MATH 681 Introductory Topology : homework assignment one

1. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. Prove that

$$\mathcal{T} \cap \mathcal{T}' := \{U \cap U' | U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$$

is also a topology on X.

2. Let $X = \mathbb{R}$ and

$$\mathcal{B} := \{[a, b) | a < b\}.$$

Prove that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$ (known as the *lower-limit topology*). Prove that the lower-limit topology is different to the standard (Euclidean) topology on $\mathbb{R}$.

3. Find a subset A of $\mathbb{R}$ which has the same limit points whether we use the lower-limit topology or the standard topology. Find a subset B of $\mathbb{R}$ which has different limit points depending on whether we use the lower-limit topology or the standard topology.

4. Prove that the boundary BdA of a subset $A \subset X$ is empty if and only if A is both open and closed in X.

5. If A is closed in X, does its boundary BdA equal $A -$ Int$A$? Prove or find a counter-example.

6. Let $X = C^0[0, 1]$ be the set of continuous functions on the unit interval. Then

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

and

$$d'(f, g) = \int_0^1 |f(x) - g(x)| dx$$

are both metrics on $C^0[0, 1]$, which induce topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Is $\mathcal{T}$ finer, coarser, or the same as $\mathcal{T}'$?