpha=0} = v(x^*, \tilde{F}) - v(x^{**}, \tilde{F}) > 0.$$

Therefore, for a small $\alpha^* > 0$, $F^{**}(\cdot, \alpha^*)$ stochastically dominates $F^*(\cdot, \alpha^*)$, yet $V\big(F^*(\cdot, \alpha^*)\big) > V\big(F^{**}(\cdot, \alpha^*)\big)$, which is a contradiction. □

Finally, we prove Theorem 4.5.

Proof of Theorem 4.5 (*(i) implies (ii) (Sketch)*)**.** As in Machina (1982), suppose for some cdf F^{**}, there are no local utility functions $v(\cdot, F^{**})$ and $v^*(\cdot, F^{**})$ such that $v(\cdot, F^{**})$ is at least as concave as $v^*(\cdot, F^{**})$. Then, we follow Machina (1982), and use our Eq. (3), thereby reaching the same contradiction as Machina, while noting that all the paths used in the proof are uniformly Lipschitz and semismooth.

(ii) implies (iii). We present this part in some detail, since Machina (1982) relies on the differentiability of the preference functional, a route we cannot directly follow. As in Machina (1982), let F^* differ from F by a simple compensated spread from the point of view of V^*. Define $\phi^+ := \max[F^* - F, 0]$, $\phi^- := \min[F^* - F, 0]$ and $F(x, \alpha, \beta) := F(x) + \alpha\phi^+(x) + \beta\phi^-(x)$.

Let $Q(\alpha, \beta) := V^*\big(F(\cdot, \alpha, \beta)\big)$. It is easily seen that Q is Lipschitz and concave. Moreover, since V^* is strictly monotone (with respect to first order stochastic dominance), $Q(\cdot, \beta)$ is strictly decreasing in β for each $\alpha \in [0, 1]$ and $Q(\alpha, \cdot)$ is strictly increasing in α for each $\beta \in [0, 1]$. Therefore, for each $\alpha \in (0, 1)$, there exists a unique β such that $Q\big(\alpha, \beta(\alpha)\big) = V^*(F)$. This implies that the subdifferential ∂Q is a matrix of full rank for each (α, β). Therefore, by a nonsmooth Implicit Function Theorem, as in Clarke (1983, Chapter 7, p 255), there exists a solution $\beta(\alpha)$ that is Lipschitz in a neighbourhood of α. By the uniqueness of the solution, we see that there is a unique function $\beta(\alpha)$ on $[0, 1]$ that is Lipschitz. The Lipschitz continuity of $\beta(\alpha)$ implies that the path $F\big(\cdot, \alpha, \beta(\alpha)\big)$ is uniformly Lipschitz and semismooth, so Theorem 4.1 applies. We follow Machina (1982) to complete the proof.

(iii) implies (i). Follow Machina (1982) without change. Under the assumptions that weak local utility functions are differentiable and that individuals are diversifiers, we can show that (ii) and (iv) are equivalent. We follow Wang (1993) for the proof. Roughly, he shows that the paths described in Machina (1982) can be approximated by smooth paths (that are consequently uniformly Lipschitz and semismooth). Continuity of the functions V and V^* then allows us to draw the required conclusions. □

Please cite this article in press as: Chatterjee, K., Vijay Krishna, R., A nonsmooth approach to nonexpected utility theory under risk. Mathematical Social Sciences (2011), doi:10.1016/j.mathsocsci.2011.08.001

Proposition A.3. *Suppose $V(\cdot;\alpha)$ and u are as above. Then, there exists a family of affine functions $v(x,t) : [0,1]^2 \to \mathbb{R}$ such that for each $\alpha \in [0,1]$ and lottery F,*

$$V(F,\alpha) = \min\left[\int \left[\alpha v(x,t) + (1-\alpha)u(x)\right] \mathrm{d}F(x) : t \in [0,1]\right]$$

