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Uniquely Representing "A Preference for Uniformity"

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Uniquely Representing "A Preference for Uniformity"*

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Abstract

In a model of decision making over sets of alternatives, we consider an agent who conceives of the different utilities she will receive (depending on the state of mind she is in, her *subjective state*) when she finally makes a choice from the set. Her uncertainty is described by multiple beliefs (or measures) over the state space. She is pessimistic about the true beliefs, and would like to hedge her bets. This results in her having a preference for uniformity of payoffs across the various subjective states. We present a utility representation for such agents and show that our representation provides a natural measure of the agent's desire for uniformity of payoffs in the subjective states. We also show that concerns for uniformity are orthogonal to concerns about flexibility or commitment. We achieve this by relaxing the Independence axiom as applied to our environment. However, this weakening of Independence means that in settings with temptation, we can no longer ensure the existence of a unique normative ranking.

KEYWORDS: preference for uniformity, hedging, subjective states

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1. Motivation

An agent has to choose between two sets of alternatives in the morning, but is unsure about how she will feel in the afternoon. Consider, for instance, the agent who has to choose a menu (say, from a restaurant) for lunch. There are only three menus (from three restaurants) available. Suppose the first restaurant offers only chicken (c) and the second only fish (f) and that she is indifferent between the two restaurants. Also suppose the third restaurant offers either chicken or fish daily, but never both on the same day (so that each is offered with positive probability), and suppose the agent is unaware of the dish being served on the day in question. We are interested in agents who strictly prefer the third restaurant to the first two.

An intuitive explanation of such preferences is the following: The agent considers possible two psychological states in the afternoon, which will govern her choice of restaurant, and believes that her taste in the afternoon will be drawn from one of a set of beliefs. Each psychological state corresponds to a different preference over lunch items. In the first state she strictly prefers chicken and in the second she strictly prefers fish. In the morning, when she has to make a choice of restaurant, she is unsure what state of mind she will be in at lunchtime. A desire for uniformity across psychological states means that she will hedge her bets about how she will feel at lunch time and therefore choose the third restaurant. We shall say that the agent's preferences correspond to her having a *preference for uniformity* in terms of payoffs she receives at lunch time in the restaurant.

In this paper, we provide an axiomatic foundation and a utility representation for such preferences by relaxing the Independence axiom. Our main axiom, *Quasiconcavity*, says that for menus A and B where $A \sim B$, it is the case that $\frac{1}{2}A + \frac{1}{2}B \succcurlyeq A$, thereby reflecting the agent's desire for uniformity in payoffs across subjective states. Her concern for uniformity in ex post payoffs is orthogonal to any concerns she might have about flexibility or commitment. For instance, suppose she is indifferent between two menus A and B. A preference where $A \cup B \succcurlyeq A \sim B$ indicates a desire for flexibility. On the other hand, $A \sim B \succcurlyeq A \cup B$ represents a preference for commitment on her part, due to self-control issues she might have. Neither flexibility nor commitment tells us how she ranks $\frac{1}{2}A + \frac{1}{2}B$ relative to A and B. A preference for uniformity, corresponds to $\frac{1}{2}A + \frac{1}{2}B \succcurlyeq A$, with her ranking of $A \cup B$ not being relevant to her desire for uniformity. Thus, a preference for flexibility/commitment and a preference for uniformity address distinct concerns of the agent.

As Dekel, Lipman and Rustichini (2001) state (p 897), on models based on the subjective states of a decision maker, "... applications of the model seem to require some measure of the agent's aversion to uncertainty regarding future contingencies, which would presumably be based on the size of [the subjective states considered possible], and loosely, on the variance of [payoffs] across states." Our axiom *Quasiconcavity* captures such concerns.

There is yet another way to interpret the agent's hedging motives described above. Epstein, Marinacci and Seo (2007) interpret such hedging as evidence that the agent only has a coarse perception about how she will feel in the afternoon, ie, she has a coarse perception about her future psychological states of mind. We discuss this paper more fully in section 4.

Since we allow for choices of lotteries, we focus on ex post choice guided by a vN-M function. The set of all possible ex post (afternoon) vN-M utility functions will be indexed by S_Z, so that a subjective, psychological state is any $s \in S_Z$, and the utility in that state is given by the vN-M function $u(\cdot, s) : \Delta \to \mathbb{R}$. (As is standard, we identify vN-M utility functions up to positive affine transformations.)

Our representation takes the form

$$[\spadesuit] \qquad V(A) := \min \left[\int_{S_Z} \max_{p \in A} u(p; s) \, d\pi(s) : \pi \in \Pi \right]$$

where Π is a weak* compact, convex set of (finite, Borel) signed measures on S_Z. (Clearly, this means that S_Z has a topology — we shall present this in section 2.) We shall refer to this as a *Preference for Uniformity (PfU) representation*. The intuition underlying the representation is simple. Faced with a menu, the agent considers some states likely to occur in the afternoon. She knows the utility she will get in each state and the measure π weighs the various ex post utilities, aggregating them into a morning (ex ante) value. The representation captures the hedging motivation ascribed to the agent, since the agent evaluates the menu A according to the lower envelope of a set of linear functionals (as each π is a linear functional on the set of ex post utilities).

1.1. *Nature of our Contributions*

Our main contribution, Theorem 2.2, is to provide axioms on preferences that completely characterise PfU representations of the form ($\spadesuit$). At a technical level, our representation significantly weakens the Independence axiom, while still delivering a tractable representation. The PfU representation (♠) is sufficiently weak that we do not require the utility functional to be linear over singletons.

The set of signed measures Π that we obtain is unique, facilitating a comparison of agents with bigger Π's (in terms of set inclusion). This leads to Theorem 2.4, where we show how a preference for uniformity can be quantified. The set of weights (or, more generally, signed measures on S_Z) given by Π provides a natural measure of the agent's desire for uniformity in ex post payoffs, in the sense that a larger Π (in terms of set inclusion) means the agent has a greater desire for uniformity. An advantage of our representation is that we do not require preferences to be monotone. Thus, we can provide a unified analysis of problems where there is both a preference for flexibility and a preference for commitment.

We do not require our PfU representation to be linear over singletons. Instead, the utility of a (singleton) lottery is given as the minimum of a collection of vN-M utility functions over the space of lotteries. Clearly, this precludes the idea of a single, coherent normative preference represented by a vN-M utility function that represents the ideal choice of the agent in the absence of temptation.

We now proceed as follows. We introduce the model and the main representation theorem in section 2. We discuss notions of comparative preference for uniformity in section 2.4. In section 3, we consider some additional axioms considered in the literature on temptation and study the more refined representations that result. An overview of the literature is provided in section 4. Proofs are deferred to section 5.

2. Model and Results

2.1. *Environment and Notation*

Let Z denote a finite set of prizes where the cardinality of Z is $n+1$. We shall require that $n > 1$. The space of probability measures on Z is denoted by Δ and $\mathcal{F}(\Delta)$ denotes the set of nonempty closed subsets of Δ. When endowed with the Hausdorff metric, $\mathcal{F}(\Delta)$ becomes a compact metric space. A *menu* is a nonempty and closed subset of Δ. Preferences over menus are complete, transitive, binary relations over $\mathcal{F}(\Delta)$, represented by $\succcurlyeq \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$. We shall denote the set of all compact, convex subsets of Δ by $\mathcal{K}(\Delta)$.

The affine hull of Δ is $\operatorname{aff}(\Delta)$, $p^* = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ is the uniform probability measure over the prizes, and for any $q \in \Delta$, $D_r(q) := \{p \in \operatorname{aff}(\Delta) :$

$\|q-p\| \leqslant r\}$ is the closed disk of radius r centered at q.

2.2. *Axioms*

We now proceed to the axioms that lie at the heart of our model. Our theory shall require the existence of two distinguished sets; a singleton q^* with full support (so that $q^* \in \mathrm{ri}\,\Delta$, where "ri" refers to the relative interior), and a closed disk $D_r(q^*) \subset \Delta$.

Axiom (Non-triviality*)**.** There exist $D, \{q^*\} \subset \Delta$ such that $q^* \in \mathrm{ri}\, D$, D is a closed disk centered at q^*, and $D \not\sim \{q^*\}$.[1]

This axiom says that preferences are non-trivial in a particular way. We note that Non-triviality* encompasses two assumptions: the first is the existence of two sets that are valued differently, and the second is that they are nested in terms of set inclusion. We can relax this second assumption, but at the cost of a weaker representation theorem. Hence this assumption has behavioural implications. The next axiom is the standard continuity axiom.

Axiom (Continuity)**.** $\succcurlyeq$ is continuous in the Hausdorff topology.

Axiom (L Continuity)**.** There exists $N > 0$ such that for all $\varepsilon \in (0, 1/N)$, for all A, B with $d_h(A,B) \leqslant \varepsilon$,

$$(1-N\varepsilon)A + N\varepsilon A^* \succcurlyeq (1-N\varepsilon)B + N\varepsilon A_*$$

where $A^*, A_* \in \big\{D, \{q^*\}\big\}$ and $A^* \succ A_*$.

We need the axiom L Continuity above, because Continuity by itself is not sufficient to give us the desired representation (even in the presence of other axioms). This axiom was first introduced in Dekel et al (2007), although our version is stronger in the sense that Dekel et al (2007) allow A^* and A_* to be arbitrary compact subsets of Δ. As in that paper, assuming $D \succ \{q^*\}$, one way to think about L Continuity is as follows: For each A and B with $B \succ A$, there exists a greatest $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D \succcurlyeq \lambda B + (1-\lambda)\{q^*\}$.

(1) A natural conjecture is that the continuity assumption below will render the assumption that D has a smooth boundary dispensable. While this may be the case, it is not something we are able to prove. Indeed, our key technical lemma which also guarantees us uniqueness of the representation, fails for D that is without a smooth boundary. Therefore, at this point, the dispensability of the smoothness assumption remains an open question.

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As $d_h(A, B) \to 0$, $\lambda \to 0$. L Continuity says that $\lambda \to 0$ smoothly. We now introduce an axiom that is a very weak version of Independence.[2]

Axiom (IR: Indifference to Randomisations[3])**.** $A \sim \text{conv}(A)\ \forall A \in \mathcal{F}(\Delta)$.

In line with our interpretation of a menu as representing a set of alternatives from which a choice will be made in the future, we provide a justification of this axiom. Suppose the agent knows that in the afternoon, she will make a choice according to one of a given set of vN-M utility functions (which, taken together, represent her subjective state space), then the (ex post) utility in each state remains unchanged if we replace a menu by its convex hull. Thus, the agent is indifferent between a menu and its convex hull.

We now introduce the behavioural axiom that captures a preference for uniformity. To reiterate, the intuition behind our axiom is that the agent prefers greater uniformity in the ex post payoffs he receives. In other words, she is averse to variation in ex post payoffs.

Axiom (Quasiconcavity)**.** $A \sim B$ implies $(1/2)A + (1/2)B \succcurlyeq B$.

Our final axiom is a weakening of the traditional Independence axiom, and is similar to the C-Independence axiom of Gilboa and Schmeidler (1989). Suppose there are two menus A^* and A_* that the agent considers to be *sure bets* in the sense that they give the agent the same utility in each subjective state. Then, for any other menus $A \succ B$, it is reasonable to assume that the agent still has $\frac{1}{2}A + \frac{1}{2}C \succ \frac{1}{2}B + \frac{1}{2}C$, where $C \in \{A_*, A^*\}$, since the mixture $\frac{1}{2}A + \frac{1}{2}D$ is easily visualised. In order to obtain a PfU representation, we require that $A_* = \{q^*\}$, and $A^* = D$. In other words, in addition to requiring A_* and A^* to be sure bets, we also require them to be nested, and insist that the latter have a smooth boundary. These last two requirements are technical and allow us to obtain a concave representation without insisting that preferences are linear over singletons.

Axiom (D-Independence)**.** For all $A, B \in \mathcal{K}(\Delta)$ and all $\lambda \in (0, 1]$, $A \succ B$ implies $\lambda A + (1 - \lambda)C \succ \lambda B + (1 - \lambda)C$ for all $C \in \big\{\{q^*\}, D\big\}$.

