MATH 681 Introductory Topology: homework assignment eleven solutions.

1\. In class we proved that if ω is exact on W and V, and if $W \cap V$ is connected, then ω is exact on $W \cup V$. $\circledast$

The result now follows by induction. Firstly, ω is exact on U_1 by hypothesis. For the inductive step, assume that ω is exact on $U_1 \cup U_2 \cup \dots \cup U_i$. Let $W = U_1 \cup U_2 \cup \dots \cup U_i$ and let $V = U_{i+1}$.

Then ω is exact on W by the inductive hypothesis, and ω is exact on V by the hypothesis of the problem. In addition

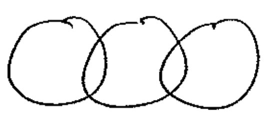
$$W \cap V = (U_1 \cup \dots \cup U_i) \cap U_{i+1}$$

is connected. Therefore, by $\circledast$ ω is exact on

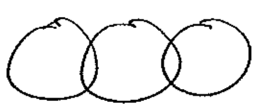
$$W \cup V = U_1 \cup \dots \cup U_i \cup U_{i+1}.$$

$\therefore$ ω is exact on $U = U_1 \cup \dots \cup U_k$ by induction.

Rmk: When applying this result, the order of the sets is important.

E.g. 

satisfies the hypothesis

but 

does not satisfy the hypothesis as $(U_1 \cup U_2) \cap U_3$ is not connected.

2. a) Without loss of generality (i.e., after a rotation, translation, and dilation) we can assume the pts P and Q lie at $(1,0)$ and $(-1,0)$ respectively. It is easy to verify that ω is closed; therefore it is exact on any half-space:

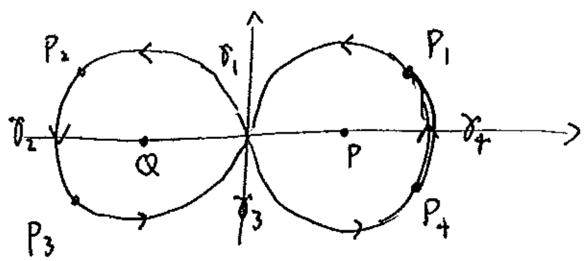
on $U_1 = \{(x,y) \mid y>0\}$ $\quad \omega = df_1$

on $U_2 = \{(x,y) \mid x<-1\}$ $\quad \omega = df_2$

on $U_3 = \{(x,y) \mid y<0\}$ $\quad \omega = df_3$

on $U_4 = \{(x,y) \mid x>1\}$ $\quad \omega = df_4$

Let $P_1 = (1+\frac{1}{\sqrt2}, \frac{1}{\sqrt2})$, $P_2 = (-1-\frac{1}{\sqrt2}, \frac{1}{\sqrt2})$, $P_3 = (-1-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2})$, $P_4 = (1+\frac{1}{\sqrt2}, -\frac{1}{\sqrt2})$ as shown, and let γ be the union of two unit circles around P and Q, broken up into sub-paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as shown:



$$\therefore \int_\gamma \omega = \int_{\gamma_1}\omega + \int_{\gamma_2}\omega + \int_{\gamma_3}\omega + \int_{\gamma_4}\omega = \int_{\gamma_1} df_1 + \int_{\gamma_2} df_2 + \int_{\gamma_3} df_3 + \int_{\gamma_4} df_4$$

$$= f_1(P_2) - f_1(P_1) + f_2(P_3) - f_2(P_2) + f_3(P_4) - f_3(P_3) + f_4(P_1) - f_4(P_4).$$

Of course, by adjusting f_2, f_3, and f_4 by constants, we can assume that

$f_1 \equiv f_2$ on $U_1 \cap U_2$ (in particular $f_1(P_2) = f_2(P_2)$)

and $f_2 \equiv f_3$ on $U_2 \cap U_3$ (——— $f_2(P_3) = f_3(P_3)$)

and $f_3 \equiv f_4$ on $U_3 \cap U_4$ (——— $f_3(P_4) = f_4(P_4)$).

$$\therefore \int_\gamma \omega = -f_1(P_1) + f_4(P_1) = C,$$

where C is the difference of f_1 and f_4 on $U_1 \cap U_4$ (which must be constant).

Finally, we must show that $c = 0$, as then f_1 will agree with f_4 on $U_1 \cap U_4$, so together f_1, f_2, f_3, f_4 will define the required function f such that $\omega = df$ on $\mathbb{R}^2 - L$.