Proof. Recall that $u : [0,1] \to \mathbb{R}$ is concave, Lipschitz and increasing. Then, for any $t \in [0,1]$, $u(t) =: t\left[u'(s)|_{s=t}\right] + c_t$ which implicitly defines $c_t \in \mathbb{R}$. Now define, for each $t \in [0,1]$, $v(x,t) := xu'(t) + c_t$, so that each $v(x,t)$ is an affine function of x. Let $\tilde{\mathcal{U}}(u) := \{v(x,t) : t \in [0,1]\}$ be a set of utility functions. By construction, $u(x) := \min\{v(x) : v \in \tilde{\mathcal{U}}(u)\}$ (and is concave). For any distribution F, denote its mean by $\mu_F := \int x\,\mathrm{d}F$.

Notice that for any lottery F, $\min\left\{\int v(\cdot,t)\,\mathrm{d}F : t \in [0,1]\right\} = u(\mu_F)$, since $\int v(\cdot,t)\,\mathrm{d}F = v(\mu_F,t)$ for each $t \in [0,1]$. Therefore, for any $\alpha \in [0,1]$,

$$\begin{aligned} V(F;\alpha) &= \min\left\{\int \left[\alpha v(x,t) + (1-\alpha)u(x)\right] : t \in [0,1]\right\} \\ &= \alpha u(\mu_F) + (1-\alpha)\int u\,\mathrm{d}F \end{aligned}$$

as desired. □

To see more clearly the connection with ε-contamination, let $\mathcal{U}_\alpha := \alpha\tilde{\mathcal{U}}(u) + (1-\alpha)u$, be a set of utility functions, where $\tilde{\mathcal{U}}(u)$ is defined in the proof above. Then, for any $\alpha \in [0,1]$, $V(F;\alpha) = \min[\int w\ \mathrm{d}F : w \in \mathcal{U}_\alpha]$.

Since each $v(x,t)$ is affine, it follows that reducing α is a concavification of $\alpha v(x,t) + (1-\alpha)u(x)$. Thus, a smaller α means all the local utility functions are more concave, so that a smaller α implies a greater risk aversion, as in Theorem 4.5 above.

A.3. A utility representation

We shall first obtain a utility representation in a general setting. Let X be a compact, metrisable and convex subset of a topological vector space, and let $d(\cdot,\cdot)$ be a metric that induces the relative topology. $\succsim$ is a preference relation on X. We shall assume that it is continuous. Let $\succsim$ satisfy the following axioms.

Axiom (*Homothety*)**.** There exists $y^* \in X$ such that for all $x_1, x_2 \in X$ and $\alpha \in (0,1)$, $x_1 \sim x_2$ implies $\alpha x_1 + (1-\alpha)y^* \sim \alpha x_2 + (1-\alpha)y^*$. In this case, we shall say that $\succsim$ is *homothetic with respect to* y^*.

Axiom (*Weak Convexity*)**.** For all $x \in X$, $\{y \in X : y \succsim x\}$ is convex.

Axiom (*Weak Monotonicity*)**.** For all $x \in X$, $x \neq y^*$ implies $x \succ y^*$.

We shall refer to these as the homothety and convexity properties of the preference $\succsim$. Following Rader (1972), consider the following definition of distance between subsets of X. (Note, this is not a metric.) For subsets $A, B \subset X$, define

$$\mathfrak{H}(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

If $A = \{x\}$ is a singleton, we shall write $\mathfrak{H}(x,B)$, dropping the set notation for x.

For any $x \in X$, let the indifference class of x be denoted by $[x]$, the upper contour set be denoted by $L^+_\succsim(x) := \{y \in X : y \succsim x\}$ and the lower contour set by $L^-_\succsim(x) := \{y \in X : x \succsim y\}$. We now turn to an axiom that delivers Lipschitz continuity of the utility function.

Axiom (*Uniform Sensitivity*)**.** Preference $\succsim$ is *uniformly sensitive* if there exists $t > 0$ such that for all $x, y \in X$,

$$d(x',y') \geqslant t\mathfrak{H}(x,[y])$$

for all $x' \in [x]$ and $y' \in [y]$.