(2) This axiom was introduced by Dekel, Lipman and Rustichini (2001) (henceforth DLR). Dekel et al (2007) show that for any preference, Independence implies IR.

(3) We should mention that IR rules out the following example, (due to Dekel, Lipman and Rustichini, 2008). Suppose the agent has to make a choice between a healthy dish and an unhealthy dish. To assuage feelings of guilt, she may prefer to add a randomisation of the two dishes to the menu. This is clearly a violation of IR.

(The "D" in D-Independence stands for "disk".) The axiom is very mild in settings where there is a preference for flexibility. However, in settings where the agent likes self control, as in section 3.3, the axiom has its limitations, since it implies that if $\succcurlyeq$ also satisfies Set Betweenness (see below), a normative preference can be identified only if the preference also satisfies Independence.

The axiom allows us to construct the set of (ex post) states S_Z so that each state has the same (maximum) utility in D and $\{q^*\}$. The natural parametrisation of such states is

$$S_Z := \left\{ s \in \mathbb{R}^{|Z|} : \sum_i s_i = 0 \ \text{ and } \ \|s\|_2 = 1 \right\}$$

Thus, the set of non-trivial vN-M utility functions on Δ is indexed by S_Z, with generic element being denoted as $u(\cdot, s)$. Ex post utility in state s is given by $u(p, s) := \langle p - q^*, s \rangle$, so that (i) $u(q^*, s) = 0$, and (ii) $\max_{p \in D} u(p, s) = R$, where R is the radius of D.

2.3. *Representations*

We begin with a simple utility representation of preferences satisfying the above axioms. First, a definition. Say that a function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ is *D-linear* if, for all $\alpha \in [0, 1]$ and $A \in \mathcal{F}(\Delta)$, $V\big(\alpha A + (1 - \alpha)C\big) = \alpha V(A) + (1 - \alpha)V(C)$ for $C \in \big\{\{q^*\}, D\big\}$.

Proposition 2.1. (a) A preference $\succcurlyeq$ satisfies Non-triviality*, Continuity, IR, and D-Independence if and only if there exists a D-linear function V that represents it and that is unique up to positive affine transformation.

(b) A D-linear utility function that represents $\succcurlyeq$ is Lipschitz continuous if and only if $\succcurlyeq$ satisfies L Continuity.

(c) Moreover, a D-linear utility function V that represents $\succcurlyeq$ is concave if and only if $\succcurlyeq$ satisfies Quasiconcavity.

Proposition 2.1 (parts (a) and (c)) provide the algebraic structure that is required to ensure the existence of a concave representation, while part (b) of the proposition ensures that the representation is Lipschitz continuous, which in turn delivers an envelope representation.

In what follows, we shall normalise V so that $V\big(\{q^*\}\big) = 0$ and $|V(D)| = R > 0$. From section 2.2, we know that there exists a collection of vN-M utility

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functions parametrised by the set S_Z, so that $u(q^*, s) = 0$ and $\max_{p \in D} u(p, s) = R$ for each $s \in S_Z$. As we shall see below, the normalisations of V that we have chosen and those of S_Z are intimately related.

For any menu A, let $u_A(s) := \max_{p \in A} u(p, s)$. By the construction of S_Z, we see that for each A, the function $u_A : S_Z \to \mathbb{R}$ is a continuous function. Moreover, if π is a signed measure on S_Z, we define $\langle u_A, \pi \rangle := \int_{S_Z} u_A(s)\, d\pi(s)$.

A *Preference for Uniformity (PfU) representation* is a function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ of the form

$$[\spadesuit] \qquad V(A) := \min \big[\langle u_A, \pi \rangle : \pi \in \Pi \big]$$

where Π is a weak* compact, convex set of signed measures on S_Z such that for all $\pi, \mu \in \Pi$, $\pi(S_Z) = \mu(S_Z)$. We will assume, without loss of generality, that $|\mu(S_Z)| = 1$ for all $\mu \in \Pi$. It is easily seen that a PfU representation is D-linear. We shall say that a PfU representation is unique if the set of measures Π is unique.

Theorem 2.2. *A function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ is D-linear, concave and Lipschitz continuous if and only if it has a unique ($\spadesuit$) PfU representation.*

The proof of the theorem is in section 5. We present here a brief sketch of the main ideas.

Sketch of Proof. The embedding of $\mathcal{F}(K)$ in $C(S_Z)$ is standard. The first step of the proof involves constructing a D-linear representation that is concave and Lipschitz continuous.

We now use the important observation that for any $f \in C^2(S_Z)$, the space of twice continuously differentiable functions on S_Z, there exist $\alpha, \beta > 0$ such that $\beta\big[f + \alpha \mathbf{1}\big]$ is the support function of some convex subset of $\Delta(Z)$. Using D-linearity and the fact that $C^2(S_Z)$ is norm ($\|\cdot\|_\infty$) dense in $C(S_Z)$ allows us to extend the preference functional uniquely to $C(S_Z)$. This extension is superlinear, and is Lipschitz continuous.

But superlinear functions that satisfy our additivity conditions have the required characterisation, as the minimum of a family of linear functionals. The Riesz Representation Theorem, in turn, tells us that a continuous linear functional on $C(S_Z)$ can be written as the integral with respect to a unique signed Borel measure on S_Z, and the representation follows immediately. $\square$

The special case where Π is a singleton is referred to as an *additive EU representation*. Such a representation is characterised by Continuity, L

Continuity and Independence, where Independence says that $A \succ B$ implies $\lambda A + (1-\lambda)C \succ \lambda B + (1-\lambda)C$ for all sets C (making clear how we have weakened Independence).

2.4. *Comparative Preference for Uniformity*

Our representation allows us to compare degrees of preference for uniformity. Before we proceed to the formalities, we recall an important theorem from DLR. Given preferences $\succcurlyeq_1$ and $\succcurlyeq_2$, say that agent 2 is *more uncertain than* agent 1 if $A \cup B \nsim_1 A$ implies $A \cup B \nsim_2 A$. Intuitively, agent 2 consider more ex post states possible than agent 1. This gives us the following characterisation.

Theorem 2.3 (Theorem 2, DLR)**.** *If $\succcurlyeq_2$ is more uncertain than $\succcurlyeq_1$, then $S_1 \subset S_2$.*

As before, the menus D and $\{q^*\}$ play a special rôle in our model, ie, $\{q^*\}$ represents the menu which gives the constant payoff of 0 in every subjective state and the menu D gives the constant payoff of R in every subjective state. Analogous, to the notion of the certainty equivalent of a gamble, we may define the *subjective certainty equivalent* of a menu A, and assuming $D \succ A \succ \{q^*\}$, as the (unique) menu $\lambda_A D + (1-\lambda_A)\{q^*\}$ such that $\lambda_A D + (1-\lambda_A)\{q^*\} \sim A$. Thus, the subjective certainty equivalent of a menu A is the menu $\lambda_A D_R + (1-\lambda_A)\{p^*\}$, that gives utility $\lambda_A R$ in every state and is indifferent to A.

This gives us a natural way to operationalize the notion of greater preference for uniformity. We say that $\succcurlyeq_2$ has a *greater desire for uniformity* than $\succcurlyeq_1$ if for each menu A, $\lambda D+(1-\lambda)\{q^*\} \sim_1 A$ implies $\lambda D+(1-\lambda)\{q^*\} \succcurlyeq_2 A$. Intuitively, an agent with a greater desire for uniformity has a smaller subjective certainty equivalent (in terms of set inclusion), leading us to the following characterisation.

Theorem 2.4. *Suppose $\succcurlyeq_1$ and $\succcurlyeq_2$ both admit PfU (♠) representations. Then, $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if and only if $\Pi_1 \subset \Pi_2$.*

3. Special Cases

Here we shall consider some additional axioms on preferences. These additional axioms have been useful in characterising certain kinds of behaviour.

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3.1. *Flexibility*

Intuitively, an agent with monotone preferences values flexibility, as in Kreps (1979). The following axiom formalises this property.

Definition 3.1. A preference $\succcurlyeq$ is *monotonic* if $A \supset B$ implies $A \succcurlyeq B$.

Monotonic preferences have an easy characterisation.

Proposition 3.2. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a PfU representation (♠). Then, V is monotonic if and only if each $\pi \in \Pi$ is a probability measure.

DLR show that it is possible to attach a sign to a state. A state $s \in S_Z$ is said to be *positive* if for each neighbourhood N of s, there exist menus A and B with $A \subset B$ and $B \succ A$, wherein $u_A(s') = u_B(s')$ for all $s' \in S_Z \setminus N$. The idea is that increasing the ex-post utility only in some parts of a neighbourhood state s of the menu A to give the menu B makes B more valuable than A. Similarly, a state $s \in S_Z$ is *negative* if for each neighbourhod N of s, there exist menus A and B with $A \subset B$ and $A \succ B$, wherein $u_A(s') = u_B(s')$ for all $s' \in S_Z \setminus N$.

As noted in proposition 3.2, if preferences are monotonic, then all states are positive and each measure in Π is a probability measure.

3.2. *Commitment*

An agent desires commitment if it is never the case that $A \cup B \succ A, B$. In particular, we shall consider the axiom Set Betweenness introduced by Gul and Pesendorfer (2001) and the following weakenings of Set Betweenness, introduced by Dekel, Lipman and Rustichini (2008).

Axiom (Betweenness)**.** A preference $\succcurlyeq$ satisfies ***Positive Set Betweenness*** if $A \succcurlyeq B$ implies $A \succcurlyeq A \cup B$. It satisfies ***Negative Set Betweenness*** if $A \succcurlyeq B$ implies $A \cup B \succcurlyeq B$. It satisfies ***Set Betweenness*** if it satisfies both Positive and Negative Set Betweenness.

To see the intuition behind these axioms, consider the case where $A \succcurlyeq B$. Then, Positive Set Betweenness says that the increased flexibility from considering $A \cup B$ is never sufficiently valuable for the agent, as compared to the menu A. In a similar vein, Negative Set Betweenness says that the increased flexibility from being able to choose from $A \cup B$ is at least valuable as B. The axioms are complementary in the sense that they provide upper and lower bounds on the value of $A \cup B$ in terms of A and B. Clearly, Positive and Negative Set Betweenness together imply Set Betweenness. Introducing these axioms imposes a lot of structure on Π, as we shall see below. Models with commitment typically assume that the set of subjective states is finite. Let $S := \bigcup_{\pi\in\Pi} \mathrm{supp}(\pi)$ denote the *subjective state space*. This allows us the following:

Definition 3.3. A *finite PfU representation* is a PfU representation (♠), where the subjective state space is finite.

For concreteness, let $S := \{s_1, \ldots, s_m\}$ be the finite state space.

Proposition 3.4. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite PfU representation (♠) and satisfies Positive Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \Pi$, there exists exactly one i such that $\pi^i > 0$. In other words, for any menu A, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where each U_π is a finite additive EU representation with one positive state.

The fact that a finite additive EU representation satisfies Positive Set Betweenness if and only if it has exactly one positive state is Lemma 1 in Dekel, Lipman and Rustichini (2008). The effect of imposing Negative Set Betweenness is similar. Again, Lemma 2 from Dekel, Lipman and Rustichini (2008) is the fact that a finite additive EU representation satisfies Negative Set Betweenness if and only if it has exactly one negative state. This gives us

Proposition 3.5. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite PfU representation (♠) and satisfies Negative Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \Pi$, there exists exactly one i such that $\pi^i < 0$. In other words, for any menu A, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where each U_π is a finite additive EU representation with one negative state.

3.3. *Local Temptation*

We now consider the consequences of assuming that $\succcurlyeq$ satisfies Set Betweenness. We shall completely characterise PfU representations that satisfy Set Betweenness. As we shall see, this results in temptation having a local character.

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Theorem 3.6. *Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite PfU representation ($\spadesuit$). Then, the following are equivalent.*

(i) V satisfies Set Betweenness.

(ii) The PfU representation has at most two subjective states, ie, $|S| \leqslant 2$. If $|S| = 2$, one state is positive (but not negative) and the other state is negative (but not positive). If $|S| = 1$, then that state is either positive or negative, but not both.

