Decision Making When Outcomes Are Uncertain
Class Notes

Outline
1. Introduction

2. Exercise One: Ranking Artificial Securities
Students will apply the concepts of expected value, variance, and "Marcowitz" risk to describe the tradeoffs implied in the choice of securities.

3. Exercise Two: Risk Aversion
a. Students will investigate the meaning of risk aversion and the reasonableness of the hypothesis that the typical agent is risk averse.
b. Students will decide on their reservation price for playing the "St. Petersburg" game.

4. Expected Utility Hypothesis
Students will work with a set of notes that introduces the expected utility hypothesis and provides an explanation of the St. Petersburg paradox.

5. Exercise Three: Diversification
Students will complete an exercise that introduces them to the concept of diversification and explains the relationship between diversification and the statistical covariance.

Introduction

Let (S_{t+1}, S_{t+2}, ...S_{t+M}) be payments that will be received at times t +1, t +2, ..., t +M and let R be the discount rate. The present value of the stream is:

$$PV_t = \frac{S_{t+1}}{(1+R)} + \frac{S_{t+2}}{(1+R)^2} + \frac{S_{t+3}}{(1+R)^3} + \dots + \frac{S_{t+M}}{(1+R)^M}$$

When choosing among streams of payments, it is often reasonable to prefer a stream with a higher present value to one with a lower present value. If the stream of payments is associated with a bond issued by the U.S. Treasury, it is reasonable to assume that there is no uncertainty associated with the size or timing of the payments.

But the payments made by many assets are uncertain. For example, when we purchase a share of common stock we do not know for sure what dividends we will receive. The purpose of these notes and exercises is to introduce concepts that are used in models describing how agents make decisions when outcomes are uncertain.

Exercise One: Decision Making When Outcomes Are Uncertain

In this exercise, students will decide which of four "securities" they prefer and learn how analysts measure the expected pay out and riskiness of securities. For simplicity, we consider securities for which there are six possible outcomes. Outcomes are equally likely and chosen by the roll of a fair die. Each security costs $1.00. Pay outs are given by the following table.

Security	Table of Pay Outs					
	1	2	3	4	5	6
A	$0.50	$0.50	$1.00	$1.00	$2.00	$2.00
B	$0.00	$0.50	$1.00	$1.00	$2.00	$2.50
C	$0.50	$0.50	$1.00	$1.00	$1.00	$3.00
D	$0.00	$0.50	$1.00	$1.00	$2.00	$3.00

A. Which security is your first choice, second choice, etc.?

Ranking: A_________ B________ C_________ D_________

Is there a security for which you would not pay a dollar even if it were the only one available?

B. Prior to modern finance theory, it was assumed that agents would rank securities on the base of their expected payout. Expected payout is the probability weighted average of the payout in each state of the world. What is the expected payout securities A through D?

Expected Payout (:): A_________ B________ C_________ D_________.

C. The chief problem with ranking securities on the basis of expected payout alone is that such a ranking completely ignores the relative riskiness of the securities. There are two measures of risk which are typically used in modern finance theory.

Variance of the Payout: $E\,(1/6)(y_i - :)^2$

Marcowitz Risk Measure: (Probability of a loss)(Average loss given a loss occurs)

What are the risk measures associated with each security?

Variance: A_________ B________ C_________ D_________

Marcowitz: A_________ B________ C_________ D_________

D. Discussion: What role does risk play in the ranking you made in part A?

Exercise Two: Risk Aversion

The purpose of this exercise is to introduce the concept of risk aversion. Risk aversion is important in understanding how securities are priced, why markets for insurance exist, and is a key insight into the modern approach to the economics of decision making in the presence of uncertainly.

Game 1 (Coin Toss): The following is a set of alternative games. You may choose to play at most one game. For each game, there will be a single toss of a fair coin. If the outcome is "heads" you win the amount in the win column. If "tails", you lose the amount in the lose column. Are you willing to play a game? If your answer is "yes", which game do you prefer?

Game	Lose	Win
A	$ 1.00	$ 1.00
B	$ 2.00	$ 2.00
C	$ 5.00	$ 5.10
D	$ 10.00	$ 10.40
E	$ 20.00	$ 21.00
F	$ 50.00	$ 53.00
G	$100.00	$110.00
H	$500.00	$525.00

I am willing to play_________________.