Let us now define a utility function on X as follows. For all $x \in X$,

$$v(x) := \mathfrak{H}\left(L^+_\succsim(x), L^-_\succsim(y^*)\right) - \mathfrak{H}\left(L^+_\succsim(y^*), L^-_\succsim(x)\right).$$

Notice first that $v(y^*) = 0$. Notice also that if $x \succ y^*$, $L^+_\succsim(x) \cap L^-_\succsim(y^*) = \varnothing$, while $L^+_\succsim(y^*) \cap L^-_\succsim(x) \neq \varnothing$, so that $v(x) > 0$. We shall now establish some properties of v.

Theorem A.4 (*Rader*)**.** *The preference $\succsim$ is continuous and uniformly sensitive if, and only if, there exists a function v that represents $\succsim$ and satisfies a Lipschitz condition.*

Proof. The proof is in Rader (1972). We merely provide some details for completeness. Recall that $v(x) := \mathfrak{H}\left(L^+_\succsim(x), L^-_\succsim(y^*)\right) - \mathfrak{H}\left(L^+_\succsim(y^*), L^-_\succsim(x)\right)$, where one of the terms in the difference is 0. As in Rader, it may be shown that for any $x, z \in X$ with $x \succsim z$, $\mathfrak{H}\left(L^+_\succsim(x), L^-_\succsim(z)\right) = \mathfrak{H}\left([x],[z]\right)$. Therefore, for all $x, z \in X$, we have

$$|v(x) - v(z)| = \left|\mathfrak{H}\left([x],[y^*]\right) - \mathfrak{H}\left([z],[y^*]\right)\right|.$$

Fact (i). For any $\varepsilon > 0$, there exist $x' \in [x]$ and $z' \in [z]$ such that

$$|v(x) - v(z)| \leqslant \mathfrak{H}\left(x',[y^*]\right) - \mathfrak{H}\left(z',[y^*]\right) + \varepsilon.$$

Fact (ii). We can choose $y' \in [y^*]$ such that

$$\mathfrak{H}\left(z',[y^*]\right) + \varepsilon \geqslant d(z',y').$$

Fact (iii). Moreover, $\mathfrak{H}\left(x',[y^*]\right) \leqslant d(x',y')$ (by definition of $\mathfrak{H}$). Hence,

$$\begin{aligned} |v(x) - v(z)| &\leqslant \mathfrak{H}\left(x',[y^*]\right) - \mathfrak{H}\left(z',[y^*]\right) + \varepsilon \quad \text{(Fact (i))} \\ &\leqslant d(x',y') - d(z',y') + 2\varepsilon \quad \text{(Facts (ii) and (iii))} \\ &\leqslant d(x',z') + 2\varepsilon \quad \text{(triangle inequality)} \\ &\leqslant \frac{1}{t}\mathfrak{H}\left([x'],[z']\right) \quad \text{(uniform sensitivity)} \\ &\leqslant \frac{1}{t}d(x,z) \end{aligned}$$

since $\mathfrak{H}\left([x'],[z']\right) = \mathfrak{H}\left([x],[z]\right)$ (by definition of $\mathfrak{H}$) and $d(x,z) \geqslant t\mathfrak{H}\left([x],[z]\right)$ by uniform sensitivity. Therefore, v is Lipschitz continuous. Now to show that v represents $\succsim$.

By definition of v, $x \succsim z$ implies $v(x) \geqslant v(z)$. We now prove the converse. Suppose $v(x) \geqslant v(x')$. There are two possible cases of x' with respect to y^*. The first case is where $x' \succ y^*$, from which it follows that $v(x') > 0$. In this case, $x \succ y$, since otherwise would imply that $v(x) \leqslant 0$, contradicting the hypothesis that $v(x) \geqslant v(x')$. We claim that $x \succsim x'$. We shall prove the contrapositive.