It follows from Propositions 3.4 and 3.5 that if $|S| = 2$ and preferences satisfy Set Betweenness, then each $\pi \in \operatorname{ext} \Pi$ can have only two components, one of which is positive (resp negative) and one is non-positive (resp non-negative). It does not follow from these propositions that a state cannot be both positive and negative. That this cannot be so is demonstrated in the appendix, thereby proving Theorem 3.6.

We now turn to the methodological implications of the above result. Notice that if we have 2 states and have $D \succ \{q^*\}$, then, V takes the following form:

$$V(A) := \min \{\langle \pi, u_A \rangle, \langle \nu, u_A \rangle\}$$

where $\pi^1 + \pi^2 = \nu^1 + \nu^2 = 1$ and $\operatorname{sgn}(\pi^s) = \operatorname{sgn}(\nu^s)$. To get at an interesting interpretation of this representation, let us first assume $\pi = \nu$ and $\pi \gg \mathbf{0}$, so that $V(A)$ can be rewritten as

$$\begin{aligned} V(A) &= \max_{p \in A} \ \big[v(p) + w(p)\big] - \max_{p \in A} w(p) \\ &= \max_{p \in A} \ \big[v(p) - c(p; A)\big] \end{aligned}$$

where $w(p) := |\pi^2|\, u_2(p)$, $(v + w)(p) := \pi^1 u_1(p)$ and $c(p; A) = \max_{p \in A} w(p) - w(p)$. Notice that for singletons, we now have $V(\{p\}) = v(p)$. We can interpret $c(p; A)$ as a cost function and the interpretation is that the agent believes that when she makes a choice in the second period, she will incur a self-control cost which, in turn, informs the agent's choices between menus. (This cost interpretation is central to the analysis of Gul and Pesendorfer (2001), Dekel, Lipman and Rustichini (2008) and Noor and Takeoka (2010a,b). Notice that requiring a PfU representation also satisfy Singleton Independence (see below for this axiom) is sufficient to ensure that $\pi = \nu$.)

Now consider the case where $\pi \neq \nu$. Then, we can write

$$V(A) = \min \left\{ \max_{p \in A} \ \big[v_1(p) - c_1(p; A)\big], \max_{p \in A} \ \big[v_2(p) - c_2(p; A)\big] \right\},$$

and $c_2(p, A) = \frac{|\nu^2|}{|\pi^2|} c_1(p, A)$.

In other words, the agent can be interpreted as being unsure about two things: firstly, her objective (or normative) preferences, given by her preference over singletons, and second the cost function she will face. This result can be viewed as saying that temptation and normative preferences are ***local*** in the sense that the normative preference and concomitant cost of temptation will depend on the menu in question. This brings into stark relief the crucial rôle that (a sufficient amount of) Independence plays in the point of view that normative welfare judgements can be made by looking at preferences over singletons.

One final observation here is that if PfU preferences are non-linear and satisfy Set Betweenness, then they cannot be linear over singletons. This is made precise below. We begin with an axiom.

Axiom (Singleton Independence)**.** For all $p, q, r \in \Delta$ and $\lambda \in (0, 1)$, $\{p\} \succ \{q\}$ implies $\lambda\{p\} + (1 - \lambda)\{r\} \succ \lambda\{q\} + (1 - \lambda)\{r\}$.

Proposition 3.7. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite PfU representation (♠). Then the following are equivalent

(i) V satisfies Set Betweenness and Singleton Independence.

(ii) V is a finite additive EU representation (ie, Π is a singleton) that satisfies Set Betweenness.

Put differently, for PfU representations, under Set Betweenness, Singleton Independence is equivalent to Independence.

4. Related Literature

That preferences over planning problems can be used to infer how an agent thinks she will behave was first pointed out by Kreps (1979). The construction of a *unique* subjective state space was carried out by Dekel, Lipman and Rustichini (2001) (henceforth DLR) by considering menus of lotteries. The subjective state space constructed by DLR is an improvement over Kreps' for the following reason. Subjective states are important only if ascribing behavioural properties to these states translates into some behavioural properties of the preference over menus. If the state space is not unique, then different subjective states will translate into different (possibly conflicting) behavioural interpretations of preferences over menus, which diminishes the value of the theory. We now briefly describe DLR's *weak EU representation*.

DLR's weak EU representation consists of a collection of vN-M utility functions S over lotteries, and a continuous function $U : \mathbb{R}^S \to \mathbb{R}$ such that the utility of a menu A is

$$V(A) := U\left(\left(\max_{p \in A} u(p,s)\right)_{s \in S}\right).$$

The set S is referred to as the subjective state space and, as mentioned above, is shown to be unique (up to closure). The function U is called an *aggregator* since it takes a collection of ex post utilities and transforms it into an ex ante utility over a menu. Notice that there are no structural requirements on U other than continuity, so that a large class of behaviour can be captured. In particular, our PfU representation (♠) is also an example of a weak EU representation. Theorem 1.A of DLR shows that (only) continuous preferences over the space of menus of lotteries that have the property that a menu is indifferent to its convex hull have a weak EU representation.

The preferences considered by DLR are also not required to be monotonic (in the sense that $A \cup B \succcurlyeq A$). Preferences that are not monotonic form an important class and have been used to model, most notably, temptation by Gul and Pesendorfer (2001). However, this and subsequent papers, for instance Dekel, Lipman and Rustichini (2008) and Chatterjee and Krishna (2007), make heavy use of the Independence axiom. A study of temptation related problems while relaxing Independence is carried out by Noor and Takeoka (2010a,b), whose environment is essentially the same as ours — we discuss these papers in greater detail below — except that they allow the set of prizes Z to be a compact metric space. Olszewski (2008) considers a model of menu-dependent temptation, but looks at menus of deterministic prizes. In their (non-axiomatic) model of self control, Fudenberg and Levine (2006) argue that Independence is not an appropriate assumption for temptation and related phenomena.

From a formal viewpoint, our PfU representation is the subjective state equivalent of the multi-prior model of Gilboa and Schmeidler (1989). (Nevertheless, while the representations are similar, our proof is substantially different.) An excellent textbook presentation of Gilboa-Schmeidler's multi-prior model is in Ok (2007), who presents a finite dimensional version of the multi-prior model. A paper that looks at self control problems in the Gilboa-Schmeidler framework is the paper by Epstein and Kopylov (2007). They draw on the aforementioned multi-prior model (and a variant of Set Betweenness, cf section 3). In their environment, a menu is a collection of Anscombe-Aummann acts with a finite objective state space.

The paper that is closest to ours is the paper by Epstein, Marinacci and Seo (2007). Their motivation is the study of agents who are not sure about the contingencies that may arise. Their environment is the same as ours and their representation is also very similar to ours. In particular, they find the value of a menu to be

$$W_{\text{ems}}(A) = \min \left[\int_{S_Z} \max_{p \in A} u(p, s) \; d\pi(s) : \pi \in \Pi_{\text{ems}} \right]$$

where Π_{ems} is a weak* compact, convex set of probability measures on S_Z. While this representation looks like our (♠) PfU representation, there are some subtle qualifications to the above representation. The first is that they assume preferences are monotone, which implies each π is a probability measure. We drop this assumption.

They also assume the existence of a prize z_* and define $N := \{s \in S_Z : u(z, s) \geqslant u(z_*, s) \text{ for all } z \in Z\}$, requiring that for each $\pi \in \Pi_{\text{ems}}$, $\operatorname{supp} \pi \subset N$. Put another way, the set of subjective states that are possible are restricted to those where one (predefined) prize is always the worst. In principle, there is nothing wrong with assuming that there is a prize that is always worst in every subjective state. For instance, consider a prize z_*, where the agent loses a very large sum of money. It is plausible that such a prize would always be the worst in every state of mind the agent can conceive of. Nevertheless, it is not clear why such an assumption is necessary to obtain the representation. Moreover, the set Π_{ems} is not unique, although there is a smallest such set (in terms of set inclusion) that satisfies the above.

Key to the EMS representation theorem is the assumption that the payoffs to the (worst) prize z_* and the set of lotteries Δ are certain, ie, they are the same in each subjective state that the agent considers relevant (ie, possible). Similarly, we assume the existence of two menus $\{q^*\}$ (where q^* is a singleton menu) and a menu D, which is a disk centered around q^*, that give the agent the same payoffs in every subjective state. (More precisely, we define the subjective state space so as to have this property.) This enables us to obtain a representation that is unique (up to normalisation of the subjective state space).[4]

(4) Actually, more is true. If we take $\{q^*\}$ and D to be any two arbitrary menus, we can still obtain a representation of preferences of the form (♠), with the proviso that the set

EMS also provide a second representation, in a setting with two-stage randomisation. In this model, ambiguity about tastes persists ex post, which leads to a violation of Indifference to Randomisation, which means that the agent is no longer indifferent between a set and its convex hull. They argue that such a model is realistic in applications since it is unlikely that ambiguity about tastes is ever completely resolved. We shall have nothing more to say about this second representation.

In sum, there are two key distinguishing features between the first representation in Epstein, Marinacci and Seo (2007) and the representation obtained here. The first is that we drop of the assumption of monotone preferences. Second, we drop the assumption that there exists a prize z_* that is the worst in any conceivable subjective state. Dropping this assumption comes at the cost of having to assume that there are two menus, the payoffs to which are certain, ie, constant in all subjective states. Uniqueness of the representation obtains when the two sets satisfy additional conditions, as in our axiom D-Independence below. This is useful as it allows us to ascribe behavioural meaning to increasing sets of measures.

The papers Noor and Takeoka (2010a,b) seek a general approach to temptation and self-control. They follow Gul and Pesendorfer (2001) and retain the assumption of Set Betweenness, but relax Independence. The generalisation then comes by allowing the cost of temptation to be non-linear. To keep matters simple, we only consider the convex model of Noor and Takeoka (2010a).[5] Let u denote the normative preference of the agent, while v denotes the temptation utility, and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and convex, with $\varphi(0) = 0$. Then, the utility from a menu A is given by

$$W_{\mathrm{nt}}(A) := \max_{p \in A} \left[u(p) - \varphi\Big(\max_{q \in A} v(q) - v(p) \Big) \right]$$

Intuitively, in any menu $\{p, q\}$ temptation occurs when $u(p) > u(q)$ but $v(q) > v(p)$. Thus, $\varphi(\cdot)$ is active, ie, is positive, only when temptation matters. In the event that there are tempting alternatives, the cost of temptation rises as a convex function of the temptation opportunity cost $w = v(q) - v(p)$, that is, the convexity of φ corresponds to the increasing marginal cost of self control. Finally, notice that the cost of self control (under Set Betweenness)

of (signed) measures, while weak* compact, is not unique. It is the assumption that q^* is in the relative interior of D that enables us to provide a unique representation.

(5) Noor and Takeoka (2010b) is a generalisation of Noor and Takeoka (2010a) that also discusses the failure of WARP in temptation-related problems.

can be smooth, whereas the PfU representation (under Set Betweenness) is only piecewise linear.

It is easy to see that the Noor-Takeoka representation is convex. As in the present study, they also relax Independence, but manage to do it in such a way as to maintain the notion of a normative preference. It is also easy to see that if the preference functional W_{nt} were required to be positively homogeneous, then the cost function φ must necessarily be linear. In other words, it seems that in a convex cost model of self-control, positive homogeneity either results in linear costs or an inability to uniquely identify normative preferences.

It is worth mentioning that the added generality of our approach comes at the expense of a multiplicity of normative rankings. For instance, the agent described in the introduction is not unreasonable, yet has multiple normative rankings, and is ruled out by the models of Noor and Takeoka. A thorough investigation of these issues is left to future research.

At a somewhat abstract level, our representation theorem can be reduced to the problem of finding a concave utility function representing preferences. This is quite difficult, since requiring that upper contour sets be convex only delivers us a *quasi*concave utility function, which need not be concave. Thus, some other condition on preferences is required to obtain a concave representation. Broadly speaking, there are two kinds of conditions that can be imposed. The first kind of condition, initially due to Gilboa and Schmeidler (1989), requires that the utility function be additive with respect to a smaller set of objects than the entire space. The second kind of condition, which follows Maccheroni, Marinacci and Rustichini (2006) requires that indifference curves be parallel on a sufficiently large subset of the domain. We follow Gilboa and Schmeidler in that we require Independence to hold for very few combinations. In particular, this means that our indifference curves need be not be parallel anywhere. Indeed, we do not even require preferences over singletons to be linear, so we relax Independence considerably.

