Economics 185 **Michael Salemi**

Notes on Present Value and Internal Rate of Return

1. Assets such as bonds and stocks are held because they give the owner the right to receive a **stream of payments**.

Examples of Streams of Payments

BOND	C_t	C_{t+1}	....	C_{t+m}	+	Maturity Value
	\$100	\$100		\$100	+	\$1000
STOCK	D_t	D_{t+1}	....	D_{t+m}	+	Sale Price at Date t+m
	\$50	\$60		\$40	+	P_{t+m}

2. **Present Value**

Because a dollar received in the future is less valuable than a dollar in hand today, we need some procedure to evaluate the current value of future payments. In these notes, we assume that all payments will occur without any risk of default. The concept we will use to compute the value of future payments is "present value."

Suppose the interest rate is expected to be constant over the M-year life of the asset and suppose the asset promises to pay S_{t+j} dollars in period t+j. Then present value is defined to be:

$$PV_t = \frac{S_{t+1}}{(1+R)} + \frac{S_{t+2}}{(1+R)^2} + \frac{S_{t+3}}{(1+R)^3} + \ldots + \frac{S_{t+M}}{(1+R)^M}$$

How to Think About Present Value

The present value of a stream of payments is an amount that would permit you to replicate the stream provided you can borrow and lend at the same rate of interest used in the PV computation.

Example:

Let R = .08. Consider a three year bond with 11% coupon and par value of \$1000. The stream of payments is: \$110 at end of years 1 and 2, and \$1110 at the end of year 3. Applying the above formula, we can compute the present value of this stream as follows.

$$PV = \frac{\$110.}{1.08} + \frac{\$110.}{(1.08)^2} + \frac{\$1110.}{(1.08)^3} = \$1077.31$$

But what does it mean for the present value to be \$1077.31? The following table shows that with \$1077.31 in hand, one has just enough to replicate the stream of payments that the bond promises to pay.

Year	Balance	Earned Interest	Year-end Payout
1	\$1077.31	86.18	\$110.
2	\$1053.49	84.28	\$110.
3	\$1027.77	82.22	\$110. + \$1000.
4	0.00		

Generalization of Present Value Formula

The previous definition of present value assumed a constant rate of interest. Next we define present value for a situation where the interest rate is **not** expected to remain constant in the future. Suppose the interest rate is expected to be R_1 in the first year (t to t+1), R_2 in the second year (t+1 to t+2) and so forth. The stream of payments is $S_{t+1}, S_{t+2}, ..., S_{t+M}$. In the present value formula, payment S_{t+j} is discounted by dividing by a product of interest rate terms. (If $R_j = R$ for all j, the following formula simplifies to the one on the previous page.)

$$PV_t = \frac{S_{t+1}}{(1+R_1)} + \frac{S_{t+2}}{(1+R_1)(1+R_2)} + \frac{S_{t+3}}{(1+R_1)(1+R_2)(1+R_3)} + \dots + \frac{S_{t+M}}{(1+R_1)\dots(1+R_M)}$$

Example: $M = 3$, $S_{t+1} = S_{t+2} = 110$, and $S_{t+3} = 1110$

$R_1 = .08$, $R_2 = .09$, $R_3 = .10$

$$PV_t = \frac{110}{1.08} + \frac{110}{(1.08)(1.09)} + \frac{1110}{(1.08)(1.09)(1.10)} = \$1052.49$$

Continuous Time Discounting

The series $\left(\frac{1}{1+R}\right), \left(\frac{1}{1+R}\right)^2, \left(\frac{1}{1+R}\right)^3, ...$

is a discrete-time discount function. Discrete time discounting is appropriate when it is reasonable to hypothesize that time is a sequence of years, months, weeks or days in which case the right R to use is a yearly, monthly, weekly, or daily interest rate respectively. In some settings it is reasonable to hypothesize that time evolves continuously. When time evolves continuously, the right discount function is $e^{-R(s-t)}$ for $s \geq t$ where R is a continuously compounded rate. If we measure time in years, the sequence corresponding to that above is e^{-R}, e^{-2R}, e^{-3R}.

