Random permutations and unique fully supported ergodicity for the Euler adic transformation

Sarah Bailey and Karl Petersen

Department of Mathematics, CB 3250, Phillips Hall, University of North Carolina, Chapel Hill, NC 27599 USA

Abstract

There is only one fully supported ergodic invariant probability measure for the adic transformation on the space of infinite paths in the graph that underlies the Eulerian numbers. This result may partially justify a frequent assumption about the equidistribution of random permutations.

Key words: Random permutations, Eulerian numbers, adic transformation, invariant measures, ergodic transformations, Bratteli diagrams, rises and falls
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1 Introduction

We give a new proof of the main result of [1], in fact we prove a stronger result by a different method. For each $n = 0, 1, 2, \dots, k = 0, 1, \dots n$, the *Eulerian number* $A(n, k)$ is the number of permutations $i_1 i_2 \dots i_{n+1}$ of $\{1, \dots, n + 1\}$ with exactly k *rises* (indices $j = 1, 2, \dots, n$ with $i_j < i_{j+1}$) and $n - k$ *falls* (indices $j = 1, 2, \dots, n$ with $i_j > i_{j+1}$). Besides their obvious combinatorial importance, these numbers are also of interest in connection with the statistics of rankings: see, for example, [2],[3],[5],[4], and [6]. In studying random permutations, it is often assumed that all permutations are equally likely, each permutation of length $n + 1$ occurring with probability $1/(n + 1)!$. Our main result says that if any two permutations of the same length which *have the same number of rises* are equally likely, and if every permutation has positive

Email addresses: sebailey@email.unc.edu (Sarah Bailey), petersen@math.unc.edu (Karl Petersen).

probability, then in fact *all* permutations of the same length are equally likely. We arrive at this conclusion by means of ergodic theory, studying invariant measures for a certain measurable transformation on the space of infinite paths on the Euler graph.

The *Euler graph* Γ is a Bratteli diagram with levels $n = 0, 1, 2, \dots$. There are $n+1$ vertices of each level n, labelled (n, k), $0 \le k \le n$, and each vertex (n, k) is connected to vertex $(n+1, k)$ by $k+1$ edges and to vertex $(n+1, k+1)$ by $n-k+1$ edges. The number of paths into any vertex (n, k) is the Eulerian number $A(n, k)$. Hence the Eulerian numbers have the recursion

$$A(n, k) = (n - k + 1)A(n - 1, k - 1) + (k + 1)A(n - 1, k), \tag{1}$$

where $A(0, 0) = 1$ and by convention $A(n, k) = 0$ for $k \notin \{0, 1, ..., n\}$.

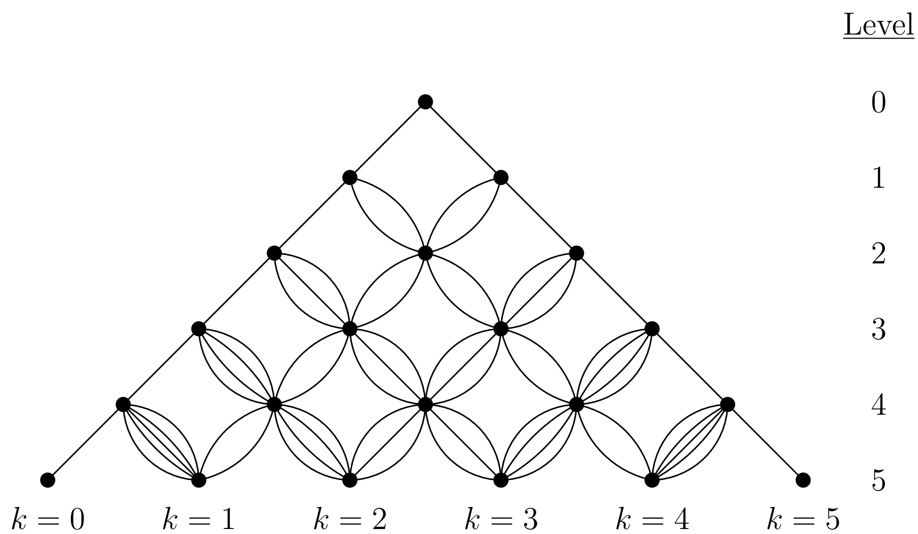

Fig. 1. The Euler graph

Define X to be the space of infinite edge paths in this graph: $X = \{x = (x_n) | n = 0, 1, 2, \dots\}$, each x_n being an edge from level n to level $n + 1$. The vertex through which x passes at level n is denoted by $(n, k_n(x))$. We say that an edge is a *left turn* if it connects vertices (n, k) and $(n + 1, k)$ and a *right turn* if it connects vertices (n, k) and $(n+1, k+1)$. X is a metric space in the usual way: $d(x, y) = 2^{-j}$, where $j = \inf\{i | x_i \ne y_i\}$. The cylinder sets, where a finite number of edges are specified, are clopen sets that generate the topology of X.

