MATH681 Introductory Topology: Solutions to exercise sheet one

1. Firstly, $\emptyset = \emptyset \cap \emptyset$ and $X = X \cap X$ both belong to $\mathcal{T} \cap \mathcal{T}'$.
Secondly, if $U_\alpha \cap U_\alpha'$ belong to $\mathcal{T} \cap \mathcal{T}'$ for all α in A, then
$$\bigcup_{\alpha \in A} (U_\alpha \cap U_\alpha') = \left(\bigcup_{\alpha \in A} U_\alpha\right) \cap \left(\bigcup_{\alpha \in A} U_\alpha'\right)$$
also belongs to $\mathcal{T} \cap \mathcal{T}'$, as $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ and $\bigcup_{\alpha \in A} U_\alpha' \in \mathcal{T}'$.
Thirdly, if $U_i \cap U_i'$ belongs to $\mathcal{T} \cap \mathcal{T}'$ for $i = 1, \dots, n$, then
$$(U_1 \cap U_1') \cap \dots \cap (U_n \cap U_n') = (U_1 \cap \dots \cap U_n) \cap (U_1' \cap \dots \cap U_n')$$
also belongs to $\mathcal{T} \cap \mathcal{T}'$, as $U_1 \cap \dots \cap U_n \in \mathcal{T}$ and $U_1' \cap \dots \cap U_n' \in \mathcal{T}'$.
Therefore $\mathcal{T} \cap \mathcal{T}'$ is a topology on X.

2. Every point in $\mathbb{R}$ belongs to a subset $[a,b)$ (e.g. if $x \in \mathbb{R}$ just take $a = x$ and $b = x+1$). If $x \in B_1 = [a_1, b_1)$ and $x \in B_2 = [a_2, b_2)$ then let $a_3 = \max\{a_1, a_2\}$ and $b_3 = \min\{b_1, b_2\}$. Then $a_3 \leq x < b_3$, so $x \in B_3 := [a_3, b_3)$ and $B_3 \subset B_1 \cap B_2$. Therefore $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

The lower-limit topology $\mathcal{T}$ is the same as the standard topology $\mathcal{T}'$ if and only if it is both finer and coarser than $\mathcal{T}'$. Recall that $\mathcal{T}'$ has basis
$$\mathcal{B}' = \{ (a,b) \mid a < b \}.$$
Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $x \in B = [a,b) \in \mathcal{B}$ there is a $B' = (a', b') \in \mathcal{B}'$ with $x \in B' \subset B$. If we choose $x = a \in [a,b)$ then clearly $B' = (a', b')$ cannot exist, as we need both $a' < a$ for $x = a \in B'$ and $a \leq a'$ for $B' \subset B$. Therefore $\mathcal{T}'$ is not finer than $\mathcal{T}$, and $\mathcal{T}$ and $\mathcal{T}'$ cannot be the same. (Note that the lower-limit topology is finer than the standard topology.)

3. Let $A = \mathbb{Q} \subset \mathbb{R}$. Then every interval $[a,b)$ and (a,b) with $a<b$ intersects $\mathbb{Q}$ in infinitely many points. Since open sets are unions of basis elements, every open set (with respect to both the lower-limit topology and the standard topology) which is non-empty will intersect $\mathbb{Q}$ in infinitely many points. Therefore every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, whether we use the lower-limit or the standard topology.

Let $B = (0,1) \subset \mathbb{R}$. The limit points of B with respect to the standard topology are $[0,1]$. However, with respect to the lower-limit topology 1 is <u>not</u> a limit point of B. For example, $[1,2)$ is an open neighbourhood of 1 which does not intersect $B = (0,1)$.

4. Suppose A is both open and closed. Then $X-A$ and A are both closed.

$$\therefore \quad \operatorname{Bd} A = \bar{A} \cap \overline{(X-A)} = A \cap (X-A) = \emptyset.$$

Conversely, suppose that $\operatorname{Bd} A = \emptyset$, ie., $\bar{A} \cap \overline{(X-A)} = \emptyset$.
Now A must equal $\bar{A}$. Otherwise there would be a point $x \in \bar{A}$ and $x \notin A$, ie., $x \in X-A$. But then $x \in \bar{A} \cap (X-A) \subset \bar{A} \cap \overline{(X-A)} = \emptyset$.
Similarly, $X-A$ must equal $\overline{X-A}$. Thus A and $X-A$ are both closed, ie., A is both open and closed.