Which game has the largest expected return? Did you prefer that game? If not, why?

Game 2 (St. Petersburg): There will be only one play of the game. Players are required to pay a price to participate. The price is the only money the player can lose. If a player pays the price, a fair coin is tossed until "heads" appears. The player is paid a prize according to the following table.

Toss when first head appears	1	2	3	4	5	m
Prize	.25	.50	1.00	2.00	4.00	$(2^{m-1}) \times (.25)$

A Reservation Price is the most a buyer would be willing to pay to obtain a good or service. It is the least a seller would be willing to accept in exchange for a good or service.

Question: What is your reservation price for one play of the game?______________________.

Discussion: What is the expected payout of the St. Petersburg game? Is it reasonable to believe that agents use only expected value calculations when making choices in the face of uncertainty?

The Expected Utility Hypothesis

The expected utility hypothesis was first put forward by Swiss mathematician Daniel Bernoulli (1700-1782). The hypothesis says that when faced with decisions involving uncertain outcomes, agents choose the strategy that maximizes their expected utility. The idea was quite novel when Bernoulli introduced it. Until Bernoulli, the standard rule of thumb was to choose the strategy that provided the highest expected value. Today, the expected utility hypothesis has become the standard hypothesis used to model decision making in the face of uncertainty.

To better understand the hypothesis, we may consider a simple example. Suppose an individual is faced with the opportunity to invest in a project that has the statistical properties of a fair coin toss. Specifically, the agent will invest \$50.00 and has a 50 percent probability that the investment is a success. If the investment succeeds, the agent receives \$100.00; if it does not succeed, she receives nothing. An expected value maximizer would be indifferent between investing and not investing because the expected value of the investment is zero.

$$\text{Expected Value} = -\$50 + (.5)\$100 + (.5)\$0 = \$0$$

The expected utility hypothesis begins with the specification of a utility of wealth function. Bernoulli argued that the typical investor was risk averse and had a utility of wealth function that was concave. A good example of such a utility function is the log function:

$$U(W) = \log(W/W_0)$$

where W is post-decision wealth and W_0 is pre-decision wealth. Since the outcome of the decision is random, W is random. W_0 is not random. Scaling wealth by W_0 permits the degree of risk aversion to depend on the wealth of the individual.

The expected utility hypothesis says that an individual will choose the option that provides the highest expected utility. Suppose our agent has initial wealth of \$100. There are two options: invest or not. What will the agent do?

$$EU_{\text{not invest}} = \log(W_0/W_0) = \log(1) = 0$$

$$\begin{aligned} EU_{\text{invest}} &= (.5)\log(50/100) + (.5)\log(150/100) \\ &= (.5)\log(.5) + (.5)\log(1.5) \\ &= (.5)(-.301) + (.5)(.176) = -.063 \end{aligned}$$

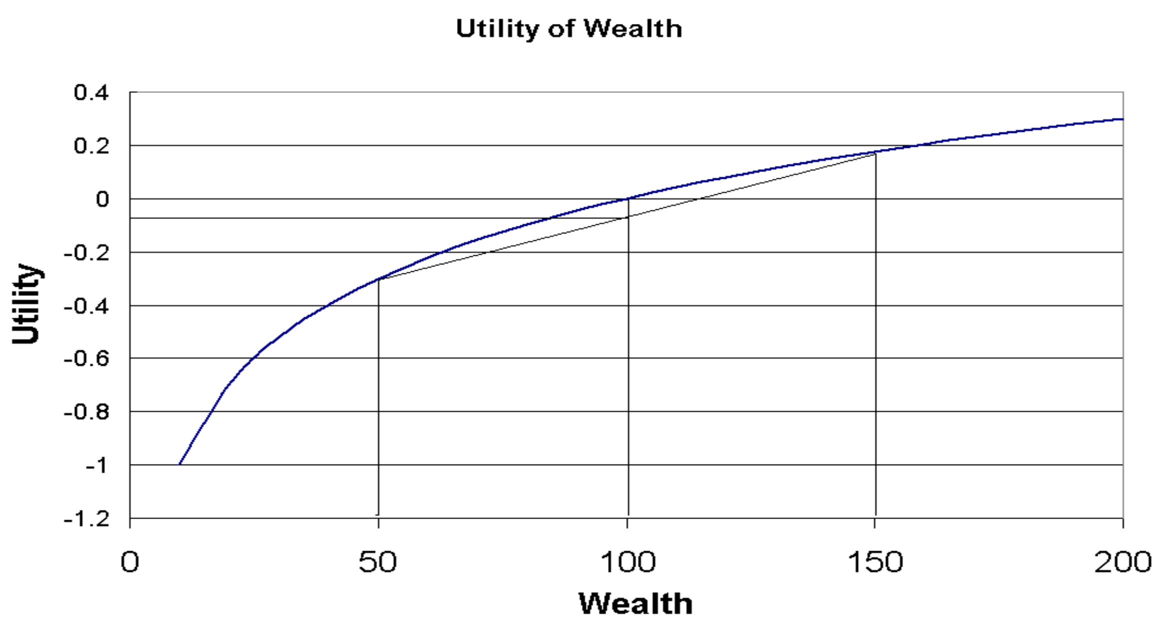
The agent will not invest because the expected utility of not investing is higher than the expected utility of investing. A graph of the utility function provides a key insight: the marginal utility associated with losing a dollar is greater than the marginal utility associated with gaining a dollar.

