MATH 681 : Introductory Topology : midterm one solutions

1. We must show that every set U in $\mathcal{T}'$ is also in $\mathcal{T}$. This is clear if U is the empty set or X, so suppose $U \neq \emptyset, X$. Then U is the complement of a finite set of pts, $U = X - \{x_1, \dots, x_n\}$. We will show that $U = X - \{x_1, \dots, x_n\} \in \mathcal{T}$. Let $x \in U$, and let
$$r = \min\{d(x, x_1), \dots, d(x, x_n)\}.$$
Note that $r > 0$ since $d(x, x_i) > 0$ for all i $(x \neq x_i)$.
Then $x_i \notin B_d(x, r)$ for all $i = 1, \dots, n$ as $d(x, x_i) \geq r$
$\therefore \quad B_d(x, r) \subset U$.
So every pt $x \in U$ is contained in an open ball $B_d(x, r)$ which is contained in U. By definition, U is open in the metric topology $\mathcal{T}$.

2. Let $X = [0,1]$ with the subspace topology induced from $[0,1] \subset \mathbb{R}$. Let $U = (0,1)$. Then $\bar{U} = [0,1]$ and $\operatorname{int}(\bar{U}) = [0,1]$. Note that we are taking the interior in X, <u>not</u> in $\mathbb{R}$.
So $$\operatorname{int}(\bar{U}) \neq U.$$

3. We must show that $U \subset Y$ is an open set iff $f^{-1}(U) \subset X$ is an open set.
If U is open then $f^{-1}(U)$ is open because f is continuous.
In the opposite direction, suppose that $U \subset Y$ is a subset such that $f^{-1}(U)$ is open. Then $X - f^{-1}(U)$ is closed. Therefore $f(X - f^{-1}(U))$ is closed because f is a closed map.
We claim that $f(X - f^{-1}(U)) = Y - U$.

$f(X - f^{-1}(U)) \subset Y - U$ holds for every map f. Conversely, if $y \in Y - U$ then there exists $x \in X$ such that $f(x) = y$ (as f is onto). Moreover, $x \notin f^{-1}(U)$, as otherwise $y = f(x)$ would lie in U.
$\therefore \quad Y - U \subset f(X - f^{-1}(U))$, proving the claim.

We conclude that $Y - U = f(X - f^{-1}(U))$ is closed, i.e., U is open.

4. We must show that if $A \subset \mathbb{R}^2$ is closed then $f(A) \subset \mathbb{R}$ is closed. Let r be a limit point of $f(A)$. We will show that $r \in f(A)$.

Case 1: $r < 0$.

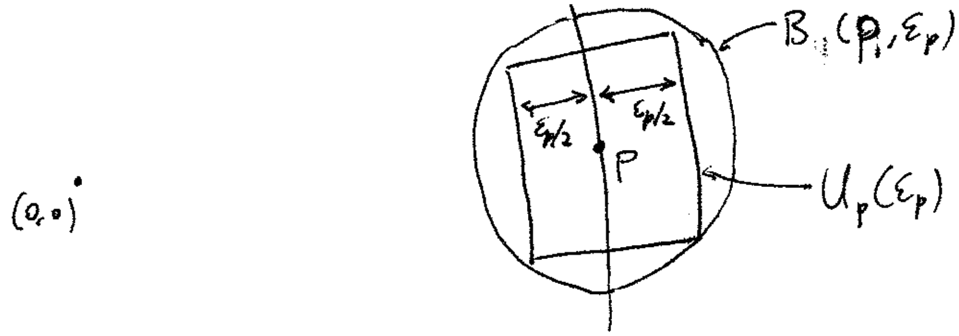
This is impossible, as $f(A) \subset [0, \infty)$, so every limit point must be contained in $[0, \infty)$ too.

Case 2: $r = 0$.

Since 0 is a limit point of $f(A)$, there must exist points $(x_n, y_n) \in A$ such that $f((x_n, y_n)) = \sqrt{x_n^2 + y_n^2}$ converges to 0. But then (x_n, y_n) must converge to $(0,0)$. Since A is closed, $(0,0) \in A$, and thus $0 = f((0,0)) \in f(A)$.

Case 3: $r > 0$.

Again there exist points $(x_n, y_n) \in A$ such that $f((x_n, y_n)) = \sqrt{x_n^2 + y_n^2}$ converges to r. This means (x_n, y_n) become arbitrarily close to the circle $S^1 \simeq \{x^2 + y^2 = r^2\}$ in $\mathbb{R}^2$. Suppose that no point of this circle is a limit point of $\{(x_n, y_n)\}$. Then for every point p on S^1 there is a nbhd $B(p, \varepsilon_p)$ which contains no point (x_n, y_n). For this problem it is more convenient to use a nbhd $U_p(\varepsilon_p)$ of the shape below:



Since $U_p(\varepsilon_p) \subset B(p, \varepsilon_p)$ it is okay to replace $B(p, \varepsilon_p)$ by $U_p(\varepsilon_p)$.
Now the circle is covered by the open sets $\{U_p(\varepsilon_p)\}_{p \in S^1}$. By compactness, there is a finite subcover. Let ε be the minimum of $\frac{\varepsilon_p}{2}$ for all $U_p(\varepsilon_p)$ in this finite subcover. Choose ε to be smaller than r, too. Then the finite subcover will actually cover the annulus $\{ r-\varepsilon \leq \sqrt{x^2+y^2} < r+\varepsilon \} \subset \mathbb{R}^2$.