5. Proofs

5.1. *The Cone of Ex-post utilities*

Recall that S^{n-1} is the canonical $(n-1)$-dimensional sphere and let $\mathcal{K}^n$ denote the space of all compact convex subsets of $\operatorname{aff}\Delta$ (which is essentially $\mathbb{R}^n$). It is easy to show that $\mathcal{K}^n$ is a complete metric space when endowed with the Hausdorff metric, d_h. Let $S_Z := \{p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1\}$. Notice that S_Z is isometrically homeomorphic to S^{n-1}. For any $A \in \mathcal{K}^n$, its *support function* $h_A : S_Z \to \mathbb{R}$ is given by $h_A(s) := \max_{p \in A} \langle p - p^*, s\rangle$, where $p^* := (1/n+1, \ldots, 1/n+1)$. Notice that for each $A \in \mathcal{K}^n$, h_A is a continuous function on S_Z. This definition of a support function has some immediate consequences. Firstly, $h_{\{p^*\}} = \mathbf{0}$ and second, for $D_r := \{p \in \operatorname{aff} \Delta : \|p - p^*\| \leqslant r > 0\}$, $h_{D_r}(s) = r$ for each $s \in S$.

It should be noted that for any $\lambda \in [0,1) \cup (1, \infty)$, $\lambda A \notin \mathcal{K}^n$. So, in what follows, when we write λA where $\lambda \geqslant 0$, we shall actually mean $\lambda(A - p^*) + p^*$. Similarly, for $A, B \in \mathcal{K}^n$, $A + B$ shall mean $(A - p^*) + (B - p^*) + p^*$, which is a compact, convex subset of $\operatorname{aff} \Delta$. Notice that for any $p \in \operatorname{aff} \Delta$ and $\lambda \geqslant 0$, $\lambda(p - p^*) + p^* = \lambda p - (\lambda - 1)p^* \in \operatorname{aff} \Delta$. Thus, λA as defined above is also a subset of $\operatorname{aff} \Delta$. Moreover, for $A, B \in \mathcal{K}^n$ and $\lambda \in (0,1)$, $\lambda A + (1 - \lambda)B \in \mathcal{K}^n$. We begin with some technical properties of support functions.

Proposition 5.1. The support function has the following properties:
(i) $h_{A+B} = h_A + h_B$,
(ii) $h_{\lambda A} = \lambda h_A$ for all $\lambda \geqslant 0$,
(iii) $h_A \wedge h_B = h_{A \cap B}$,
(iv) $h_A \cup h_B = h_{\operatorname{conv}(A \cup B)}$, and
(v) $\|h_A - h_B\|_\infty = d_h(A, B)$.
(vi) The (unique) extension $\bar{h}$ of a support function h to $\operatorname{span} S_Z$ by positive homogeneity is sublinear and the restriction to S_Z of a sublinear function on $\operatorname{span} S_Z$ is a support function.
(vi) Moreover, the following duality relation holds: $h_{K_h} = h$ and $K_{h_K} = K$.

Proof. See sections 5.18 and 5.19 of Aliprantis and Border (1999). $\square$

Thus, there is an isometry between the space of compact, convex sets of $\mathbb{R}^n$ ($\simeq \operatorname{span} S_Z$) and the space of sublinear functions on $\mathbb{R}^n$. Let $K^* \subset C(S_Z)$ denote the cone of functions on S_Z whose unique extensions to $\operatorname{span} S_Z$ by positive homogeneity are sublinear. Then, $K^* \simeq \mathcal{K}^n$.

We shall be interested in the embedding of $\operatorname{aff} \Delta$ in K^*. To this end, we define $F := \left\{h_{\{p\}} : p \in \operatorname{aff} \Delta\right\}$. Thus, F represents the space of support functions of singletons (which are also compact and convex). We show below that F is an n-dimensional subspace of K^*.

Proposition 5.2. Let $S_0 \subset S_Z$ be a linearly independent set. Then,

$$F_0 := \left\{ \left(h_{\{p\}}(s)\right)_{s \in S_0} : p \in \operatorname{aff} \Delta \right\}$$

is a (closed) $|S_0|$-dimensional subspace of K^*. Therefore, if S_0 is a maximal linearly independent set, then $F_0 = F$.

Proof. Let S_0 be a linearly independent subset of S_Z. Notice first that for any $p \in \text{aff}\,\Delta$, there exists (a unique) $p' \in \text{aff}\,\Delta$ such that $(p+p')/2 = p^*$. Therefore $p-p^* = -(p'-p^*)$, so that $h_{\{p\}}(s) = \langle p-p^*, s\rangle = -\langle p'-p^*, s\rangle = -h_{\{p'\}}(s)$ for all $s \in S_0$. This shows that for any $p \in \text{aff}\,\Delta$ and any $\lambda \in \mathbb{R}$, $\lambda h_{\{p\}}(s) = h_{\lambda\{p\}}(s)$ for all $s \in S_0$ (where $\lambda\{p\}$ is actually $\lambda(p-p^*)+p^*$. Also, for any $p, q \in \text{aff}\,\Delta$, $h_{\{p\}+\{q\}}(s) = h_{\{p\}}(s) + h_{\{q\}}(s)$ for all $s \in S_0$. Thus, F_0 is a vector subspace of K and it is easily seen to be of dimension $|S_0|$. We can similarly show that F is a vector subspace of K^* from which it follows that F_0 is a vector subspace of F.

Suppose now that S_0 spans S_Z. To show that $F = F_0$, notice that for any $s \in S_Z \setminus S_0$, there exist (unique numbers) $\lambda_1, \ldots, \lambda_k$ such that $s = \sum_{i=1}^k \lambda_i s_i$. Then, $h_{\{p\}}(s) = \langle p-p^*, s\rangle = \left\langle p-p^*, \sum_{i=1}^k \lambda_i s_i \right\rangle = \sum_{i=1}^k \lambda_i \langle p-p^*, s_i\rangle = \sum_{i=1}^k \lambda_i h_{\{p\}}(s_i)$. Thus, $F = F_0$. □

As mentioned in the text, the subjective state space can be interpreted as the set of (expected) utility functions that the agent believes can be used to make a choice from a set in the second stage. For a given menu, she will receive some utility in each state. If the subjective state space is finite, then we can say a little more about the space defined above. The following proposition is stated without proof.

Proposition 5.3. Let $S \subset S_Z$ be finite with cardinality m. Then, $K_S := \{(h_A(s))_{s\in S} : A \in \mathcal{K}^n\}$ is a closed convex cone that spans $\mathbb{R}^m$.

In general, we cannot guarantee that K_S is a vector space. To see this, suppose $S = \{u, -u\}$. Then, for any $x > 0$, $(-x, -x) \notin K_S$. We now show that K_S has a non-empty interior.

Proposition 5.4. K_S has a topological interior. Therefore, $\text{span}\, K_S = \mathbb{R}^m$.

Proof. Let $r > 0$ be such that $D_r \subset \text{ri}\,\Delta$. Since S is finite, there exists $\varepsilon > 0$ such that for any $s \in S$, there exist A^s and B^s, compact convex subsets of (the relative interior of) Δ such that (i) $h_{A^s}(s) = r - \varepsilon$ and $h_{B^s}(s) = r + \varepsilon$ and (ii) $h_{A^s}(s') = r$ and $h_{B^s}(s') = r$ for any $s' \neq s$. Thus, there is neighbourhood of $r\mathbf{1}$ in K_S, so K_S has an interior. Finally, since $0 \in K_S$, $\text{span}\, K_S = \mathbb{R}^m$. □

The following corollary is immediate.

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Corollary 5.5. $\mathbf{1} \in \operatorname{int} K_S$. Therefore, for each $w \in \mathbb{R}^m$, there exists $\lambda_w > 0$ such that $w + \lambda \mathbf{1} \in \operatorname{int} K_S$.

Proof. Since K_S is a convex cone, we see that if $w \in \operatorname{int} K_S$, $\lambda w \in \operatorname{int} K_S$ for all $\lambda > 0$. Therefore, $\mathbf{1} \in \operatorname{int} K_S$. Now take $w \in \mathbb{R}^m$. Then, there exists $\mu > 0$ such that $\mu w + (1-\mu)\mathbf{1} \in \operatorname{int} K_S$. This implies $\frac{1}{\mu}(\mu w + (1-\mu)\mathbf{1}) =: w + \lambda_w \mathbf{1} \in \operatorname{int} K_S$. $\square$

5.2. *A Utility Representation*

We shall now construct a D-linear utility representation for our preferences. Essentially, the rest of this section is devoted to proving proposition 2.1. Recall that one of our main axioms is D-Independence. Now consider the following strengthening of D-Independence. Throughout, we shall write $D_r(q^*)$ as D_r.

Axiom (Strong D-Independence)**.** $A \succ B$ implies $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$ for all $\lambda \in (0,1)$ and $r > 0$ and $D_r \subset \Delta$.

It is clear that Strong D-Independence implies D-Independence. It is useful to note that the two are actually equivalent. This is shown below.

Proposition 5.6. For a preference relation $\succcurlyeq \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$, strong D-Independence is equivalent to D-Independence.

Proof. As noted above, it follows from the definitions that Strong D-Independence implies D-Independence. To prove the converse, suppose $A \succ B$. We will first prove that for any $r \in (0,R)$ and any $\lambda \in (0,1]$, $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$. Fix such an r and λ and let $\mu := {}^r/_R$, so that $D_r = \mu D_R + (1-\mu)\{p^*\}$. Then, $\frac{\lambda}{\lambda+(1-\lambda)\mu}A + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R \succ \frac{\lambda}{\lambda+(1-\lambda)\mu}B + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R$. This implies $\gamma\left[\frac{\lambda}{\lambda+(1-\lambda)\mu}A + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R\right] + (1-\gamma)\{p^*\} \succ \gamma\left[\frac{\lambda}{\lambda+(1-\lambda)\mu}B + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R\right] + (1-\gamma)\{p^*\}$ where $\gamma := \lambda + (1-\lambda)\mu$. Notice that $1-\gamma = 1-\lambda-(1-\lambda)\mu = (1-\lambda)(1-\mu)$, so that we can rewrite the above relations as $\lambda A + (1-\lambda)[\mu D_R + (1-\mu)\{p^*\}] \succ \lambda B + (1-\lambda)[\mu D_R + (1-\mu)\{p^*\}]$, ie, $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$ as required.

The case where $r > R$ and $D_r \subset \Delta$ is similar. Fix such an $r > R$ and notice that there exists $\nu > 0$ such that $\nu r < R$. Also fix a $\lambda \in (0,1]$. By D-Independence, $\nu A + (1-\nu)\{p^*\} \succ \nu B + (1-\nu)\{p^*\}$. Notice also that $D_{\nu r} := \nu D_r + (1-\nu)\{p^*\}$. By what we have proved in the paragraph above, we get $\lambda[\nu A + (1-\nu)\{p^*\}] + (1-\lambda)D_{\nu r} \succ \lambda[\nu B + (1-\nu)\{p^*\}] + (1-\lambda)D_{\nu r}$

which can be rewritten as $\nu\lambda A + (1-\nu)\lambda\{p^*\} + (1-\lambda)[\nu D_r + (1-\nu)\{p^*\}] \succ \nu\lambda B + (1-\nu)\lambda\{p^*\}$ which in turn can be written as $\nu[\lambda A + (1-\lambda)D_r] + (1-\nu)\{p^*\} \succ \nu[\lambda B + (1-\lambda)D_r] + (1-\nu)\{p^*\}$, which holds if and only if $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$, which is what we wanted to prove. $\square$

By DLR's Theorem 1.A, there exists a closed subset $S \subset S_Z$ and a utility function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ such that V is continuous and

$$V(A) := U(u_A).$$

This V can be chosen such that $V(p^*) = 0$ (where $p^* = (1/n+1, \ldots, 1/n+1)$ is the uniform lottery). Notice that since V represents preferences, and $V(D_R)$ is positive or negative if (and only if) $D_R \succ \{p^*\}$ or $\{p^*\} \succ D_R$. Our second normalisation will set $V(D_R) \in \{-R, R\}$. Unfortunately, even with these normalisations, we do not have a utility representation with any structure.