If $x' \succ x$, $\mathfrak{H}\left([x],[x']\right) =: \varepsilon > 0$ by uniform sensitivity. Consider $\bar{x}' \in [x']$ and $\bar{y} \in [y^*]$. Since $x' \succ x$, there exists $\alpha \in (0,1)$ such that $\alpha\bar{x}' + (1-\alpha)\bar{y} =: x_0 \sim x$. Therefore,

$$d(\bar{x}',\bar{y}) = d(\bar{x}',x_0) + d(x_0,\bar{y}) \geqslant d(x_0,\bar{y}) + \varepsilon$$

from which it follows that $\mathfrak{H}\left([x'],[y^*]\right) \geqslant \mathfrak{H}\left([x],[y^*]\right) + \varepsilon$. This implies $v(x') > v(x)$, as desired. The case where $y^* \succ x'$ is treated similarly. □

Notice that the proof above did not require that X be compact. We shall now show that homothety will make v suitably affine and weak convexity will make v concave (and not just quasiconcave).

We shall say that a function $v : X \to \mathbb{R}$ is *homothetic with respect to* y^* (or simply homothetic) if $v\left((1-\lambda)x + \lambda y^*\right) = (1-\lambda)v(x) + \lambda v(y^*)$ for all $\lambda \in (0,1)$.

Proposition A.5. *Let X be compact and v defined as above. Then, v is homothetic with respect to y^* if and only if $\succsim$ is homothetic with respect to y^*.*

Proof (*Sufficiency*). Let $x \in X$ and let us suppose $x \succsim y^*$. Then, $v(x) = \mathfrak{H}([x], [y^*])$. Moreover, since X is compact, there exist $\bar{x} \in [x]$ and $\bar{y} \in [y^*]$ such that $d(\bar{x}, \bar{y}) = \mathfrak{H}([x], [y^*])$, which implies $v(x) = d(\bar{x}, \bar{y})$.

Now fix $\alpha \in (0, 1)$. By homothety of $\succsim$ it follows that $[(1-\alpha)x + \alpha y^*] = (1-\alpha)[x] + \alpha y^*$ and that $[y^*]$ is convex. Moreover, $d((1-\alpha)\bar{x} + \lambda y^*, (1-\alpha)\bar{y} + \alpha y^*) = (1-\alpha)d(\bar{x}, \bar{y})$, since d is induced by a norm.

Then, $\mathfrak{H}([(1-\alpha)x + \alpha y^*], [y^*]) \leqslant d((1-\alpha)\bar{x} + \lambda y^*, (1-\alpha)\bar{y} + \alpha y^*) = (1-\alpha)d(\bar{x}, \bar{y})$, where the inequality follows from the definition of $\mathfrak{H}$. Again, since X is compact, there exist $\tilde{x}_\alpha, \tilde{y}_\alpha \in X$ such that

$$d(\tilde{x}_\alpha, \tilde{y}_\alpha) = \mathfrak{H}([(1-\alpha)x + \alpha y^*], [y^*]).$$

Let $\tilde{x}, \tilde{y} \in X$ be such that $\tilde{x}_\alpha = (1-\alpha)\tilde{x} + \alpha y^*$ and $\tilde{y}_\alpha = (1-\alpha)\tilde{y} + \alpha y^*$. But $d(\tilde{x}_\alpha, \tilde{y}_\alpha) = (1-\alpha)d(\tilde{x}, \tilde{y}) \geqslant (1-\alpha)d(\bar{x}, \bar{y})$. Therefore,

$$\mathfrak{H}([(1-\alpha)x + \alpha y^*], [y^*]) = (1-\alpha)d(\bar{x}, \bar{y}).$$

But this implies $v((1-\alpha)x + \alpha y^*) = (1-\alpha)v(x) + \alpha v(y^*)$ since $v(y^*) = 0$. This proves that v is homothetic. The case where $y^* \succsim x$ is handled similarly. Necessity is straightforward. □

In what follows, say that v has a worst element if there exists a unique minimiser of v on X. We can collect all the observations above in the following proposition.

