MATH681 Introductory Topology: solutions to homework assignment nine

1. The generator $\bar{1} \in \mathbb{Z}/n\mathbb{Z}$ acts by $(z,w) \mapsto (e^{2\pi i/n}z, e^{2\pi ik/n}w)$.

$\therefore$ $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ acts by $(z,w) \longmapsto (e^{2\pi ia/n}z, e^{2\pi ika/n}w)$.

We first show that this action has no fixed points unless $\bar{a} = \bar{0}$. Suppose

$$(z,w) = (e^{2\pi ia/n}z, e^{2\pi ika/n}w).$$

Then $z = e^{2\pi ia/n}z \implies$ either $\bar{a} = \bar{0}$ or $z = 0$.

If $z = 0$, then $w \neq 0$ (since $|z|^2 + |w|^2 = 1$), and thus

$w = e^{2\pi ika/n}w$ will imply $1 = e^{2\pi ika/n}$.

Since k and n are coprime, this implies $\bar{a} = \bar{0}$ again.

Now suppose that there is no nbhd U of (z,w) such that:

$$gU \cap U = \emptyset \quad \text{unless } g \text{ is trivial.}$$

Using the Euclidean metric on $\mathbb{R}^4$ (and induced metric on S^3), we let U_n be a sequence of decreasing nbhds of (z,w) of radius $\frac{1}{n}$. For all n,

$$g_n U_n \cap U_n \neq \emptyset \quad \text{for some non-trivial } g_n \in G.$$

Since G is finite, we can choose a subsequence $\{n_k\}$ s.t. $g_{n_k} = g$ for some fixed element $g \in G$. Then

$$gU_{n_k} \cap U_{n_k} \neq \emptyset$$

implies $\exists\ x_{n_k} \in U_{n_k}$ and $y_{n_k} \in U_{n_k}$ s.t. $g(x_{n_k}) = y_{n_k}$.

Finally, letting $n_k \to \infty$, both x_{n_k} and y_{n_k} must converge to the point $(z,w) \in S^3$.

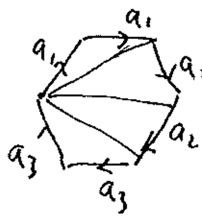
$$\therefore \quad (z,w) = \lim_{n_k \to \infty} y_{n_k} = \lim_{n_k \to \infty} g(x_{n_k}) = g\left(\lim_{n_k \to \infty} x_{n_k}\right) = g((z,w)).$$

and (z,w) is a fixed pt of g, a contradiction.

$\therefore$ the action is properly discontinuous.

2. a) The simplest way to identify $\tilde{X}$ is to calculate its Euler characteristic.

$$\chi(X) = \chi(P_3) = 1 - 6 + 4 = -1$$

as after gluing $P_3 =$ 

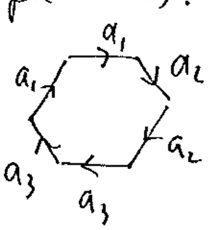
has one vertex, ~~three~~ six edges, and ~~one~~ four faces.

$\therefore$ the double cover $\tilde{X}$ has $\chi(\tilde{X}) = 2.\chi(X) = -2$.

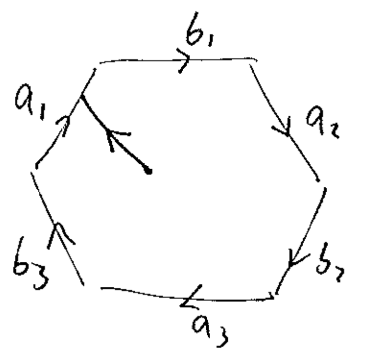
Now the orientation double cover $\tilde{X}$ is always orientable. (How do we choose a "side" of the surface? At $(x,+)$ we choose the "positive" side, at $(x,-)$ we choose the "negative" side. Either way, there is always a canonical ~~side~~ choice of side of $\tilde{X}$.)

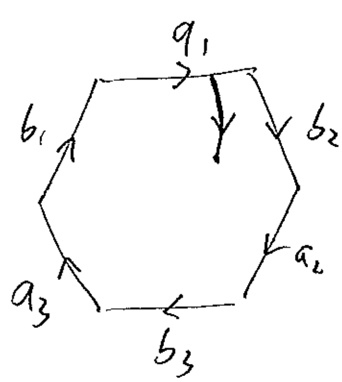
$\therefore$ $\tilde{X}$ must be Σ_2, the genus two surface.