As with other Bratteli diagrams, we define a partial order on the edges in the diagram which extends to the entire space of paths. Two edges are comparable if they terminate in the same vertex. For each vertex, totally order the set of edges that terminate at that vertex. This is pictured by agreeing that the minimal edge between two vertices is the leftmost edge, and the edges increase from left to right. Two paths, x, y, are comparable if they agree after some level. Then x is less than y if the last edges of x and y that do not agree, x_i and y_i, are such that x_i is less than y_i in the edge ordering. We define the *adic transformation* on the set of non-maximal paths in X to map a path to the next largest path according to this induced partial order. Then two paths are in the same orbit if and only if they are comparable.

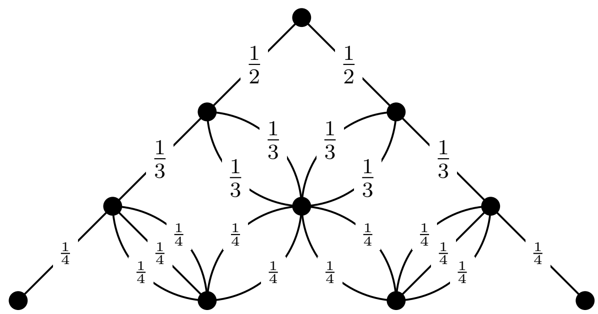

Fig. 2. The symmetric measure given by weights on edges

The *symmetric measure*, η, on the infinite path space X is the Borel probability measure that for each n gives every cylinder of length n starting at the root vertex the same measure. Clearly η is T-invariant. The measure of any cylinder set can be computed by multiplying *weights* on the edges, each weight on an edge connecting level n to level $n+1$ being $1/(n+2)$. We can think of the weights as assigning equal probabilities to all the allowed steps for a random walker who starts at the root and descends step by step to form an infinite path $x \in X$. The main result of [1], proved by probabilistic methods, is that the symmetric measure is ergodic.

There is a bijective correspondence between paths (or cylinders) of length n_0 starting at the root vertex and terminating at vertex (n_0, k_0) and permutations of $\{1, 2, \ldots, n_0+1\}$ with k_0 rises. Consider the cylinder set defined by the single edge connecting the vertex $(0,0)$ to the vertex $(1,0)$. This cylinder set is of length 1 with 1 left turn, and we assign to it the permutation 21, which has one fall. Likewise, the cylinder set defined by the single edge connecting the vertex $(0,0)$ to the vertex $(1,1)$ is of length 1 with one right turn, and we assign to it the permutation 12, which has one rise. When a cylinder F of length n, corresponding to the permutation $\pi(F)$ of $\{1, 2, \ldots, n+1\}$, is extended by an edge from level n to level $n+1$, we extend $\pi(F)$ in a unique way to a permutation of $\{1, 2, \ldots, n+2\}$, as follows. If F is extended by a left turn down the i'th edge connecting (n,k) to $(n+1,k)$, insert $n+2$ into $\pi(F)$ in the i'th place that adds an additional fall to $\pi(F)$. Likewise, if F is extended by a right turn down the i'th edge connecting (n,k) to $(n+1,k+1)$, insert $n+2$ into $\pi(F)$ in the i'th place that adds an additional rise to $\pi(F)$. This correspondence produces a labeling of infinite paths in the Euler graph starting at the root; then the path space X corresponds to the set of all linear orderings of $\mathbb{N} = \{1, 2, 3, \dots\}$, and the adic transformation T can be thought of as moving from an ordering to its successor.