In the rest of this subsection, we shall construct a D-linear utility function V, ie, a utility function that satisfies (i) $V(\lambda A + (1-\lambda)\{p^*\}) = \lambda V(A)$ and (ii) $V(\lambda A + (1-\lambda)D_R) = \lambda V(A) + (1-\lambda)V(D_R)$. For the rest of this section (and the remainder of the proof), we shall assume that $D_R \succ \{p^*\}$, so that $V(D_R) = R$. (The other case will be derived from a simple duality result.) We shall proceed in a number of simple steps. We shall first show that there is a useful class of sets in our domain that form a mixture space.

For each $A \in \mathcal{K}(\Delta)$, let $\Xi_A := \{\lambda A + (1-\lambda)\{p^*\} : \lambda \in [0,1]\}$. Notice that each Ξ_A is a mixture space. To see this, let $\alpha, \beta, \lambda \in [0,1]$ and define $A_\alpha := \alpha A + (1-\alpha)\{p^*\}$. Then, $\lambda A_\alpha + (1-\lambda)A_\beta = \big(\lambda\alpha + (1-\lambda)\beta\big)A + \big(\lambda(1-\alpha) + (1-\lambda)(1-\beta)\big)\{p^*\} \in \Xi_A$. The following proposition is useful to record.

Proposition 5.7. If $A \succ \{p^*\}$, then for all $\mu \in (0,1)$, $A \succ \mu A + (1-\mu)\{p^*\}$. Similarly, if $\{p^*\} \succ A$, then for all $\mu \in (0,1)$, $\mu A + (1-\mu)\{p^*\} \succ A$.

Proof. We shall only prove the first part. Suppose, by way of contradiction, that $A_1 := \mu A + (1-\mu)\{p^*\} \succcurlyeq A$. Then, by D-Independence, $A_2 := \mu A_1 + (1-\mu)\{p^*\} \succcurlyeq \mu A + (1-\mu)\{p^*\} = A_1$. Defining A_n inductively, we see that $A_{n+1} \succcurlyeq A_n$. But $A_n \to \{p^*\}$, which contradicts the continuity of $\succcurlyeq$. The case where $\{p^*\} \succ A$ is proved similarly. $\square$

The proposition above can be restated as follows.

Corollary 5.8. Let $1 \geqslant \alpha > \beta > 0$. If $A \succ \{p^*\}$, then $\alpha A + (1-\alpha)\{p^*\} \succ \beta A + (1-\beta)\{p^*\}$. If $\{p^*\} \succ A$, then $\beta A + (1-\beta)\{p^*\} \succ \alpha A + (1-\alpha)\{p^*\}$.

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We shall now show that preferences restricted to each Ξ_A satisfy not only D-Independence, but also Independence.

Proposition 5.9. The naturally induced preference over the mixture space Ξ_A satisfies Independence.

Proof. Let $A_\alpha, A_\beta, A_\gamma \in \Xi_A$, where $\alpha A + (1-\alpha)\{p^*\} =: A_\alpha \succ A_\beta$ and let $\lambda \in (0,1]$. Notice that $\lambda A_\alpha + (1-\lambda)A_\gamma = \big(\lambda\alpha + (1-\lambda)\gamma\big)A + \big(\lambda(1-\alpha) + (1-\lambda)(1-\gamma)\big)\{p^*\}$ and $\lambda A_\beta + (1-\lambda)A_\gamma = \big(\lambda\beta + (1-\lambda)\gamma\big)A + \big(\lambda(1-\beta) + (1-\lambda)(1-\gamma)\big)\{p^*\}$. If $A \succ \{p^*\}$, it must be that $\alpha > \beta$, so that (by the Corollary above) $\lambda A_\alpha + (1-\lambda)\{p^*\} \succ \lambda A_\beta + (1-\lambda)\{p^*\}$ as desired. If, on the other hand, $\{p^*\} \succ A$, then $\alpha < \beta$, so that $\lambda A_\alpha + (1-\lambda)\{p^*\} \succ \lambda A_\beta + (1-\lambda)\{p^*\}$. The proof is completed with the observation that if $A \sim \{p^*\}$, then $A_\alpha \sim \{p^*\}$ for each $A_\alpha \in \Xi_A$. (This follows from D-Independence.) $\square$

We now proceed to constructing the desired utility representation. Recall that we are only considering the case where $D_R \succ \{p^*\}$. We shall proceed via a number of simple steps.

Step 1. For all $D_r \subset \Delta$, $V(D_r) = r$. This represents preferences since Ξ_{D_r} is a mixture space for each D_r and preferences on this domain satisfy Independence.

Step 2. Now suppose $A \succ \{p^*\}$. We claim that if $A \succ D_r$ (for some D_r, and since preferences are continuous, such a D_r must exist), there exists a unique $\lambda > 0$ such that $\lambda A + (1-\lambda)\{p^*\} \sim D_r$. To see that this is the case, notice that since preferences are continuous, there exists at least one λ such that $\lambda A + (1-\lambda)\{p^*\} \sim D_r$. If there are $\lambda_1 > \lambda_2 > 0$ such that $\lambda_i A + (1-\lambda_i)\{p^*\} \sim D_r$ for $i = 1, 2$, we contradict Step 1. Hence the λ must be unique. Similarly, for each A such that $D_R \succcurlyeq A \succcurlyeq \{p^*\}$, there exists a unique $\lambda \in [0,1]$ such that $A \sim \lambda D_R + (1\lambda)\{p^*\}$. For such an A, let $V(A) := \lambda$ and if $A \succ D_R$, let $V(A) := R/\lambda$. Notice that for $A_\alpha \in \Xi_A$, let $V(A_\alpha) = \alpha V(A)$.

Step 3. For $A \sim \{p^*\}$, let $V(A) = 0$ and also $V(A_\alpha) = 0$, where $A_\alpha := \lambda A + (1-\lambda)\{p^*\}$.

Step 4. Suppose $\{p^*\} \succ A$. We claim that there exists a unique $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. If this is true, define $V(A)$ such that $\lambda V(A) + (1-\lambda)V(D_r) = V(\{p^*\}) = 0$, ie, $V(A) = -r\frac{1-\lambda}{\lambda}$. Now extend V linearly to all Ξ_B that contain A. Clearly, this represents preferences restricted to Ξ_B.

To see that the claim is true, notice first that, since preferences are continuous, there exists a $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. Suppose λ is not unique, assume there exist λ_1, λ_2 such that $\lambda_i A + (1-\lambda_i)D_r \sim \{p^*\}$, with $\lambda_1 > \lambda_2$. As before, let $\mu := \lambda_2/\lambda_1 < 1$. Then, by D-Independence, $\{p^*\} \sim \lambda_2 A+(1-\lambda_2)D_r = \mu\big(\lambda_1 A+(1-\lambda_1)D_r\big)+(1-\mu)D_r \sim \mu\{p^*\}+(1-\mu)D_R$, which contradicts Step 1.

We shall now verify that V is well defined.

Lemma 5.10. V is well defined.

Proof. Consider first the case where $A \succ \{p^*\}$. Then, for some D_r, $\lambda A + (1 - \lambda)\{p^*\} \sim D_r$, giving us $V(A) = r/\lambda$. We need to show that $V(A)$ is independent of the D_r chosen. Suppose $D_s \subset D_r$, so that $D_r \succ D_s$. Letting $\eta := {}^s\!/\!_r$, we see that $\eta\big[\lambda A + (1-\lambda)\{p^*\}\big] \sim \eta D_r + (1-\eta)\{p^*\} = D_s$. Consistency would require that ${}^s\!/\!_{\eta\lambda} = {}^r\!/\!_\lambda$ which is definitely the case. Similar arguments show this for the case where $s > r$, and also show that the value of $V(A)$ is independent of the Ξ_B it is contained in.

Now to the case where $\{p^*\} \succ A$. Recall that by finding $V(A)$, we can assign values to all A' in any Ξ_B that contains A. So suppose $\lambda A+(1-\lambda)D_r \sim \{p^*\}$ for some $\lambda \in (0,1)$, D_r, giving us $V(A) = -r(1-\lambda)/\lambda$. Suppose D_s is such that $s < r$ and let $\eta := {}^s\!/\!_r$ so that $A' := \eta A + (1-\eta)\{p^*\}$ and $V(A') = \eta V(A)$. It will suffice to show that $\lambda A' + (1-\lambda)D_s \sim \{p^*\}$. To see that this is the case, notice that $\{p^*\} \sim \eta\big[\lambda A + (1-\lambda)D_r\big] + (1-\eta)\{p^*\} = \lambda\big[\eta A+(1-\eta)\{p^*\}\big]+(1-\lambda)\big[\eta D_r+(1-\eta)\{p^*\}\big] = \lambda A'+(1-\lambda)D_s$, as required. We have thus shown that V is well defined. $\square$

All that remains is to check that the V we have constructed is, in fact, D-linear.

Lemma 5.11. V is D-linear.

Proof. By the definition of V, all we need to verify is that $V\big(\lambda A + (1-\lambda)D_r\big) = \lambda V(A) + (1-\lambda)V(D_r)$ for all A, D_r and $\lambda \in (0,1)$. Assume first that $A \succ \{p^*\}$ and assume, without loss of generality, that $A \sim D_t$. Then, $\lambda A+(1-\lambda)D_r \sim \lambda D_t+(1-\lambda)D_r$ so that $V\big(\lambda A+(1-\lambda)D_r\big) = \lambda t+(1-\lambda)r = \lambda V(A) + (1-\lambda)V(D_r)$.

Now for the case where $\{p^*\} \succ A$. For each D_r, there exists a $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. Consider first the case where $1 > \mu > \lambda$, so that $\eta := {}^\lambda\!/\!_\mu$. Then, $\eta\big[\mu A + (1-\mu)D_r\big] + (1-\eta)D_r = \lambda A + (1-\lambda)D_r \sim \{p^*\}$ which means $V\big(\mu A + (1-\mu)D_r\big) = -r(1-\eta)/\eta = -r\frac{\mu-\lambda}{\lambda}$. But $\mu V(A) + (1-\mu)r = -\mu r\frac{1-\lambda}{\lambda} + (1-\mu)r = -r\frac{\mu-\lambda}{\lambda} = V\big(\mu A + (1-\mu)D_r\big)$, as desired. Finally consider the case where $\mu < \lambda$ and once again let $\eta := {}^\mu\!/\!_\lambda$. Recalling that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$ which means that $V(A) = -r\frac{1-\lambda}{\lambda}$, we see that

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$\mu A + (1-\mu)D_r = \eta\big[\lambda A + (1-\lambda)D_r\big] + (1-\eta)D_r \sim \eta\{p^*\} + (1-\eta)D_r$, which implies $V\big(\mu A + (1-\mu)D_r\big) = (1-\eta)r = r\frac{\lambda-\mu}{\lambda} = \mu V(A) + (1-\mu)V(D_r)$. $\square$

In sum, we have constructed a function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ that represents $\succcurlyeq$ and is D-linear. We shall now show that if $\succcurlyeq$ also satisfies L continuity, V is actually Lipschitz continuous. It should be noted that the proof of the following lemma exactly the same as the proof of Lemma 1 in Dekel et al (2007). The key difference is that we have a stronger version of L continuity in that we require A^* and A_* to be one of D_R and $\{p^*\}$. Thus, the linearity that is so crucial in the proof of the lemma in Dekel et al (2007), is still present in our setting since V is D-linear.

Definition 5.12. A function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ is Lipschitz continuous if there is an $\bar{N}$ such that $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$ for all A, B.

Lemma 5.13. Let $\succcurlyeq$ have a D-linear representation V where $V(D_R) \neq V(\{p^*\})$. Then V is Lipschitz continuous if and only if $\succcurlyeq$ satisfies L continuity.

