MATH 681 Introductory Topology : homework assignment nine

1. Let n and k be coprime and let the generator of $G = \mathbb{Z}/n\mathbb{Z}$ act on

$$S^3 = \{(z, w) || z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$$

by taking (z, w) to $(e^{2\pi i/n} z, e^{2\pi ik/n} w)$. Show that this action is properly discontinuous. The quotient space $\mathrm{L}(n, k) = S^3/G$ is known as a *lens space*.

2. Let X be a non-orientable (compact) surface. We can always choose an orientation locally: near each point p there are two sides to the surface. Define

$$\tilde{X} = \{(p, \text{local orientation in near } p) | p \in X\}.$$

The projection $\pi : \tilde{X} \to X$ which "forgets" the local orientation is a two-to-one covering map.

a) Identify the *orientation double cover* $\tilde{X}$ of $X = P_3$.

b) Describe the induced map π_* on fundamental groups (when $X = P_3$).

3. Let $U = \{(x, y) | x > 0\} \subset \mathbb{R}^2$ and define the function θ on U to be given by $\arctan(y/x)$, with $-\pi/2 < \theta < \pi/2$.

a) Calculate $d\theta$ and note that it extends to a one-form ω on $\mathbb{R}^2 - \{(0, 0)\}$.

b) Using the expression for ω from part a), show that ω is closed. Is ω also exact (on $\mathbb{R}^2 - \{(0, 0)\}$)?

4. Calculate the pull-back of the "volume form" $dxdy$ under the following maps:

a)

$$\begin{aligned} f : (0, \infty) \times (0, \pi/2) &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x\mathrm{cos}y, x\mathrm{sin}y) \end{aligned}$$

b)

$$\begin{aligned} g : (0, \infty) \times (0, \infty) &\to \mathbb{R}^2 \\ (x, y) &\mapsto \left(-\frac{1}{2} \log \left(\frac{y}{x} \right), \sqrt{xy} \right) \end{aligned}$$