5. If A is closed then $\operatorname{Bd} A = A - \operatorname{Int} A$.

Firstly, let $x \in \operatorname{Bd} A = \bar{A} \cap \overline{(X-A)} = A \cap \overline{(X-A)}$ as A is closed.

$\therefore$ $x \in A$ and $x \in \overline{X-A}$.

We must show that x is not in $\operatorname{Int} A$. If $x \in \operatorname{Int} A$, then there is a neighbourhood U with $x \in U \subset A$. Since U is open, $X-U$ is closed, and $X-A \subset X-U$ which implies

$$\overline{X-A} \subset \overline{X-U} = X-U.$$

Since $x \notin X-U$, then x cannot lie in $\overline{X-A}$, a contradiction.

Therefore x is not in $\operatorname{Int} A$, so $x \in A - \operatorname{Int} A$.

We have shown $\operatorname{Bd} A \subset A - \operatorname{Int} A$.

Conversely, suppose $x \in A - \operatorname{Int} A$. Therefore $x \in A = \bar{A}$ and it remains to show that $x \in \overline{X-A}$. If $x \notin \overline{X-A}$, then there is a closed set B with $X-A \subset B$ and $x \notin B$. Therefore $U = X-B$ is open, $U = X-B \subset X-(X-A) = A$, and $x \in U = X-B$.

In other words, x is in the interior $\operatorname{Int} A$ of A, a contradiction.

Therefore x must lie in $\overline{X-A}$, and we conclude that

$$A - \operatorname{Int} A \subset \bar{A} \cap \overline{(X-A)} = \operatorname{Bd} A.$$

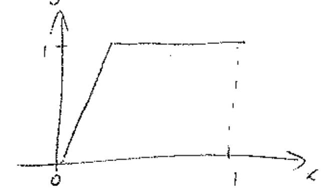
All together, $\operatorname{Bd} A = A - \operatorname{Int} A$ if A is closed.

6. $$d'(f,g) = \int_0^1 |f(x)-g(x)|\,dx \le \int_0^1 \sup_{x'\in[0,1]} |f(x')-g(x')|\,.dx$$
$$= \int_0^1 d(f,g)\,.dx$$
$$= d(f,g).$$

$$\therefore \quad B_d(f,\varepsilon) = \{g \in C^\circ[0,1] \mid d(f,g) < \varepsilon\}.$$
$$\subset \{g \in C^\circ[0,1] \mid d'(f,g) < \varepsilon\} = B_{d'}(f,\varepsilon).$$

Since $B_d(f,\varepsilon)$ and $B_{d'}(f,\varepsilon)$ are the basis elements of $\mathcal{T}$ and $\mathcal{T}'$, it follows that $\mathcal{T}$ is finer than $\mathcal{T}'$. We still need to check whether $\mathcal{T}$ and $\mathcal{T}'$ are the same topology.

Let $$f_n(x) := \begin{cases} nx & x \in [0, \tfrac{1}{n}] \\ 1 & x \in [\tfrac{1}{n}, 1]. \end{cases}$$



and let $A = \{f_n \mid n = 1, 2, \dots\} \subset C^\circ[0,1]$.

Claim 1: $f \equiv 1$ is a limit point of A with respect to the topology $\mathcal{T}'$.

It is enough to observe that any open ball
$$B_{d'}(f,\varepsilon) = \left\{ g \in C^\circ[0,1] \mid \int_0^1 |f(x)-g(x)|\,dx < \varepsilon \right\}$$
around f contains f_n for n sufficiently large.

$$\left(d'(f,f_n) = \int_0^1 |f(x) - f_n(x)|\,dx = \frac{1}{2n} < \varepsilon \quad \text{for } n > \frac{1}{2\varepsilon}. \right)$$

Claim 2: $f \equiv 1$ is not a limit point of A with respect to the topology $\mathcal{T}$.

In this case $d(f,f_n) = \sup_{x\in[0,1]} |f(x) - f_n(x)| = |f(0) - f_n(0)| = 1$ for all n, so $f_n \notin B_d(f, \tfrac{1}{2})$ and $B_d(f,\tfrac{1}{2}) \cap A = \emptyset$.

It follows that $\mathcal{T}$ and $\mathcal{T}'$ cannot be the same topology.