As mentioned above, the proof is essentially the same as the proof of Lemma 1 in Dekel et al (2007). Therefore, we only provide a sketch with some details missing, for which the reader is referred to the original.

Sketch of Proof. Suppose $\succcurlyeq$ satisfies L continuity. Fix an $N > 0$ and $M \in (0, 1/N)$ and fix A, B such that $\delta := d_h(A, B) \leqslant M$. Suppose $\delta > 0$ (if not, we are done) so that L continuity implies

$$(1 - N\delta)A + N\delta A^* \succcurlyeq (1 - N\delta)B + N\delta A_*.$$

Using the D-linearity of V, we can rewrite this as

$$V(B) - V(A) \leqslant \frac{N}{1 - N\delta}\big[V(A^*) - V(A_*)\big] d_h(A, B).$$

Let $\bar{N} := \frac{N}{1-NM}\big[V(A^*) - V(A_*)\big]$, so that for all A, B with $d_h(A, B) \leqslant M$, we have $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$.

To show the same is true for arbitrary A and B, fix an A and B and a sequence $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_J < \lambda_{J+1} = 1$ such that $(\lambda_{j+1} - \lambda_j)d_h(A, B) \leqslant M$. Defining $B_j : \lambda_j B + (1 - \lambda_j)A$, so that $d_h(B_{j+1}, B_j) = (\lambda_{j+1} - \lambda_j)d_h(A, B)$. This implies, $V(B_{j+1}) - V(B_j) \leqslant \bar{N}(\lambda_{j+1} - \lambda_j)d_h(A, B)$ and summing over j from 0 to J gives $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$, ie, Lipschitz continuity of V.

To see the converse, suppose there is an $\bar{N}$ such that $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$ for all A, B. Let $N := \bar{N} / |V(D_R) - V(\{p^*\})|$. Thus, for all A, B with $d_h(A, B) \leqslant 1/N$ and for every $\varepsilon \in [d_h(A, B), 1/N)$,

$$V(B) - V(A) \leqslant \frac{N\varepsilon}{1 - N\varepsilon} \left| V(D_R) - V(\{p^*\}) \right|.$$

Rearranging terms gives us the L continuity of $\succcurlyeq$. $\square$

It is easy to see that extending V to $\mathcal{K}(\operatorname{aff} \Delta)$ by positive homogeneity preserves the above properties. Abusing notation, let the extension be denoted by $V : \mathcal{K}(\operatorname{aff} \Delta) \to \mathbb{R}$. (Rather than prove V is concave if preferences satisfy Quasiconcavity, we shall show that the extension is concave. This is done in the next section.) We shall now use the embedding of $\operatorname{aff} \Delta$ in $C(S_Z)$ and extend V to a dense linear subspace of $C(S_Z)$. It is easy to see that the extension is also Lipschitz continuous.

5.3. *Extension to a Dense Linear Subspace*

Recall that $S_Z := \{p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1\}$ is isometrically homeomorphic to S^{n-1} and that $\mathcal{K}(\operatorname{aff} \Delta)$ is isometrically embedded as $K^* \in C(S_Z)$ (which has the sup norm). For each $A \in \mathcal{K}(\operatorname{aff} \Delta)$, $h_A \in K^*$. (Also, a function $f \in K^*$ if and only if the unique extension of f to $\mathbb{R}^n$, which is isometrically homeomorphic to $\operatorname{aff} \Delta$, by positive homogeneity is superlinear. Thus, for each $f \in K^*$, there exists an $A \in \mathcal{K}(\operatorname{aff} \Delta)$ such that $f = h_A$.) Let $C^2(S_Z)$ be the space of all twice continuously differentiable functions on S_Z viewed as a subspace of $C(S_Z)$ and define $K_2^* := K \cap C(S_Z)$. We begin by defining a functional on K^*.

Define $\varphi : K^* \to \mathbb{R}$ as $\varphi\big(h_A\big) = V(A)$. Notice that since V is Lipschitz continuous, φ is too. The following property is a weakening of C-additivity introduced by Gilboa and Schmeidler (1989).

Definition 5.14. Let $K \subset C(S_Z)$ be a cone such that $\mathbf{1} \in K$. A function $\varphi : K \to \mathbb{R}$ is C$^+$-*additive* if $\varphi(f + \alpha\mathbf{1}) = \varphi(f) + \varphi(\alpha\mathbf{1})$ for all $f \in K$ and $\alpha \in \mathbb{R}_+$.

Notice that φ is, by definition, positively homogeneous. We shall now show that φ is C$^+$-additive and superlinear.[6]

(6) Recall that a function is superlinear if it positively homogeneous and concave.

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Lemma 5.15. Let $\varphi : K \to \mathbb{R}$ be defined as above. Then φ is C+-additive.

Proof. Recall that if $v \in K$, $v + \alpha \mathbf{1} \in K$ for all $\alpha \in \mathbb{R}_+$. Now consider $\alpha \in [0, 1)$ and let w such that $(1 - \alpha)w = v$. Then,

$$\begin{aligned}\varphi(v + \alpha\mathbf{1}) &= \varphi\big((1-\alpha)w + \alpha\mathbf{1}\big)\\ &= V\big((1-\alpha)A_w + \alpha D\big)\\ &= V\big((1-\alpha)A_w\big) + V(\alpha D)\\ &= \varphi\big((1-\alpha)w\big) + \varphi(\alpha\mathbf{1})\\ &= \varphi(v) + \varphi(\alpha\mathbf{1})\end{aligned}$$

where $A_w \in \mathcal{K}^n$ is such that $w = u_A$. To prove the general case, we shall use the fact that φ is positively homogeneous. Now let $\alpha \in [1, \infty)$. For any such α, there exists $\lambda > 0$ such that $\lambda\alpha < 1$. Then, $\varphi(v + \alpha\mathbf{1}) = ({}^1\!/\!\lambda)\varphi(\lambda v + \lambda\alpha\mathbf{1}) = ({}^1\!/\!\lambda)\big(\varphi(\lambda v) + \lambda\varphi(\alpha\mathbf{1})\big) = \varphi(v) + \varphi(\alpha\mathbf{1})$ which gives us the desired result. □

And now to finally show that φ is superlinear.

Lemma 5.16. The functions φ as defined above is superlinear.

Proof. We have already established that φ is positively homogeneous. It only remains to establish that φ is concave. To see this, suppose $v, w \in K$ such that $\varphi(v) = \varphi(w)$. Then, there exist $A_v, A_w \in \mathcal{K}^n$ such that $V(A_v) = V(A_w)$ and $u_{A_v} = v$ and $u_{A_w} = w$. Then, $V(A_v + A_w) = 2V\left(\frac{1}{2}(A_v + A_w)\right) \geqslant 2V(A_v)$ by Quasiconcavity. Also $V(A_v) = \frac{1}{2}\left(V(A_v) + V(A_w)\right)$ so that $V(A_v + A_w) \geqslant V(A_v) + V(A_w)$. Thus, $\varphi(v + w) \geqslant \varphi(v) + \varphi(w)$.

If $v, w \in K$ are such that $\varphi(v) \neq \varphi(w)$, let us suppose $\varphi(v) > \varphi(w)$. Let $\alpha := \big(\varphi(v) - \varphi(w)\big)/\varphi(\mathbf{1})$ and define $z := w + \alpha\mathbf{1}$. Then $\varphi(z) = \varphi(w + \alpha\mathbf{1}) = \varphi(w) + \alpha\varphi(\mathbf{1}) = \varphi(w) + \varphi(v) - \varphi(w)$ by the C+-additivity of φ. Using the C+-additivity again, we see that $\varphi(v+w)+\varphi(\alpha\mathbf{1}) = \varphi(v+w+\alpha\mathbf{1}) = \varphi(v+z) \geqslant \varphi(v) + \varphi(z) = \varphi(v) + \varphi(w) + \varphi(\alpha\mathbf{1})$, which gives us $\varphi(v + w) \geqslant \varphi(v) + \varphi(w)$, the desired result. □

We now extend the function to $C^2(S_Z)$ in such a way that it preserves C+-additivity and superlinearity. Define $\psi : K_2^* \to \mathbb{R}$ as $\psi(f) := \varphi(f)$ for all $f \in K_2^*$. The first step we need to take is to observe that $\operatorname{span} K_2^* = C^2(S_Z)$. Let us state this formally.

Theorem 5.17. $\operatorname{span} K_2^* = C^2(S_Z)$. *In particular, for each $f \in C^2(S_Z)$, there exists a convex body K and $r > 0$ such that $f = h_K - h_{D_r}$.*

This is merely Lemma 1.7.9 from Schneider (1993). Another way of stating this is that for each $f \in C^2(S_Z)$, there exists $\alpha > 0$ such that $f + \alpha\mathbf{1} \in K_2^*$.[7] Moreover, for each $\alpha' > \alpha$, $f + \alpha'\mathbf{1} \in K_2^*$. This is because $f + \alpha'\mathbf{1} = (f + \alpha\mathbf{1}) + (\alpha' - \alpha)\mathbf{1}$, the sum of two convex sets, which is also convex.

We now show φ has a unique extension to $C^2(S_Z)$ that preserves the relevant features of φ.

Lemma 5.18. The function $\varphi : K_2^* \to \mathbb{R}$ has a unique C⁺-additive, superlinear and Lipschitz continuous extension to $C(S_Z)$, $\psi : C(S_Z) \to \mathbb{R}$.

Proof. We shall first prove that there exists unique C⁺-additive, superlinear and Lipschitz continuous extension of φ to $C^2(S_Z)$. We shall then extend this extension to $C(S_Z)$.

First, define $\psi(-\mathbf{1}) := -\psi(\mathbf{1})$. For $\alpha > 0$, define $\psi(-\alpha\mathbf{1}) := \alpha\psi(-\mathbf{1})$. More generally (from the theorem above), for each $f \in C^2(S_Z)$, there exists $\alpha > 0$ such that $f + \alpha\mathbf{1} \in K_2^*$. Then define,

$$\psi(f) := \psi(f + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}).$$

Now extend ψ by positive homogeneity. To see that ψ is still well defined, let $\lambda > 0$, and notice that $\lambda f + \lambda\alpha\mathbf{1} \in K_2^*$ and that $\psi\big(\lambda f + \lambda\alpha\mathbf{1}\big) = \lambda\psi(f + \alpha\mathbf{1})$.

We now show that the definition of ψ is independent of the particular α chosen for each f. Suppose $\alpha_1 > 0$ is such that $g + \alpha_1\mathbf{1} \in K_2^*$. As before, define $\psi(g) := \psi(g+\alpha_1\mathbf{1}) - \alpha_1\psi(\mathbf{1})$. Then, for $\alpha_2 > \alpha_1$, we have $\psi(g+\alpha_2\mathbf{1}) - \alpha_2\psi(\mathbf{1}) = \psi\big(g+\alpha_1\mathbf{1}+(\alpha_2-\alpha_1)\mathbf{1}\big) - \alpha_2\psi(\mathbf{1}) = \psi(g+\alpha_1\mathbf{1}) + (\alpha_2-\alpha_1)\psi(\mathbf{1}) - \alpha_2\psi(\mathbf{1}) = \psi(g)$ (where we have used the fact that ψ is C⁺-linear on K^*). Thus, the definition of ψ doesn't depend on the choice of $\alpha > 0$.

We also want to check that ψ is C⁺-additive everywhere on its domain. To see this, let $f \in C^2(S_Z)$ and let $\beta > 0$ be such that $f + \beta\mathbf{1} \in K_2^*$. Clearly, for all $\alpha \geqslant \beta$, $\psi(f + \alpha\mathbf{1}) = \psi(f) + \alpha\psi(\mathbf{1})$. Now suppose $\alpha \in (0, \beta)$. As before, $\psi(f) = \psi(f + \beta\mathbf{1}) - \beta\psi(\mathbf{1})$. Thus, $\psi(f + \alpha\mathbf{1}) = \psi(f + \beta\mathbf{1}) - (\beta - \alpha)\psi(\mathbf{1}) = \psi(f) + \alpha\psi(\mathbf{1})$, which shows that ψ is C⁺-additive.