Proposition A.6. *Let X be compact, and $\succsim$ a preference relation on X. The following are equivalent.*

(i) *$\succsim$ satisfies Homothety, Weak Convexity, Weak Monotonicity and Uniform Sensitivity.*
(ii) *There exists a function $v : X \to \mathbb{R}$ that represents $\succsim$ such that v is homothetic with respect to y^*, concave, Lipschitz and has a unique worst element.*

Proof. It is easy to see that (b) implies (a). To see that (a) implies (b), notice that by Proposition A.5, there exists a utility representation v that is Lipschitz and homothetic with respect to y^*. By Weak Convexity, v is quasiconcave. Therefore, all that remains to be shown is that v is concave and has y^* as its worst element.

Translate X by $-y^*$, so that we now consider the set $X - y^*$. Letting K be the cone generated by X, we can extend v by positive homogeneity to K. Abusing notation, this extension will also be denoted v. By Weak Monotonicity, it is the case that $0 = v(\theta) < v(x)$ for all $x \in K \setminus \theta$. Theorem 2 of Korablev (1998) says that v is, in fact, concave. In particular, this means that v restricted to X is also concave. □

We now relate the propositions above to the objects in our model. A functional $V : D[0, 1] \to \mathbb{R}$ is said to be *G_0-affine* if $V(\alpha F + (1-\alpha)G_0) = \alpha V(F) + (1-\alpha)V(G_0)$ for all $F \in D[0, 1]$, $\alpha \in [0, 1]$.

Proposition A.7. *A preference $\succsim$ on $D[0, 1]$ satisfies Continuity, Uniform Sensitivity and Z-Independence if, and only if, there exists a function $V : D[0, 1] \to \mathbb{R}$ that is Lipschitz continuous and is G_0-affine. Moreover, V is unique up to positive affine transformation.*

Proof. In terms of the axioms introduced in this section of the Appendix, let $X = D[0, 1]$ and $y^* = G_0$. Then, by Proposition A.5 above, there exists a utility representation that is Lipschitz continuous and G_0-affine. The fact that the utility representation is unique up to positive affine transformation follows from arguments in Maccheroni (2002). □

Proof of Theorem 5.1. Let V be the preference functional obtained above. As before, take $X = D[0, 1]$ and $G_0 = y^*$. Then, the desired conclusion follows from Proposition A.6 above. □

A.4. Differentiability

The usefulness of differentiability hypotheses on the preference functional $V : D[0, 1] \to \mathbb{R}$ was first noticed by Machina (1982). However, the notion of Fréchet differentiability used by Machina is too strong to accommodate Rank Dependent Expected Utility, as observed by Chew et al. (1987), who instead use the weaker notion of Gâteaux differentiability of the preference functional, and demonstrate that local utility analysis is nonetheless possible. An extensive and extremely thorough discussion of Gâteaux and Fréchet differentiable preference functionals in L^p spaces, and their relation to various models, can be found in Wang (1993). The notion of Hadamard derivative is used by Chew and Nishimura (1992), again for the purposes of local utility analysis.

To understand the various notions of differentiability, let X be a Banach space, X^* the dual space of X, and $\Phi : X \to \mathbb{R}$ a mapping. The function Φ is *Gâteaux* differentiable at x if there is an element $x^* \in X^*$ – also denoted $D\Phi(x)$, the *Gâteaux* derivative at x – such that for all $v \in X$,

$$\lim_{t \downarrow 0} \frac{\Phi(x + tv) - \Phi(x)}{t} = \langle v, D\Phi(x) \rangle.$$

Notice that the convergence is uniform with respect to v in finite sets (by definition). The requirement that convergence be uniform with respect to v in compact sets defines the *Hadamard* derivative; on bounded sets defines the *Fréchet* derivative. (There is another notion of a differential due to Bourbaki, known as the *strict derivative*, which is closely related to Clarke's notion of generalised subgradient—see Clarke, 1983 for details.) In finite dimensional spaces, Hadamard and Fréchet differentiability are equivalent (as a consequence of the Heine–Borel theorem), and strictly stronger than Gâteaux differentiability. In infinite dimensional spaces, the three notions are distinct, and progressively more stringent.