$\therefore$ the annulus contains no points (x_n, y_n).

$\therefore$ for all n, $f(x_n, y_n) = \sqrt{x_n^2+y_n^2} \geq r+\varepsilon$ or $\leq r-\varepsilon$.

$\therefore$ $\sqrt{x_n^2+y_n^2}$ cannot converge to r, a contradiction.

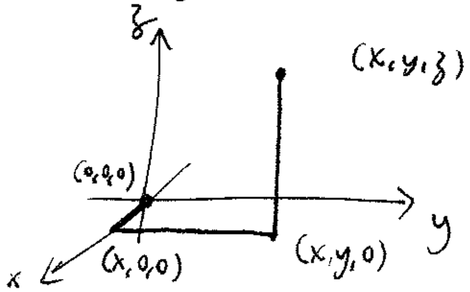
It follows that some point p on the circle is a limit point of $\{(x_n, y_n)\}$, and of A. Since A is closed, $p \in A$.

$\therefore$ $r = f(p) \in f(A)$.

Since $f(A)$ contains all of its limit points, $f(A)$ is closed.

5. We will prove that X is path connected, and therefore connected.
Note that X is a union of lines: Fix $x, y \in \mathbb{Q}$. Then $z \in \mathbb{R}$ is arbitrary. This gives a line parallel to the z-axis. Similarly, we get lines parallel to the y-axis and x-axis.
It suffices to prove that every point $(x, y, z) \in X$ is connected

to the origin $(0,0,0) \in X$ by a path. Suppose x and y are rational. Then the segment joining (x,y,z) to $(x,y,0)$ lies in X. Then x and 0 are rational, so the segment joining $(x,y,0)$ to $(x,0,0)$ lies in X. Finally, the segment joining $(x,0,0)$ to $(0,0,0)$ lies in X. These segments combine to form a path from (x,y,z) to $(0,0,0)$. A similar argument applies if x and z, or y and z are rational.



6. a) • $\emptyset$ is open in X so $U = \emptyset$ is in T.

$\emptyset$ is closed and compact in X, so $U = Y = \{p\} \cup X$ is in T.

• arbitrary unions:

Let $U_\alpha \in T$ for $\alpha \in A$. If $p \notin U_\alpha$ for any α, then U_α is open in X for all α. Then $\bigcup_{\alpha \in A} U_\alpha$ is open in X, and thus $\bigcup_{\alpha \in A} U_\alpha \in T$.

If $p \in U_\beta$ for some β, then $p \in \bigcup_{\alpha \in A} U_\alpha$. Now

$$Y - \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} (Y - U_\alpha)$$

In this intersection, $Y - U_\alpha$ is either closed and compact in X (if $p \in U_\alpha$) or equal to $\{p\} \cup C_\alpha$ where C_α is closed in X (if $p \notin U_\alpha$).

$\therefore$ $\bigcap_{\alpha \in A} (Y - U_\alpha)$ is the intersection of ~~arbitrary~~ closed sets in X, at least one of which $(Y - U_\beta)$ is compact.

We therefore get a closed subset of a compact set $(Y-U_\beta)$, which is then also compact.

i.e. $Y - \bigcup_{\alpha\in A} U_\alpha = \bigcap_{\alpha \in A} (Y-U_\alpha)$ is a closed compact subset of X.

$\therefore \bigcup_{\alpha\in A} U_\alpha \in T$.

- finite intersections (it is enough to intersect two sets)

Let $U_1, U_2 \in T$. If $p \notin U_1, U_2$ then U_1 and U_2 are open in X, and thus $U_1 \cap U_2$ is open in X, so $U_1 \cap U_2 \in T$.

If $p \in U_1$, $p \notin U_2$, then $p \notin U_1 \cap U_2$. Since $Y - U_1$ is closed in X, $U_1 = \{p\} \cup U_3$ where U_3 is open in X.

$\therefore \quad U_1 \cap U_2 = U_3 \cap U_2$ is open in X, and $U_1 \cap U_2 \in T$.

If $p \in U_1$ and $p \in U_2$, then $p \in U_1 \cap U_2$. Then

$$Y - U_1 \cap U_2 = (Y-U_1) \cup (Y-U_2)$$

is the union of two closed compact subsets of X.

$\therefore \quad (Y-U_1) \cup (Y-U_2)$ is closed and compact (if A_1 and A_2 are compact, then an open cover of $A_1 \cup A_2$ contains finite subcovers of A_1 and A_2; the union of these finite subcovers is then a finite subcover for $A_1 \cup A_2$).

$\therefore \quad Y - U_1 \cap U_2$ is a closed compact subset of X, and $U_1 \cap U_2 \in T$.

b) Let $\{U_\alpha\}$ be a cover of Y. Clearly p must belong to U_β for some β. Then $Y - U_\beta$ is a closed compact subspace of X. Let $\{U_1, \dots, U_n\}$ be a finite subcover which covers $Y - U_\beta$ (which exists since $Y - U_\beta$ is compact). Then $\{U_\beta, U_1, \dots, U_n\}$ is a finite subcover of Y.