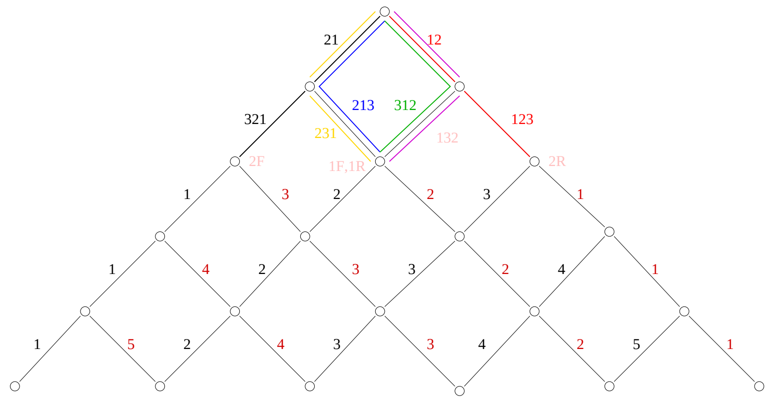

Fig. 3. Some cylinders and their corresponding permutations

2 Uniqueness of the symmetric measure

The *dimension* $\dim(n, k)$ of a vertex (n, k) is defined to be the number of paths connecting the root vertex $(0, 0)$ to the vertex (n, k). For any cylinder F define $\dim(F, (n, k))$ to be the number of paths in F that connect the root vertex $(0, 0)$ to the vertex (n, k). If F is a cylinder starting at the root vertex $(0, 0)$, then $\dim(F, (n, k))$ is the number of paths from the terminal vertex of F to (n, k). Denote by $\mathcal{I}$ the σ-algebra of T-invariant Borel measurable subsets of X. The following is a well-known result about the measures of cylinders in an adic situation.

Lemma 1 (Vershik,[7],[8]) *Let μ be an invariant probability measure for the Euler adic transformation. Then for every cylinder set F and μ-almost every $x \in X$,*

$$\lim_{n\to\infty} \frac{\dim(F, (n, k_n(x)))}{\dim((n, k_n(x)))} = E_\mu(\chi_F|\mathcal{I}).$$

It is clear from the Markov property and the patterns of weights on the edges that with respect to η almost every path $x \in X$ has infinitely many left and right turns.

Lemma 2 *Let μ be an invariant fully-supported ergodic probability measure for the Euler adic transformation. For μ-almost every $x \in X$, there are infinitely many left and right turns (i.e., $k_n(x)$ and $n - k_n(x)$ are unbounded a.e.).*

Proof. For each $K = 1, 2, \ldots,$ let

$$A_K = \{x \in X | x \text{ has no more than } K \text{ right turns}\}.$$

Then A_K is a proper closed T-invariant set. Since μ is ergodic and fully supported, $\mu(A_K) = 0$. Similarly, the measure of the set of paths with a bounded number of left turns is 0. □

Proposition 3 *If μ is an invariant probability measure for the Euler adic transformation such that $k_n(x)$ and $n - k_n(x)$ are unbounded a.e., then $\mu = \eta$.*

Proof. For any string w on an ordered alphabet denote by $r(w)$ the number of rises in w and by $f(w)$ the number of falls in w (defined as above). Let F and F' be cylinder sets in X specified by fixing the first n_0 edges, and let $\pi(F)$ and $\pi(F')$ be the permutations assigned to them by the correspondence described in the preceding section. Suppose that the paths corresponding to F and F' terminate in the vertices (n_0, k_0) and (n_0, k_0') respectively. Fix $n \gg n_0$ and $k \gg k_0$.

Each path s in Γ from (n_0, k_0) to (n, k) corresponds to a permutation σ_s of $\{1, 2, \ldots, n+1\}$ with k rises in which $1, 2, \ldots, n_0 + 1$ appear in the order $\pi(F)$. Counting $\dim(F, (n, k))$ is equivalent to counting the number of distinct such σ_s. Each such permutation σ_s has associated to it a permutation $t(\sigma_s)$ of $\{n_0 + 2, \ldots, n + 1\}$ obtained by deleting $1, 2, \ldots, n_0 + 1$ from σ_s. Taking a reverse view, one obtains σ_s from $\rho = t(\sigma_s)$ by inserting $1, 2, \ldots, n_0 + 1$ from left to right, in the order prescribed by $\pi(F)$, into ρ.

We define a *cluster* in σ_s to be a subset of $\{1, 2, \ldots, n_0+1\}$ whose members are found consecutively in σ_s, with no elements of $\{n_0 + 2, \ldots, n + 1\}$ separating them, in the order prescribed by $\pi(F)$. The set M_s of clusters in σ_s is an ordered partition of the permutation $\pi(F)$, and we define

$$r(M_s) = \sum_{c \in M_s} r(c).$$

For example, if $\pi(F) = 12\ldots(n_0 + 1)$, we could have

$$M_s = \{12, 34, 56, \ldots, n_0(n_0 + 1)\},$$

in which case $r(M_s) = |M_s|$. In general, $1 \leq |M_s| \leq n_0+1$ and $0 \leq r(M_s) \leq k_0$.