Our assumption that the preference functional is concave and Lipschitz ensures that the subdifferential exists at each point on the domain. In general, if the (concave or convex) preference functional is Gâteaux, Hadamard or Fréchet differentiable at some point, the derivative corresponds to the subgradient.

One of the difficulties with the various differentiability hypotheses is that they require information about the values of the preference functional outside the domain $D[0, 1]$. Indeed, in the case of Fréchet differentiability, for each point in the domain, the Fréchet derivative provides a linear approximation of the functional in an open neighbourhood of the domain. But $D[0, 1]$ does not have an interior point, and it is not clear how the values of the functional are to be ascertained outside the domain. A way to interpret the envelope representation (1) is as follows: For any preference functional $V : D[0, 1] \to \mathbb{R}$ that is concave and Lipschitz, there is a unique greatest proper concave (and hence Lipschitz) function that extends V to the entire domain. Put differently, in our setting we assume that the preference functional is concave and Lipschitz, and then show that it can be extended to the entire space. Fréchet differentiability is tantamount to the assumption that the preference functional can be extended uniquely to an open neighbourhood of the domain.

A.5. Lipschitz continuity

The assumption of Lipschitz continuity of the preferences (and the corresponding preference functional) serves two purposes: the *intrinsic* and the *extrinsic*. Rader (1972, p 171) uses it for the intrinsic purpose, which is to ensure that difference in utility levels at any two points is the integral of the marginal utility along Lipschitzian paths connecting these two points. Of course, the paths that are relevant for us are defined implicitly by the preference functional. Here too, we use the Lipschitz continuity

Please cite this article in press as: Chatterjee, K., Vijay Krishna, R., A nonsmooth approach to nonexpected utility theory under risk. Mathematical Social Sciences (2011), doi:10.1016/j.mathsocsci.2011.08.001

of the preference functional and a nonsmooth Implicit Function Theorem to show that the induced paths over which we integrate are, in fact, Lipschitzian.

The extrinsic purpose is to ensure that the preference functional does not increase too quickly which, in turn, ensures that if the preference functional is concave, for instance, a subgradient exists. This is closely related to the cone condition of Mas-Colell (1986).

We should mention that Mas-Colell (1986) works with preferences on a Banach lattice, and his condition is named *uniform properness*. It is informally referred to as a cone condition since it implies that existence of a cone with a nonempty interior that can be suitably translated. This nomenclature will be clarified next, where we will look at the analogue of the cone condition for preference functionals.

Suppose E is a (locally convex) topological vector space, $C \subset E$ is convex and $\varphi : C \to \mathbb{R}$ is concave. Then, φ is subdifferentiable at x if, and only if, there exists a seminorm $p : E \to \mathbb{R}$ such that $\varphi(y) - \varphi(x) \leqslant p(y-x)$. The analogue of Mas-Colell's cone (or uniform properness) condition is to assume that such a seminorm p exists, and is the same for each $x \in C$.

In particular, let X be a normed space, $C \subset X$ be convex and $\varphi : C \to \mathbb{R}$ be concave. Then, φ is subdifferentiable at x if, and only if, there exists $M > 0$ such that $\varphi(y) - \varphi(x) \leqslant M \|y - x\|$. Lipschitz continuity ensures that such an M exists and is the same for each $x \in C$. It is clear that $M \|\cdot\|$ is a seminorm, which establishes the connection to the general case.

For each $x \in C$, let $H(x) := \{(y, t) \in X \oplus \mathbb{R} : f(x) + p(x-y) < t\}$. It is clear that $H(x)$ and $\text{hypo}(f) := \{(x, t) \in X \oplus \mathbb{R} : t \leqslant f(x)\}$ are disjoint. Moreover, $H(x)$ is an open convex cone. Therefore, there is a (closed) hyperplane that separates the two sets, which allows us to claim that there is a hyperplane that supports $\text{hypo}(f)$ at x. The *cone condition* says that the same set H works (to provide a separating hyperplane) for each $x \in C$, which is equivalent to the assumption above that there exists a seminorm p such that $\varphi(y) - \varphi(x) \leqslant p(y - x)$ for all $x, y \in C$.

