MATH681 Introductory Topology : solutions to midterm two

1. Consider first the map
$$g : S^1 \times [0,1] \longrightarrow B^2$$
$$(\vartheta, r) \longmapsto ((1-r)\cos\vartheta, (1-r)\sin\vartheta)$$

If A is a closed subset of $S^1 \times [0,1]$, then A is compact (as $S^1 \times [0,1]$ is compact). Then $g(A)$ will also be compact, and therefore $g(A)$ will also be closed. This shows that g is a closed map. Since g is also continuous and surjective, we conclude that g is a quotient map.

Now suppose that $h : S^1 \longrightarrow Y$ is homotopic to a constant map, and let $H : S^1 \times [0,1] \longrightarrow Y$ be the homotopy. Since $H|_{S^1 \times 1}$ is a constant map, it follows that H will factor through the quotient map g ($S^1 \times 1$ is the only non-trivial equivalence class, and H is constant on it). In other words, there exists a continuous map $f : B^2 \longrightarrow Y$ s.t. $H = f \circ g$. Then f is the required extension of $h : S^1 \longrightarrow Y$, as
$$f((\cos\vartheta, \sin\vartheta)) = f \circ g(\vartheta, 0) = H(\vartheta, 0) = h(\vartheta).$$

Conversely, suppose that h extends to $f : B^2 \longrightarrow Y$. Then define
$$H : S^1 \times [0,1] \longrightarrow Y$$
$$(\vartheta, t) \longmapsto f \circ g(\vartheta, t).$$

Then H is a continuous homotopy from $H(\vartheta, 0) = f \circ g(\vartheta, 0) = f((\cos\vartheta, \sin\vartheta)) = h(\vartheta)$ to $H(\vartheta, 1) = f \circ g(\vartheta, 1) = f((0,0))$, i.e., to a constant map.

2. First consider the induced map $f_* : \pi_1(\mathbb{RP}^n, x_0) \longrightarrow \pi_1(S^1, y_0)$ on fundamental groups (for some choice of basepoints x_0, y_0).

There is no non-trivial group homomorphism from $\pi_1(\mathbb{RP}^n, x_0) \simeq \mathbb{Z}/2\mathbb{Z}$ to $\pi_1(S^1, y_0) \simeq \mathbb{Z}$, so f_* must be trivial.

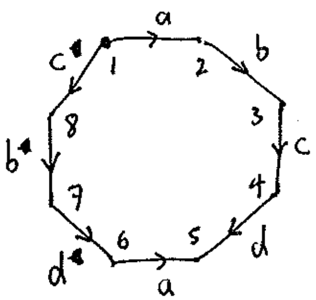
Now S^1 has cover
$$p : \mathbb{R} \longrightarrow S^1$$
$$\theta \longmapsto (\cos\theta, \sin\theta).$$

Since $f_* \pi_1(\mathbb{RP}^n, x_0) = \text{trivial} \subseteq p_* \pi_1(\mathbb{R}, e_0) = \text{trivial}$ (again, for some appropriate choice of basept e_0), we can lift the map $f : \mathbb{RP}^n \to S^1$ to a map to $\mathbb{R}$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Since $\mathbb{R}$ is contractible, every map into $\mathbb{R}$ must be homotopic to the constant map. Let $\tilde{H} : \mathbb{RP}^n \times [0,1] \longrightarrow \mathbb{R}$ be a homotopy from $\tilde{f}$ to the constant map to e_0. Then $H := p \circ \tilde{H} : \mathbb{RP}^n \times [0,1] \longrightarrow S^1$ will be a homotopy from f to the constant map to y_0, as required.

3. a)



Label the vertices as shown.

Gluing the edges labelled a identifies 1 with 6.

——— d ——— 6 — 5.

——— a ——— 5 — 2.

——— b ——— 2 — 8.

——— c ——— 8 — 4.

——— d ——— 4 — 7.

——— b ——— 7 — 3.

$\therefore$ all vertices are identified with a single pt. x_0 in X.

b) Gluing just the boundary gives a wedge on 4 circles, with fundamental group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta, \gamma, \delta \rangle$ = the free gp. on 4 generators.

