MATH 681 Introductory Topology : midterm two

Allow yourself **two hours** to sit this exam. It should be attempted under usual exam conditions, i.e., do not consult the textbook or your notes, and do not discuss the problems with anybody. Please hand in your solutions by **Wednesday 31st March.**

1. Prove that a continuous map $h : S^1 \to Y$ is homotopic to a constant map if and only if it extends to a continuous map $f : B^2 \to Y$, where B^2 is the closed disc

$$\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

   with boundary S^1. *(Pay particular attention to the continuity of f.)*

2. Show that if $n > 1$, then every continuous map $f : \mathbb{RP}^n \to S^1$ is homotopic to the constant map.

3. Let X be the quotient space obtained from an octagon by gluing the edges according to the labelling $abcda^{-1}d^{-1}b^{-1}c^{-1}$.

   a) Verify that *all* vertices of the octagon are identified with a single point $x_0 \in X$ after we have glued the edges.

   b) Calculate the fundamental group $\pi_1(X, x_0)$ and the first homology group $H_1(X)$.

   c) Identify the surface X.

   d) Find an explicit cut-and-paste operation that will convert the labelling $abcda^{-1}d^{-1}b^{-1}c^{-1}$ into the standard labelling.

4. Let $X \subset \mathbb{R}^3$ be the union of the unit sphere S^2 and the straight line segment from $(0,0,1)$ to $(0,0,-1)$. Using the Seifert-van Kampen theorem, or otherwise, find $\pi_1(X, x_0)$. *(The choice of basepoint x_0 is largely irrelevant in this problem.)*

5. Let Σ_2 and Σ_3 denote genus two and three surfaces, and let $p : \Sigma_3 \to \Sigma_2$ be the two-to-one covering map depicted in the picture below. Calculate the induced map p_* on fundamental groups by explicitly describing the image of each generator of $\pi_1(\Sigma_3, x_0)$ and verifying that the appropriate relation is preserved.