Example: Let R = .10

The following table compares the discrete and continuous time discount factors for horizons of between one and five years.

K	1	2	3	4	5
$\left(\frac{1}{1+R}\right)^K$	.909	.826	.751	.683	.621
e^{-KR}	.905	.819	.741	.670	.607

3. Internal Rate of Return

Internal rate of return (IRR) is another concept used to evaluate a stream of payments. When one knows the price of the asset and the stream of payments that it promises, it is standard to ask how much the asset yields. Internal rate of return provides the answer to this question. If the asset is a bond, internal rate of return is called the yield to maturity of the bond.

Consider the stream S_{t+1}, S_{t+2}, ..., S_{t+M} with market price PS_t. The internal rate of return (IRR) is that number such that the market price is equal to the present value using IRR as a discount rate. Note that the definition of IRR is implicit rather than explicit.

$$PS_t = \frac{S_{t+1}}{(1+IRR)} + \frac{S_{t+2}}{(1+IRR)^2} + \ldots + \frac{S_{t+M}}{(1+IRR)^M}$$

Example

$$PS_t = 900, \quad M = 3, \quad S_{t+1} = S_{t+2} = \$100, \quad S_{t+3} = \$1100$$

$$900 = \frac{100}{(1+IRR)} + \frac{100}{(1+IRR)^2} + \frac{1100}{(1+IRR)^3}$$

The formula may be rewritten as a cubic equation:

$$(1+IRR)^3\,900 = (1+IRR)^2\,100 + (1+IRR)\,100 + 1100$$

Typically, computer algorithms are used to solve for internal rate of return However, there is an IRR approximation formula. Let a bond with M years to maturity have an annual coupon of C and a market price of PS

$$IRR \approx \frac{C + \frac{1000-PS}{M}}{\frac{1000+PS}{2}} = \frac{100 + \frac{1000-900}{3}}{\frac{1000+900}{2}} = \frac{\$133.33}{950} = .140$$

4. Present Value with Constant Periodic Payments

There is a simplification of the present value formula that holds if the annual coupon is constant. If the coupons is C per year and par value is F and maturity in M years, present value is:

$$PV = \frac{C}{1+R} + \frac{C}{(1+R)^2} + \ldots + \frac{C}{(1+R)^M} + \frac{F}{(1+R)^M}$$

$$= \frac{C}{R} + (F - C/R)\left(\frac{1}{1+R}\right)^M$$

Let F = \$1000 and C = \$100. The following table gives present value for various values of R and M.

Present Value					
R\M	1	2	5	30	4
.08	\$1018.52	\$1035.67	\$1079.85	\$1225.16	\$1250.00
.10	\$1000.00	\$1000.00	\$1000.00	\$1000.00	\$1000.00
.12	\$982.14	\$966.20	\$927.90	\$838.39	\$833.33

5. Return of a Financial Asset

The return of a financial asset is a very different concept from the present value of an asset although the two concepts are related. Return on an asset can be defined retrospectively or prospectively.

Retrospective Return

$$\text{Return}(t-1,t) = \frac{C + P_t - P_{t-1}}{P_{t-1}}$$

Prospective Return

$$\text{Return}(t,t+1) = \frac{C + P_{t+1} - P_t}{P_t}$$

Questions

1. Explain how the above present value table shows that long-maturity bonds are riskier than short-maturity bonds.
2. How accurate is the approximation formula for IRR for a five year note with a \$100 coupon and a price of \$1225.16.
3. How can one use present value to compute an prospective return for a ten year bond that you buy in 2004 and expect to seel in 2005.
4. You buy a ten year bond in 2004 and sell it in 2005. Under what conditions will the retrospective return you compute in 2005 turn out to be lower than the prospective return you compute in 2004?