MATH 681 Introductory Topology : homework assignment one (correction)

Problem 1 was stated incorrectly:

1. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. Prove that

$$\mathcal{T} \cap \mathcal{T}' := \{U \cap U' | U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$$

is also a topology on X.

The intersection of two topologies should have been defined as

$$\mathcal{T} \cap \mathcal{T}' := \{U | U \in \mathcal{T} \text{ and } U \in \mathcal{T}'\}.$$

Then it is fairly easy to show that $\mathcal{T} \cap \mathcal{T}'$ *is* a topology on X.

Note that in general

$$\{U \cap U' | U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$$

is not a topology on X. For example, let $X = \{1, 2, 3\}$,

$$\mathcal{T} := \{\emptyset, \{1\}, X\},$$

and

$$\mathcal{T}' := \{\emptyset, \{2\}, X\}.$$

Then

$$\{U \cap U' | U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\} = \{\emptyset, \{1\}, \{2\}, X\}$$

which is *not* a topology on X, as it does not contain $\{1\} \cup \{2\} = \{1, 2\}$.

The error in the solution is the statement

$$\bigcup_{\alpha \in A} (U_\alpha \cap U'_\alpha) = (\bigcup_{\alpha \in A} U_\alpha) \cap (\bigcup_{\alpha \in A} U'_\alpha).$$

The left hand side is contained in the right hand side, but in general they are *not* equal.