Here is another solution which will be more useful in answering part b). The orientation double cover is really two copies of

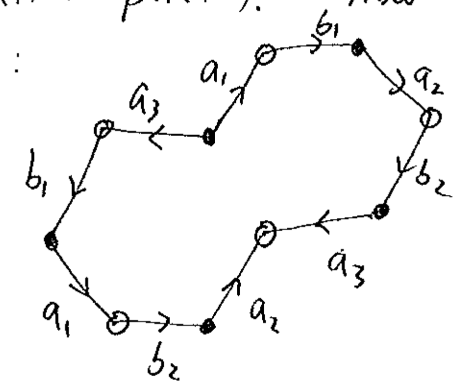


glued in the "appropriate" way. Thus if we follow a path across the edge a_1 we emerge not from the other a_1, but rather we emerge in the other hexagon. Altogther, the gluing scheme is:

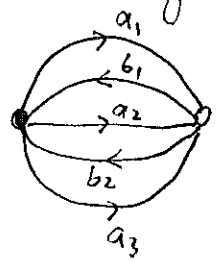




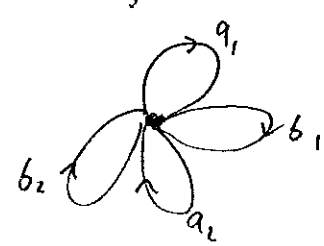
Notice that each time we go from one hexagon to the other, the orientation reverses. Therefore any closed loop is orientation-preserving (the orientation will change an even number of times if we return to the initial point). Now we can glue along b_3 to obtain a decagon:



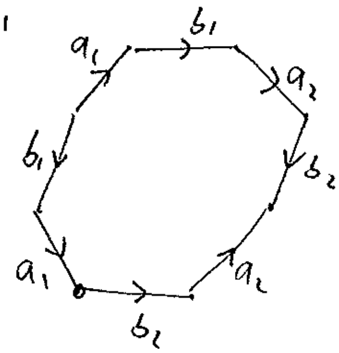
The gluing scheme does <u>not</u> identify all vertices, so when we glue the boundary we get:



We 'collapse' a_3 with a deformation retraction to get a wedge on four circles:



The ~~dag~~ decagon becomes an octagon:

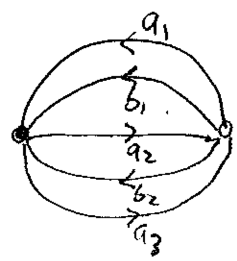


Up to relabelling, this is precisely the gluing scheme for Σ_2.

b) Notice that the labelling of the octagon is <u>not</u> quite the standard one for Σ_2 (some orientations are different). We obtain a presentation of $\pi_1(\Sigma_2)$ which looks like:

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid a_1^{-1} b_1^{-1} a_1 b_1 a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle.$$

Now we trace these back to see where they map to in $\pi_1(P_3)$. First we need to reverse the deformation retraction, which results in putting a_3 back in. Thus the loop



a_1 becomes $a_1 a_3^{-1}$,

b_1 — $a_3 b_1$,

a_2 — $a_2 a_3^{-1}$,

and b_2 — $a_3 b_2$.

So the four loops $a_1 a_3^{-1}$, $a_3 b_1$, $a_2 a_3^{-1}$, $a_3 b_2$ are the generators of π_1 of the surface obtained from the decagon. This surface is still a double cover of P_3. To map down to P_3, we just need to identify a_1 with b_1, a_2 with b_2, and a_3 with b_3.

$$\therefore \quad \pi_*(a_1) = a_1 a_3^{-1},$$
$$\pi_*(b_1) = a_3 a_1,$$
$$\pi_*(a_2) = a_2 a_3^{-1},$$
$$\pi_*(b_2) = a_3 a_2.$$

This describes the map:

$$\pi_* : \pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid a_1^{-1} b_1^{-1} a_1 b_1 a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle$$
$$\longrightarrow \pi_1(P_3) = \langle a_1, a_2, a_3 \mid a_1^2 a_2^2 a_3^2 = 1 \rangle.$$

We can check that the relation is preserved:

$$\pi_*(a_1^{-1} b_1^{-1} a_1 b_1 a_2 b_2 a_2^{-1} b_2^{-1}) = (a_3 a_1^{-1})(a_1^{-1} a_3^{-1})(a_1 a_3^{-1})(a_3 a_1)(a_2 a_3^{-1})(a_3 a_2)(a_3 a_2^{-1})(a_2^{-1} a_3^{-1})$$
$$= a_3 a_1^{-2} a_3^{-1} a_1^2 a_2^2 a_3 a_2^{-2} a_3^{-1}$$
$$= a_3 a_1^{-2} a_3^{-1} a_3^{-2} a_3 a_2^{-2} a_3^{-1} \qquad \text{as } a_1^2 a_2^2 = a_3^{-2}$$
$$= a_3 a_1^{-2} a_3^{-2} a_2^{-2} a_3^{-1}$$
$$= a_3 a_1^{-2} a_1^2 a_2^2 a_2^{-2} a_3^{-1} \qquad \text{again, } a_3^{-2} = a_1^2 a_2^2$$
$$= a_3 a_3^{-1} = 1.$$