To calculate $c = \int_\gamma \omega$ we break γ into two circles:

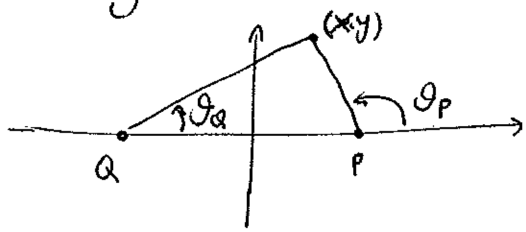
$$\int_\gamma \omega = \int_{\text{left circle}} \omega + \int_{\text{right circle}} \omega$$

Now $\displaystyle\int_{\text{left circle}} \omega = \int_{\text{left circle}} \omega_P - \omega_Q = -\int_{\text{left circle}} \omega_Q$ because ω_P is exact on $\{(x,y) \mid x < 1\}$, which contains the left circle, so $\displaystyle\int_{\text{left circle}} \omega_P = 0$. Moreover, we've seen that $\displaystyle\int_{\text{left circle}} \omega_Q = 2\pi$.

Similarly, $\displaystyle\int_{\text{right circle}} \omega = \int_{\text{right circle}} \omega_P = 2\pi$.

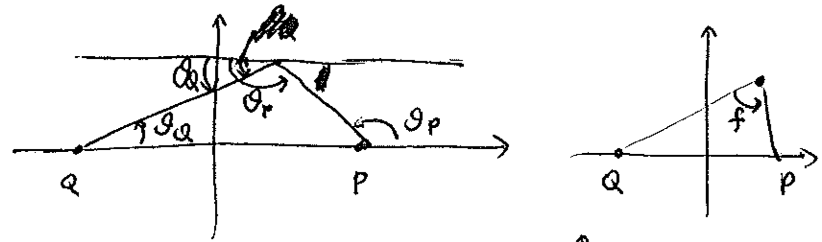
$$\therefore \quad c = \int_\gamma \omega = \int_{\text{left circle}} \omega + \int_{\text{right circle}} \omega = -2\pi + 2\pi = 0.$$

b) On $\mathbb{R}^2 - \{(x,0) \mid x \geq 1\}$, $\omega_P = d\theta_P$ where θ_P is the angle between 0 and 2π that the segment from P to (x,y) makes with the x-axis. Similarly, $\omega_Q = d\theta_Q$ on $\mathbb{R}^2 - \{(x,0) \mid x \geq -1\}$.

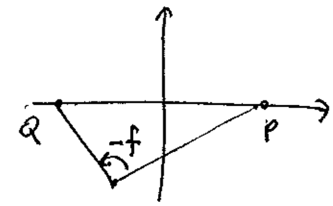


When we cross the positive x-axis, θ_P decreases by 2π, but θ_Q also decreases by 2π. So their difference $f = \theta_P - \theta_Q$ is well-defined and smooth on $\mathbb{R}^2 - L$, and $\omega = \omega_P - \omega_Q = d\theta_P - d\theta_Q = df$.

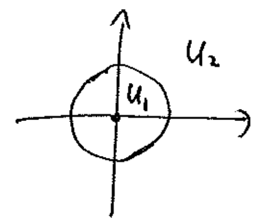
Rmk: $f \equiv 0$ on the x-axis. Some elementary Euclidean geometry shows that $f =$ the angle made by P, (x,y), and Q if (x,y) is in the upper half-plane :



In the lower half-plane, f is minus this angle :



3. U consists of two connected components, U_1 and U_2 :



So $H^1(U) = H^1(U_1) \oplus H^1(U_2)$.

Let $\omega = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 - \{(0,0)\}$.

Then $\omega|_{U_1}$ generates $H^1(U_1)$ (the argument is the same as the argument for $H^1(\mathbb{R}^2 - \{(0,0)\})$ that we saw in class).

Moreover $\omega|_{U_2}$ generates $H^1(U_2)$ (and again the argument is essentially the same as what we saw in class for $H^1(\mathbb{R}^2 - \{(0,0)\})$, i.e. if μ is a closed oneform on U_2, we can show that it is exact on various "quadrants" such as $U_2 \cap \{(x,y) \mid x > 0, y > 0\}$ by integrating along paths, then we can show that the resulting functions f_i will fit together into a single smooth function on U_2 iff $\int_\gamma \mu = 0$ where γ is the circle of radius two around the origin).

$\therefore$ define $\alpha = \begin{cases} \omega|_{U_1} & \text{on } U_1 \\ 0 & \text{on } U_2 \end{cases}$ and $\beta = \begin{cases} 0 & \text{on } U_1 \\ \omega|_{U_2} & \text{on } U_2. \end{cases}$

Then $\{\alpha, \beta\}$ is a basis for $H^1(U)$, and $H^1(U) \cong \mathbb{R}^2$.