Attaching the 2-cell (ie., the octagon with its interior) adds just one relation, giving

$$\pi_1(X, x_0) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta\alpha^{-1}\delta^{-1}\beta^{-1}\gamma^{-1} = 1 \rangle.$$

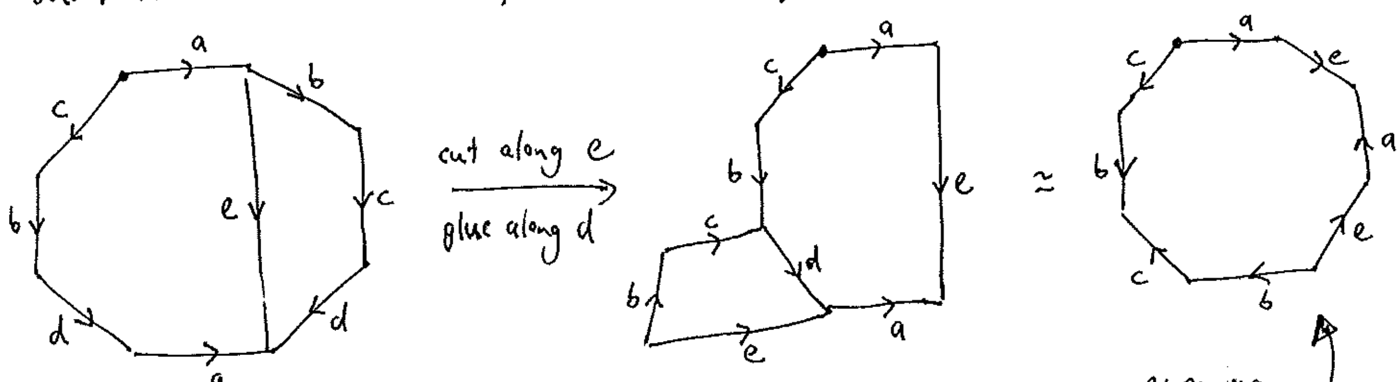
Now $H_1(X) = \pi_1(X, x_0)^{ab}. = \langle \alpha, \beta, \gamma, \delta \mid \text{everything commutes} \rangle$

$= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$

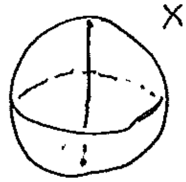
(Note: The relation $\alpha\beta\gamma\delta\alpha^{-1}\delta^{-1}\beta^{-1}\gamma^{-1} = 1$ is automatically satisfied if everything commutes.)

c) X is homeomorphic to Σ_2, the genus two surface (the only surface with $H_1(\Sigma_2) \cong \mathbb{Z}^{\oplus 4}$).

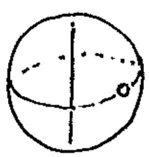
d)



This is the standard labelling for Σ_2 after ~~relabelling~~ renaming.

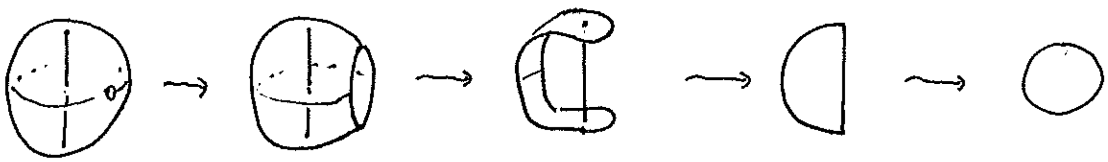
4.  X

Let $U = X - (1,0,0)$ and let V be a small disc on X around $(1,0,0)$. Then $U \cap V$ is a punctured disc.