A risk-averse agent who owned an investment like the one of our example would actually be prepared to buy insurance in order to avoid the uncertain outcome. If the agent owns the investment, her expected utility level is -.063. For what value of x is log(x/100) = -.063? The answer is x = \$86.50. The agent would prefer any certain amount over \$86.50 to a random drawing with an expected value of \$100.00. There is an opportunity for a company to sell our agent insurance.

The following table gives the utility of various levels of wealth for an individual whose initial wealth is \$100. The utility function is the log function proposed by Daniel Bernoulli, $U(W) = \log(W/W_0)$. When wealth is below initial wealth, W/W_0 is less than one and utility is negative. When wealth is above initial wealth, W/W_0 is greater than one and utility is positive.

The Expected Utility Hypothesis of Daniel Bernoulli				
			Wealth	Utility
Initial Wealth			10	-1
100			20	-0.699
Utility of Initial Wealth			30	-0.523
0			40	-0.398
Probability of Win			50	-0.301
0.5			60	-0.222
Size of Gamble			70	-0.155
50			80	-0.097
Expected Value of Gamble			90	-0.046
0			100	0
			110	0.041
Expected Utility of Gamble			120	0.079
-0.062			130	0.114
			140	0.146
			150	0.176
			160	0.204
			170	0.230
			180	0.255
			190	0.279
			200	0.301

The graph of the utility of wealth function is presented in the following figure. The figure also illustrates Bernoulli's result that an expected utility mazimizer will prefer \$100 to a lottery that has an expected outcome of \$100.



Exercise Three: Diversification and the Capital Asset Pricing Model

Diversification

Suppose you have $1.00 to invest. You may choose one option from the table.

Option 1 is to invest all of your funds in asset one. Option 2 is to invest all of your funds in asset 2. Option 3 is to split your funds equally between asset one and two given that the pay outs are independent. Option 4 is to split your funds equally between asset one and two given that the pay outs are positively correlated. Option 5 is to split your funds equally between asset one and two given that the pay outs are negatively correlated.

	Expected Return and Risk of Five Investment Options				
	Option 1 Hold only Asset One	Option 2 Hold Only Asset Two	Option 3 Split Funds between One and Two Pay outs are independent	Option 4 Split Funds between One and Two Pay outs are pos. correlated	Option 5 Split Funds between One and Two Pay outs are neg. correlated
	$2.00 prob .5 0.50 prob .5	$1.75 prob .5 0.50 prob .5	$1.875 prob .25 1.125 prob .25 1.250 prob .25 0.500 prob .25	$1.875 prob .40 1.125 prob .10 1.250 prob .10 0.500 prob .40	$1.875 prob .10 1.125 prob .40 1.250 prob .40 0.500 prob .10
Expected Return					
Marcowitz Risk					

Questions

1. Which option do you prefer?

2. Does diversification lower risk? How do you know?

3. Is there a cost to lowering risk through a strategy of diversification?

4. How do the costs and benefits of diversification depend on the co-variation of asset returns?

5. Option 4 specifies that the returns of two assets are positively correlated. What real world investment options are likely to have returns that are positively correlated?

6. Option 5 specifies that the returns of two assets are negatively correlated. What real world investment options are likely to have returns that are negatively correlated?