Now to check that ψ is superlinear. It is clear that $\varphi : K_2^* \to \mathbb{R}$ is superlinear, hence for all $f, g \in K_2^*$, $\psi(f + g) \geqslant \psi(f) + \psi(g)$. Suppose $f, g \in C^2(S_Z)$ and $\psi(f) = \psi(g)$. Then there exists an $\alpha > 0$ such that $f + \alpha\mathbf{1}, g + \alpha\mathbf{1} \in K_2^*$. Then $\psi(f + g) = \psi(f + g + 2\alpha\mathbf{1}) - 2\alpha\psi(\mathbf{1}) \geqslant \psi(f + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}) + \psi(g + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}) = \psi(f) + \psi(g)$.

(7) Here, $\mathbf{1}$ is the constant function with value 1 everywhere.

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Suppose $\psi(f) > \psi(g)$. Let $\alpha\psi(\mathbf{1}) := \psi(f) - \psi(g)$ and define $\tilde{f} := g + \alpha\psi(\mathbf{1})$. Then, $\psi(\tilde{f}) = \psi(g + \alpha\mathbf{1}) = \psi(g) + \alpha\psi(\mathbf{1}) = \psi(f)$. Therefore, $\psi(f + g) + \alpha\psi(\mathbf{1}) = \psi(f + g + \alpha\mathbf{1}) = \psi(f + \tilde{f}) \geqslant \psi(f) + \psi(\tilde{f}) = \psi(f) + \psi(g) + \alpha\psi(\mathbf{1})$, ie, $\psi(f + g) \geqslant \psi(f) + \psi(g)$. Thus, ψ is superlinear.

We conclude with the simple observation that since φ is Lipschitz continuous, so is ψ. Thus, there is a unique extension of ψ to $C(S_Z)$ that we shall also denote ψ. Moreover, this extension is also superlinear and C+-additive. □

5.4. *Proof of Theorem 2.2*

A useful axiom for the proof of the Theorem is

Axiom (Quasiconvexity). $A \sim B$ implies $B \succcurlyeq \frac{1}{2}A + \frac{1}{2}B$.

An allied representation is one that captures *Preference for Diversity (PfD)*

[♣] $$V(A) := \max\left[\langle u_A, A\rangle : \pi \in \Pi\right]$$

Now that we constructed a D-linear utility representation for our preference and extended this utility representation to a Banach space, we are ready to proceed to the final steps of our proof.

We start with the observation that since $\psi : C(S_Z) \to \mathbb{R}$ is a superlinear function, there exists a nonempty convex set $\mathcal{L}$ of continuous linear functions on $C(S_Z)$ such that

$$\psi(f) = \min\left\{L(f) : L \in \mathcal{L}\right\} \text{ for all } f \in C(S_Z).$$

By the Riesz Representation Theorem, for each continuous linear functional L, there exists a regular, countably additive, signed measure π such that $L(f) = \langle f, \pi\rangle$. Thus, we can represent ψ as

$$\psi(f) = \min\left\{\langle f, \pi\rangle : \pi \in \Pi\right\}$$

where Π is a weak* compact, convex set in $M(S_Z)$, the space of all signed, finite Borel measures on S_Z. It is possible to show directly that Π is unique, but we shall employ the following indirect method which will be useful in the sequel.

Recall that $C(S_Z)$ is a Banach space and $M(S_Z)$ is its dual. Indeed, $\langle C(S_Z), M(S_Z)\rangle$ is a dual pair. By Theorem 6.27 of Aliprantis and Border

(1999), we know that the norm topology in $C(S_Z)$ is also the Mackey topology in $C(S_Z)$ (more precisely, in $C(S_Z)$ for the dual pair $\langle C(S_Z), M(S_Z)\rangle$). Thus, ψ is Mackey continuous.

There is the obvious correspondence between sublinear and superlinear functions. Since $\psi(f) = \min\{\langle f, \pi\rangle : \pi \in \Pi\}$ is superlinear, $-\psi$ is sublinear. Moreover, $-\psi(f) = \max\{\langle f, \pi\rangle : \pi \in -\Pi\}$. Finally, by Theorem 5.102 of Aliprantis and Border (1999), we see that there is a bijection between the space of Mackey continuous sublinear functions on $C(S_Z)$ and the space of weak* compact, convex subsets of $M(S_Z)$, so that Π is unique. Indeed, $-\psi$ is the support function of $-\Pi$.

The *hypograph* of a function ψ is the set $\{(f,t) \in C(S_Z) \times \mathbb{R} : \psi(f) \geqslant t\}$. It is easy to see that since ψ is superlinear, the hypograph has a nonempty interior, so that for each $f \in C(S_Z)$, there exists $\pi_f \in \Pi$ such that $\psi(f) = \langle f, \pi_f\rangle$. Moreover, for each $\pi \in \operatorname{ext}\Pi$ (where $\operatorname{ext}\Pi$ represents the set of extreme points of Π), there exists an $f_\pi \in C(S_Z)$ such that $\psi(f_\pi) = \langle f_\pi, \pi\rangle$.

Recall that ψ is C$^+$-additive. Then, for any $\pi \in C(S_Z)$, let $f_\pi =: g + \alpha\mathbf{1}$, where $\alpha > 0$ and f_π is such that $\psi(f_\pi) = \langle f, \pi\rangle$. Then, $\psi(g + \alpha\mathbf{1}) = \psi(g) + \alpha\psi(\mathbf{1}) = \psi(f) = \langle f, \pi\rangle = \langle g, \pi\rangle + \alpha\langle \mathbf{1}, \pi\rangle$. Since $\psi(g) \leqslant \langle g, \pi\rangle$ and $\psi(\mathbf{1}) \leqslant \langle \mathbf{1}, \pi\rangle$, it must be the case that $\psi(\mathbf{1}) = \langle \mathbf{1}, \pi\rangle$.

Since we have assumed $D_R \succ \{p^*\}$, we can set $\varphi(\mathbf{1}) = 1$. With the observation that assuming Quasiconvexity instead of Quasiconcavity leads to a change in the direction of some inequalities above, we get both the (♠) and (♣) representations for this case.

We now consider the case where $\{p^*\} \succ D_R$. Define the preference relation $\succcurlyeq^* \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$ as follows: $A \succcurlyeq^* B$ if and only if $B \succcurlyeq A$. Then, $D_R \succ^* \{p^*\}$ and $\succcurlyeq^*$ satisfies IR and is continuous. Now suppose $A \succ^* B$, so that $B \succ A$. Then, by D-Independence, for all $\lambda \in (0,1]$, $\lambda A + (1-\lambda)D_R \succ \lambda B + (1-\lambda)D_R$, ie, $\lambda B + (1-\lambda)D_R \succ^* \lambda A + (1-\lambda)D_R$, with a similar condition holding for the second part of D-Independence. Thus, $\succcurlyeq^*$ satisfies D-Independence. Finally, notice that $\succcurlyeq$ satisfies Quasiconvexity (resp Quasiconcavity) if and only if $\succcurlyeq^*$ satisfies Quasiconcavity (resp Quasiconvexity).

Thus, given preference relation $\succcurlyeq$ that satisfies IR, Continuity and L Continuity, D-Independence and Quasiconvexity (resp Quasiconcavity) and has $\{p^*\} \succ D_R$, there exists a preference relation $\succcurlyeq^*$ that satisfies IR, D-Independence and Quasiconcavity (resp Quasiconvexity) and has $D_R \succ \{p^*\}$ with the property that $A \succcurlyeq B$ if and only if $B \succcurlyeq^* A$. But by the proof above, we know that $\succcurlyeq^*$ has a utility representation $V^* : \mathcal{F}(\Delta) \to \mathbb{R}$ that admits a

(♠) representation:

$$V^*(A) := \min \left\{ \langle u_A, \pi \rangle : \pi \in \Pi \right\}.$$

Now define, $V : \mathcal{F}(\Delta) \to \mathbb{R}$ as follows: $V(A) = -V^*(A)$. Then, V represents $\succcurlyeq$, and can be written as

$$V(A) := \max \left\{ \langle u_A, \mu \rangle : \mu \in -\Pi \right\},$$

which is the (♠) representation for $\succcurlyeq$. Similarly, if $\succcurlyeq$ satisfies Quasiconcavity, using the arguments above we can show it has a (♠) representation.

This demonstrates the 'only if' part of the proof. The 'if' part is easily seen. The only fact that needs checking is that the preference functional in (♠) is Lipschitz. Recall first that the set of signed measures Π is weak* compact, so that there exists $\Lambda > 0$ such that $\sup_{\pi \in \Pi} \|\pi\| \leqslant \Lambda$ (where $\|\pi\|$ is the total variation norm). Also, for any menu $A \in \mathcal{F}$, let $\pi_A \in \Pi$ be such that $V(A) = \langle u_a, \pi_A \rangle$, and notice that $\langle u_A, \pi_A \rangle \leqslant \langle u_A, \pi \rangle$ for all $\pi \in \Pi$. Notice then that $|V(A) - V(B)| = |\langle u_A, \pi_A \rangle - \langle u_B, \pi_B \rangle| \leqslant |\langle u_A, \pi_B \rangle - \langle u_B, \pi_B \rangle| \leqslant |\langle u_A - u_B, \pi_B \rangle| \leqslant \|u_A - u_B\|_\infty \|\pi_B\| \leqslant \Lambda \|u_A - u_B\|_\infty$. It follows that V is Lipschitz since the distance between A and B in the Hausdorff metric is exactly $\|u_A - u_B\|_\infty$. It follows immediately that $\succcurlyeq$ satisfies L Continuity.

This concludes the proof of Theorem 2.2. Before we proceed, we state, without proof, a simple proposition that follows from the construction of the set of support functions Π.

Proposition 5.19. For each $\pi \in \operatorname{ext} \Pi$, there exists a menu A such that $V(A) = \langle u_A, \pi \rangle$.

5.5. *Proof of Theorem 2.4*

As in the text, say that $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if, for each A where $D_R \succ_1 A \succ_1 \{p^*\}$, $\lambda D_R + (1-\lambda)\{p^*\} \sim_1 A$ implies $\lambda D_R + (1-\lambda)\{p^*\} \succcurlyeq_2 A$. Let V_i represent $\succcurlyeq_i$ and be such that $V_i(\{p^*\}) = 0$ and $V(D_R) = R$. Clearly, V_i is D-linear. We shall first show that $\succcurlyeq_2$ is more pessimistic that $\succcurlyeq_1$ if and only if for each menu A, $V_1(A) \geqslant V_2(A)$.

Proposition 5.20. The preference $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if and only $V_1(A) \geqslant V_2(A)$ for each menu A.

Proof. Notice that for any A such that $D_R \succ_1 A \succ_1 \{p^*\}$, $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if and only if $V_1(A) \geqslant V_2(A)$, since $V_1(A) = V_1(\lambda_A D_R + (1-\lambda_A)\{p^*\}) = V_2(\lambda_A D_R + (1-\lambda_A)\{p^*\}) \geqslant V_2(A)$.

Now consider an A where $\{p^*\} \succcurlyeq_1 A$. For such an A, there exists a μ such that $D_R \succ_1 \mu A + (1-\mu)D_R \succ_1 \{p^*\}$ and for such a μ, $V_1(\mu A + (1-\mu)D_R) \geqslant V_2(\mu A + (1-\mu)D_R)$ which is true if and only if $V_1(A) \geqslant V_2(A)$ (using the D-linearity of V_i).