 U

V  $U \cap V$ 

There is a deformation retraction of U onto S^1:



$\therefore \quad \pi_1(U) \simeq \mathbb{Z}.$

Since V is contractible, $\pi_1(V)$ is trivial. There is a deformation retraction of $U \cap V$ onto S^1, so $\pi_1(U \cap V) \simeq \mathbb{Z}$. Now the Seifert-van Kampen theorem says there is a surjective homomorphism

$$\pi_1(U) * \pi_1(V) \simeq \mathbb{Z} * 1 \simeq \mathbb{Z} \longrightarrow \pi_1(X)$$

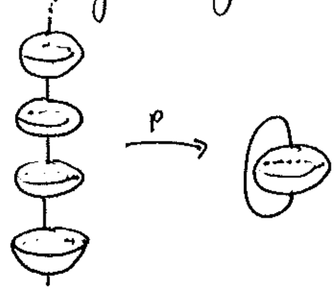
with kernel the smallest normal subgp containing

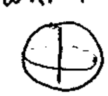
$$k_*(g)\, h_*(g)^{-1} \qquad \forall\, g$$

where $g \in \pi_1(U \cap V)$, and $k_* : \pi_1(U \cap V) \longrightarrow \pi_1(U)$ and $h_* : \pi_1(U \cap V) \longrightarrow \pi_1(V)$. But k_* and h_* are both trivial as the generator of $\pi_1(U \cap V)$ can be homotoped to a constant map in either U or V.

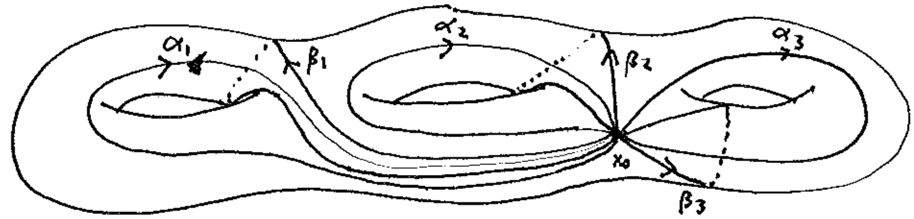
$$\therefore \qquad \pi_1(X) \simeq \mathbb{Z}.$$

Another solution is given by describing the universal cover of X, which looks like

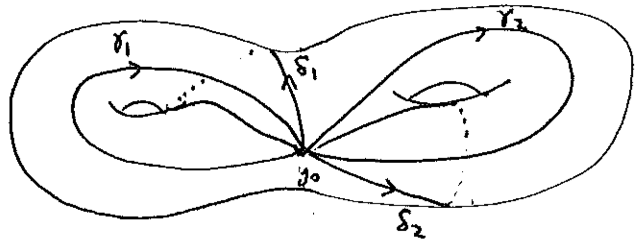


(which is homeomorphic to X $\approx$ , but turned "inside-out")

5. Label the generators of $\pi_1(\Sigma_3, x_0)$ and $\pi_1(\Sigma_2, y_0)$ as shown:



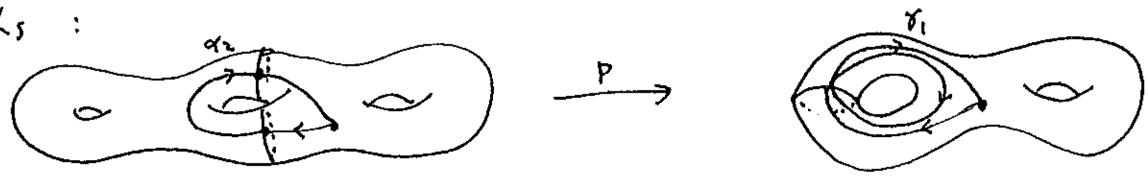
$$\pi_1(\Sigma_3, x_0) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1} = 1 \rangle.$$



$$\pi_1(\Sigma_2, y_0) = \langle \gamma_1, \delta_1, \gamma_2, \delta_2 \mid \gamma_1\delta_1\gamma_1^{-1}\delta_1^{-1}\gamma_2\delta_2\gamma_2^{-1}\delta_2^{-1} = 1 \rangle.$$