References

Aliprantis, C.D., Border, K.C., 1999. Infinite Dimensional Analysis: A Hitchhiker's Guide, 2nd ed. Springer, New York.

Cerreia-Vioglio, S., 2009. Maxmin expected utility on a subjective state space: convex preferences under risk. Technical Report. Università Bocconi.

Chateauneuf, A., Cohen, M., 1994. Risk seeking with diminishing marginal utility in a non-expected utility model. Journal of Risk and Uncertainty 9 (1), 77–91.

Chew, S.H., Karni, E., Safra, Z., 1987. Risk aversion in the theory of expected utility with rank dependent probabilities. Journal of Economic Theory 42, 370–381.

Chew, S.H., Nishimura, N., 1992. Differentiability, comparative statics and non-expected utility preferences. Journal of Economic Theory 52, 294–312.

Clarke, F.H., 1983. Optimization and Nonsmooth Analysis. John Wiley and Sons, New York.

Correa, R., Jofre, A., 1989. Tangentially continuous directional derivatives in nonsmooth analysis. Journal of Optimization Theory and Applications 61 (1), 1–21.

Ergin, H., Sarver, T., 2010a. A unique costly contemplation representation. Econometrica 78 (4), 1285–1339.

Ergin, H., Sarver, T., 2010b. The unique minimal dual representation of a convex function. Journal of Mathematical Analysis and Applications 370 (2), 600–606.

Gale, D., 1967. A geometric duality theorem with economic applications. Review of Economic Studies 34 (1), 19–24.

Gul, F., 1991. A theory of disappointment aversion. Econometrica 59 (3), 667–686.

Keeney, R.L., Raiffa, H., 1993. Decisions with Multiple Objectives: Preferences and Value Trade-Offs. Cambridge University Press.

Korablev, A.I., 1998. Directional derivatives of quasiconvex functions. Journal of Mathematical Sciences 41 (6), 1425–1428. Translated from: Issledovaniya po Prikladnoi Matematike, No. 3, 1975, pp. 92–96.

Krishna, V., Maenner, E., 2001. Convex potentials with an application to mechanism design. Econometrica 69 (4), 1113–1119.

Maccheroni, F., 2002. Maxmin under risk. Economic Theory 19, 823–831.

Machina, M.J., 1982. Expected utility analysis without the independence axiom. Econometrica 50 (2), 277–323.

Machina, M.J., 1984. Temporal risk and the nature of induced preferences. Journal of Economic Theory 33, 199–231.

Mas-Colell, A., 1977. The recoverability of consumers' preferences from market demand behavior. Econometrica 45 (6), 1409–1430.

Mas-Colell, A., 1986. The price equilibrium existence problem in topological vector lattices. Econometrica 54 (5), 1039–1054.

Milgrom, P., Segal, I., 2002. Envelope theorems for arbitrary choice sets. Econometrica 70 (2), 583–601.

Ok, E.A., 2007. Real Analysis with Economic Applications. Princeton University Press, Princeton, NJ.

Quiggin, J., 1982. A theory of anticipated utility. Journal of Economic Behavior and Organization 3, 225–243.

Rader, T., 1972. Theory of Microeconomics. Academic Press, New York.

Safra, Z., Segal, U., 1998. Constant risk aversion. Journal of Economic Theory 83 (1), 19–42.

Schlee, E.E., 2001. The value of information in efficient risk-sharing arrangements. American Economic Review 91 (3), 509–524.

Wang, T., 1993. L^p-Fréchet differentiable preference and 'local utility' analysis. Journal of Economic Theory 61, 139–159.

Wakker, P., 1993. Separating marginal utility and probabilistic risk aversion. Theory and Decision 36 (1), 1–44.

Yaari, M., 1987. The dual theory of choice under risk. Econometrica 55 (1), 95–115.

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