Similarly, for a menu A with $A \succcurlyeq_1 D_R$, there exists a μ such that $D_R \succ_1 \mu A + (1-\mu)\{p^*\} \succ_1 \{p^*\}$. Once again using the D-linearity of the V_i, we see that $V_1(A) \geqslant V_2(A)$ and that this is equivalent to saying that $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$. □

Recall from the proof of theorem 2.2, the existence (and construction) of a unique function $\psi : C(S_Z) \to \mathbb{R}$ such that for any menu A, $\psi(h_A) = V(A)$. Moreover, ψ is C⁺-additive, sublinear and continuous (and so is Lipschitz continuous). We now show that if $\succcurlyeq_2$ is more pessimistic than $\succcurlyeq_1$, it is the case that $\psi_1 \geqslant \psi_2$. To see this, let $f \in C^2(S_Z)$. Therefore, there is an $\alpha > 0$ such that $f + \alpha\mathbf{1}$ is sublinear and therefore the support function of some compact, convex set. Thus, for some $\lambda > 0$, there exists a menu A with support function $h_A = (1/\lambda)(f + \alpha\mathbf{1})$. Using the C⁺-additivity of the ψ_i's, we see that $\psi_1(f) \geqslant \psi_2(f)$. Using the density of $C^2(S_Z)$ and the continuity of ψ_i, we reach the desired conclusion.

As established in section 5.4, ψ_i is Mackey continuous and sublinear, hence $\psi_1 \geqslant \psi_2$ implies $\Pi_1 \subset \Pi_2$ (see section 5.19, Aliprantis and Border, 1999). It follows immediately that $S_1 \subset S_2$. We have thus shown that if $\succcurlyeq_2$ is more pessimistic than $\succcurlyeq_1$, then $\Pi_1 \subset \Pi_2$. The converse follows from the fact that $\Pi_1 \subset \Pi_2$ implies $\psi_1 \geqslant \psi_2$ so that $V_1(A) \geqslant V_2(A)$ for any menu A.

5.6. *Special Cases*

Here we shall prove the claims from section 3.

Proof of Proposition 3.2. The "if" part is clear. To see the "only if" part, let us suppose that there exists $\pi \in \Pi$ that is *not* a probability measure. Then, there exists a decomposition $\pi := \pi_+ - \pi_-$, where π_+ and π_- are both positive measures. Since the space of all (signed) measures on S_Z with total weight 1 is metrisable, and since the set of probability measures is a closed subset of this space, it follows that there exists a smooth function $f \in \mathbb{R}^{S_Z}$ with $f \geqslant \mathbf{0}$

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such that $\int f\pi < 0 \leqslant \int f\mu$ for all $\mu \in \Pi$ that *are* probability measures.[8] But there exist $\alpha, \beta > 0$ such that $\beta\big[f + \alpha\mathbf{1}\big]$ is the support function of some menu that contains the disk D. (Recall that the support function of D is $h_D = \mathbf{1}$.) Then, by an appropriate choice of β, we can show that $\beta\big[f + \alpha\mathbf{1}\big] \geqslant \mathbf{1}$ but $1 = \int \mathbf{1}\pi > \beta\big[\int (f + \alpha\mathbf{1})\pi\big]$. Thus, $R = V(D) > V(A)$, which contradicts Monotonicity. Thus, every $\pi \in \Pi$ must be a probability measure. □

Proof of Proposition 3.4. Suppose not, so suppose there is $\pi_A \in \operatorname{ext}\Pi$ such that $\pi_A^1, \pi_A^2 > 0$. Then, by proposition 5.19, there exists a menu A with a relative interior such that $V(A) = \langle \pi_A, u_A \rangle$. Now, for $j = 1, 2$, define B_j such that $u_{B_j}^i = u_A^i$ for $i \neq j$ and $u_{B_j}^j = u_A^j - \varepsilon > 0$ for some appropriate $\varepsilon > 0$. Then, $V(B_j) = \big\langle \pi_{B_j}, u_{B_j} \big\rangle \leqslant \big\langle \pi_A, u_{B_j} \big\rangle < \langle \pi_A, u_A \rangle = V(A)$. Thus, $V(A) = V(B_1 \cup B_2) > V(B_1), V(B_2)$, contradicting Positive Set Betweenness. □

The proof of Proposition 3.5 follows from a simple transformation. Consider a preference relation $\succcurlyeq$ that admits a finite PfU representation, V. Now define the relation $\succcurlyeq^*$ so that for any two menus $A, B \in \mathcal{F}(\Delta)$, $A \succcurlyeq B$ if and only if $B \succcurlyeq^* A$. Then, $\succcurlyeq^*$ admits a finite PfU representation, namely $-V$.

Now suppose $\succcurlyeq$ satisfies Negative Set Betweenness, ie, $A \succcurlyeq B$ implies $A \cup B \succcurlyeq B$. Then, $\succcurlyeq^*$ satisfies Positive Set Betweenness, ie, $B \succcurlyeq^* A$ implies $B \succcurlyeq^* B \cup A$. The proof of proposition 3.5 now follows from the observation that for any finite additive EU representation U_π that has exactly one positive state, $-U_\pi$ has exactly one negative state.

Proof of Theorem 3.6. We shall only prove that (i) implies (ii). That (ii) implies (i) is trivial.

The above proofs tell give us part of the characterisation theorem 3.6. To show that a state cannot be both positive and negative, we construct a counterexample.

Notice that if preferences satisfy Set Betweenness, it must be the case that $|S| = 2$. Now suppose a state is both positive and negative. In other words, $\Pi = \operatorname{conv}\{\pi, \mu\}$ where $\pi^1 + \pi^2 = \mu^1 + \mu^2 \in \{-1, 1\}$. For simplicity, let us assume that $\pi^1 + \pi^2 = 1$. Suppose, $\pi = (-\pi_1, 1 + \pi_1)$ and $\mu = (\mu_1, 1 - \mu_1)$ and suppose utilities to two singletons are as given below:

(8) The details of the construction may be found in Chatterjee and Krishna (2009). The main idea is that π_+ and π_- are concentrated on disjoint sets, and the function f can be chosen to be defined almost exclusively on the support of π_-.

	u^1	u^2
s	$\alpha+\varepsilon$	0
c	0	α

where $\alpha, \varepsilon > 0$. Notice that by propositions 3.4 and 3.5, it must be that $\pi_1, \mu_1 > 1$.

It is easily seen that for any menu A, $\langle \pi, u_A \rangle \leqslant \langle \mu, u_A \rangle$ if and only if $(\pi^1 - \mu^1)(u_A^1 - u_A^2) \leqslant 0$. Therefore, $V(\{s\}) = \langle \pi, u_{\{s\}} \rangle = -\pi_1(\alpha+\varepsilon 0$, $V(\{c\}) = \langle \mu, u_{\{c\}} \rangle = (1-\mu_1)\alpha$ and $V(\{s,c\}) = \langle \pi, u_{\{s,c\}} \rangle = -\pi_1(\alpha+\varepsilon) + (1+\pi_1)\alpha$.

Clearly, $V(\{s,c\}) > V(\{s\})$ if and only if $-\pi_1\varepsilon + \alpha > -\pi_1\alpha$ and $V(\{s,c\}) > V(\{c\})$ if and only if $-\pi_1\varepsilon + \alpha > (1-\mu_1)\alpha$. The first inequality holds if and only if $\pi_1(\alpha - \varepsilon) + \alpha > 0$ and the second inequality holds if and only if $-\pi_1\varepsilon + \mu_1\alpha > 0$. It is easy to see that for any $\pi_1, \mu_1 > 1$, we can satisfy the two inequalities by taking α sufficiently large and ε sufficiently small, giving us the desired violation of Set Betweenness. $\square$

We now provide a proof of proposition 3.7.

Proof of Proposition 3.7. We shall only prove that (i) implies (ii). The other implication is trivial. Let us assume that V is a PfU representation and satisfies Set Betweenness and Singleton Independence. From theorem 3.6, we see that $|S| \leqslant 2$. Clearly, the only case of interest is where $|S| = 2$.

We shall first assume that for $S := \{s_1, s_2\}$, $s_1 \neq -s_2$. Let

$$K_S := \big\{u \in \mathbb{R}^2 : u = \big(u_A(s_1), u_A(s_2)\big) \text{ for some } \quad A \in \mathcal{K}(\text{aff } \Delta)\big\}$$

and let

$$F_S := \big\{u \in \mathbb{R}^2 : u = \big(u_{\{p\}}(s_1), u_{\{p\}}(s_2)\big) \text{ for some } \quad p \in \text{aff } \Delta\big\}\,.$$

Clearly, $F_S \subset K_S$. But since $F_S = \mathbb{R}^2$ (see proposition 5.2), it follows that $F_S = K_S$. But if V satisfies Singleton Independence, then it must be linear on F_S and consequently on K_S, which proves our claim.

We now deal with the case where $S := \{s, -s\}$. In this case, it is easy to see that $F_S := \{u \in \mathbb{R}^2 : u^1 + u^2 = 0\}$ and $K_S := \{u \in \mathbb{R}^2 : u^1 + u^2 \geqslant 0\}$. Since V is linear on F_S, it must be the case that for all $\pi, \mu \in \Pi$, $\langle u, \pi - \mu \rangle = 0$, where $u \in F_S$ that is not the zero vector. But this implies that $\pi_1 u^1 + (1-\pi_1)(-u^1) = \mu_1 u^1 + (1-\mu_1)(-u^1)$ which holds if and only if $\pi_1 = \mu_1$, ie, if and only if $\pi = \mu$. Thus, Π must be a singleton which proves our claim. $\square$

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References

Ahn, D (2008). "Ambiguity without a State Space," *Review of Economic Studies*, **75**, 3–28.

Aliprantis, C D and K C Border (1999). *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 2nd ed., Springer, New York.

Chatterjee, K and R V Krishna (2007). "Menu Choice, Environmental Cues and Temptation: A "Dual-Self" Approach to Self-control," working paper.

Chatterjee, K and R V Krishna (2008). "A 'Dual Self' Representation for Stochastic Temptation," forthcoming, *American Economic Journal – Microeconomics.*

Chatterjee, K and R V Krishna (2009). "On Preferences with Infinitely many Subjective States," forthcoming, *Economic Theory.*

Dekel, E, B Lipman and A Rustichini (2001). "Representing Preferences with a Unique Subjective State Space," *Econometrica*, **69(4)**, 891–934.

Dekel, E, B Lipman and A Rustichini (2008). "Temptation Driven Preferences," forthcoming, *Review of Economic Studies.*

Dekel, E, B Lipman, A Rustichini and T Sarver (2007). "Representing Preferences with a Unique Subjective State Space: A Corrigendum," *Econometrica*, **75(2)**, 591–600.

Epstein, L and I Kopylov (2007). "An Axiomatic Model of 'Cold Feet'," **2(3)**, *Theoretical Economics.*

Epstein, L, M Marinacci and K Seo (2007). "Coarse Contingencies and Ambiguity," **2(4)**, *Theoretical Economics.*

Fudenberg, D and D Levine (2006). "A Dual Self Model of Impulse Control," *American Economic Review*, **96**, 1449–1476.

Gilboa, I and D Schmeidler (1989). "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics*, **18**, 141–153.

Gul, F and W Pesendorfer (2001). "Temptation and Self-Control," *Econometrica*, **69**, 1403–1435.

KREPS, D (1979). "A Representation Theorem for 'Preference for Flexibility'," *Econometrica*, **47**, 565–576.

MACCHERONI F, M MARINACCI AND A RUSTICHINI (2006). "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, **74**, 1447–1498.

NOOR, J AND N TAKEOKA (2010). "Uphill Self-Control," *Theoretical Economics*, **5(2)**, 127–158.

NOOR, J AND N TAKEOKA (2010). "Menu Dependent Self-Control," working paper, Boston University.

OK, E (2007). *Real Analysis with Economic Applications*, Princeton University Press, Princeton, NJ.

OLSZEWSKI, W (2007). "Preferences over Sets of Lotteries," *Review of Economic Stidues*, **74**, 567–595.

OLSZEWSKI, W (2008). "A Model of Temptation," working paper.

SCHELLING, T (1978). "Egonomics, or The Art of Self-management," *American Economic Review*, **68(2)**, 290–294.

SCHELLING, T (1984). "Self-command in Practice, in Policy and in a Theory of Rational Choice," (Richard T. Ely lecture), *American Economic Review*, **74(2)**, 1–11.

SCHNEIDER, R (1993). *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press.

STINCHCOMBE, M (2007). "Consequentialist Ambiguity: Choice under Ambiguity as Choice Between Sets of Probabilities," working paper, University of Texas, Austin.

VAN TIEL, J (1984). *Convex Analysis: An Introductory Text*, Wiley.

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