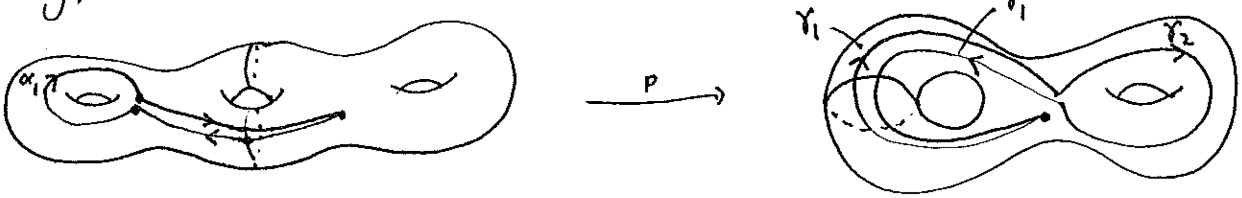
Note that the relation in $\pi_1(\Sigma_2, y_0)$ implies $\gamma_2\delta_2\gamma_2^{-1}\delta_2^{-1} = \delta_1\gamma_1\delta_1^{-1}\gamma_1^{-1}$, which we will use later.

Now under the projection p_*, α_3 and β_3 map directly to γ_2 and δ_2. Also, β_2 maps to δ_1, whereas α_2 can be broken into two paths:



Each of these paths maps to γ_1, so altogether α_2 maps to γ_1^2.

Finally, α_1 can be broken into three paths:



These paths map to γ_1, γ_2, and γ_1^{-1} respectively, so α_1 maps to $\gamma_1\gamma_2\gamma_1^{-1}$.

In a similar way, we see that β_1 maps to $\gamma_1\delta_2\gamma_1^{-1}$.

To summarize:

$$\begin{aligned}
\alpha_1 &\longmapsto \gamma_1 \gamma_2 \gamma_1^{-1} \\
\beta_1 &\longmapsto \gamma_1 \delta_2 \gamma_1^{-1} \\
\alpha_2 &\longmapsto \gamma_1^2 \\
\beta_2 &\longmapsto \delta_1 \\
\alpha_3 &\longmapsto \gamma_2 \\
\beta_3 &\longmapsto \delta_2 .
\end{aligned}$$

Now we check the relation is preserved:

$$\begin{aligned}
& p_*(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}) \\
&= \gamma_1\gamma_2\underline{\gamma_1^{-1}.\gamma_1}\delta_2\underline{\gamma_1^{-1}.\gamma_1}\gamma_2^{-1}\underline{\gamma_1^{-1}.\gamma_1}\delta_2^{-1}\gamma_1^{-1}.\gamma_1^2.\delta_1.\gamma_1^{-2}.\delta_1^{-1}.\gamma_2.\delta_2.\gamma_2^{-1}.\delta_2^{-1} \\
&= \gamma_1.\underline{\gamma_2\delta_2\gamma_2^{-1}\delta_2^{-1}}.\gamma_1^{-1} \quad . \quad \gamma_1^2\delta_1\gamma_1^{-2}\delta_1^{-1}.\underline{\gamma_2\delta_2\gamma_2^{-1}\delta_2^{-1}} \\
&= \gamma_1.\delta_1\gamma_1\delta_1^{-1}\underline{\gamma_1^{-1}.\gamma_1^{-1} \quad . \quad \gamma_1^2}\delta_1\gamma_1^{-2}\underline{\delta_1^{-1}.\delta_1}\gamma_1\delta_1^{-1}\gamma_1^{-1} \\
&\qquad (\text{substituting } \gamma_2\delta_2\gamma_2^{-1}\delta_2^{-1} = \delta_1\gamma_1\delta_1^{-1}\gamma_1^{-1}) \\
&= \gamma_1\delta_1\gamma_1\underline{\delta_1^{-1} \quad . \quad \delta_1}\gamma_1^{-2}.\gamma_1\delta_1^{-1}\gamma_1^{-1} \\
&= \gamma_1\delta_1\underline{\gamma_1.\gamma_1^{-2}.\gamma_1}\delta_1^{-1}\gamma_1^{-1} \\
&= \gamma_1\underline{\delta_1 \quad . \quad \delta_1^{-1}}\gamma_1^{-1} \\
&= \gamma_1 . \gamma_1^{-1} \\
&= 1 \\
&= p_*(1).
\end{aligned}$$

i.e., the relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1} = 1$ is preserved.