3. a) $$d\theta = d\arctan\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\, d\left(\frac{y}{x}\right)$$
$$= \frac{x^2}{x^2+y^2}\left(\frac{dy}{x} - y\frac{dx}{x^2}\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$$

Clearly $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ is defined on all of $\mathbb{R}^2 - \{(0,0)\}$.

b) $$d\omega = d\left(\frac{x}{x^2+y^2}\right)dy - d\left(\frac{y}{x^2+y^2}\right)dx$$
$$= \left(\frac{dx}{x^2+y^2} - \frac{x\,d(x^2+y^2)}{(x^2+y^2)^2}\right)dy - \left(\frac{dy}{x^2+y^2} - \frac{y\,d(x^2+y^2)}{(x^2+y^2)^2}\right)dx$$
$$= \left(\frac{dx}{x^2+y^2} - \frac{x(2x\,dx + 2y\,dy)}{(x^2+y^2)^2}\right)dy - \left(\frac{dy}{x^2+y^2} - \frac{y(2x\,dx+2y\,dy)}{(x^2+y^2)^2}\right)dx$$
$$= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right)dx\,dy - \left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)dy\,dx$$
$$= \left(\frac{2}{x^2+y^2} - \frac{2x^2+2y^2}{(x^2+y^2)^2}\right)dx\,dy$$
$$= 0.$$

ω is <u>not</u> exact on all of $\mathbb{R}^2 - \{(0,0)\}$: If it were exact, say equal to $d\alpha$ for some function α on $\mathbb{R}^2 - \{(0,0)\}$, then
$$d(\alpha - \theta) = d\alpha - d\theta$$
$$= \omega - \omega$$
$$= 0 \quad \text{on} \quad U \subset \mathbb{R}^2.$$

$\therefore$ α and θ differ by a constant on U.

This argument extends also to $\{(x,y) \mid y > 0\}$, $\{(x,y) \mid x < 0\}$, and $\{(x,y) \mid y < 0\}$.

Altogether, we see that α and the angle θ differ by a constant on $\mathbb{R}^2 - \{(0,0)\}$. But θ is not well-defined on <u>all</u> of $\mathbb{R}^2 - \{(0,0)\}$ as it increases by 2π if we follow a loop around $(0,0)$. So α cannot exist.

[We will make this argument more precise when we integrate forms over 1-cycles.]

4\. a) $$f^*(dx\,dy) = df_1\,df_2$$
$$= d(x\cos y)\,d(x\sin y)$$
$$= (\cos y\,dx - x\sin y\,dy)(\sin y\,dx + x\cos y\,dy)$$
$$= x\cos^2 y\,dx\,dy - x\sin^2 y\,dy\,dx$$
$$= x(\cos^2 y + \sin^2 y)\,dx\,dy$$
$$= x\,dx\,dy$$ (recall $r\,dr\,d\theta$?)

b) $$g^*(dx\,dy) = dg_1\,dg_2$$
$$= d\left(-\tfrac{1}{2}\log\left(\tfrac{y}{x}\right)\right) d\left(\sqrt{xy}\right)$$
$$= \left(-\frac{1}{2}\,\frac{d\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)}\right)\left(\frac{1}{2}\sqrt{y}\,\frac{1}{2}\frac{1}{\sqrt{x}}\,dx + \frac{1}{2}\sqrt{x}\,\frac{1}{2}\frac{1}{\sqrt{y}}\,dy\right)$$
$$= -\frac{1}{8}\left(\frac{\frac{dy}{x} - \frac{y\,dx}{x^2}}{\frac{y}{x}}\right)\left(\sqrt{\frac{y}{x}}\,dx + \sqrt{\frac{x}{y}}\,dy\right)$$
$$= -\frac{1}{8}\left(\frac{x\,dy - y\,dx}{xy}\right)\left(\frac{y\,dx + x\,dy}{\sqrt{xy}}\right)$$
$$= -\frac{1}{8}\,\frac{1}{xy\sqrt{xy}}\left(xy\,dy\,dx - yx\,dx\,dy\right)$$
$$= \frac{1}{8}\,\frac{1}{xy\sqrt{xy}}\left(xy\,dx\,dy + xy\,dx\,dy\right)$$
$$= \frac{1}{4}\,\frac{dx\,dy}{\sqrt{xy}}.$$ (these are "hyperbolic coordinates").