Given a permutation ρ of $\{n_0 + 2, \ldots, n + 1\}$, $0 \leq m \leq |\rho| + 1$, and an ordered partition M of $\pi(F)$ with $|M| = m$, there are $C(|\rho|, m)$ (the binomial coefficient) choices for how to insert the members of M as clusters into the permutation ρ in order to form a permutation σ_s. But not all of these choices yield valid permutations σ_s, which have exactly k rises. Looking more closely, we see that placing a cluster $c \in M$ at the tail end of ρ or into a rise in ρ produces a permutation $\overline{\rho}$ whose number of rises is $r(\overline{\rho}) = r(\rho) + r(c)$, while placing c at the beginning or into a fall produces $\overline{\rho}$ with $r(\overline{\rho}) = r(\rho) + r(c) + 1$. So we must have

$$k = r(\sigma_s) = r(\rho) + r(M) + \#\{c \in M | c \text{ is placed into a fall or at the beginning of } \rho\}.$$

Therefore the number of ways to place the members of M into ρ in such a way as to form a valid permutation σ_s, with k rises and $n-k$ falls, is

$$C(n-k+1, k-r(\rho)-r(M))\, C(k+1, m-(k-r(\rho)-r(M))).$$

For each $m = 1, 2, \ldots, n_0+1$ denote by $P_m(F)$ the set of ordered partitions M of $\pi(F)$ such that $|M| = m$. For each $r = 0, 1, \ldots, n-n_0-1$ denote by $Q(n, n_0, r)$ the set of permutations of $\{n_0+2, \ldots, n+1\}$ with exactly r rises, so that $|Q(n, n_0, r)| = A(n-n_0-1, r)$. We sum over all $m = 1, \ldots, n_0+1$, $M \in P_m(F)$, $r = k-r(M)-m, \ldots, k-r(M)$, and $P \in Q(n, n_0, r)$. Letting $s = k-r-r(M)$, so that $0 \le s \le m$,

$$\dim(F, (n,k)) = \sum_{m,M,s,P} C(n-k+1, s)\, C(k+1, m-s).$$

The numerator of each factor in each individual term in this sum consists of m factors chosen from $[n-k-m, n-k] \cup [k-m, k]$, and each denominator is $j!$ for some $j \in [1, n_0+1]$, so that each factor is comparable to either k or $n-k$.

Regrouping the terms, rewrite the sum as

$$\dim(F, (n,k)) = \sum_{r=k-(n_0+1)}^{k} \left(\sum_{m=1}^{n_0+1} \alpha(F, r, m) \right) A(n-n_0-1, r),$$

in which each of the coefficients

$$\beta(F, r) = \sum_{m=1}^{n_0+r} \alpha(F, r, m)$$

is a polynomial in k and $n-k$, with constant coefficients. Clearly the dominant term when k and $n-k$ are large is the one of maximum degree, which occurs when $m = n_0+1$. This corresponds to the partition M of $\pi(F)$ into singletons, so that $r(M) = 0$, when all the elements of $\{1, \ldots, n_0+1\}$ are placed, as singleton clusters, into the rises or falls, or at the end or beginning, of a permutation ρ of $\{n_0+2, \ldots, n+1\}$ which has exactly r rises. The *main point* is that this term is the same for $\pi(F)$ as for any other permutation $\pi(F')$ of $\{1, 2, \ldots, n_0+1\}$:

$$\alpha(F, r, n_0+1) = \alpha(F', r, n_0+1)$$

for all r and all F'. When $k-r$ of $\{1,\dots,n_0+1\}$ are put into falls in ρ or at the beginning, and the rest are put into rises or at the end, no matter which elements are placed in which slots we always produce a permutation σ_s with exactly k rises.

Let us now consider the ratio $\dim(F,(n,k))/\dim(F',(n,k))$ when n and k are both very large. Divide top and bottom by the sum on r of the dominant terms (taking maximum degree coefficients in k and $n-k$ for each r),

$$\sum_{r=k-(n_0+1)}^{k} \alpha(F,r,n_0+1)A(n-n_0-1,r),$$

which is the *same* for F and F'. This shows that the ratio is very near 1 when k and $n-k$ are both very large.

Thus if $k_n(x),\ n-k_n(x)\to\infty$ a.e. $d\mu$, we have for any two cylinders F and F' starting at the root vertex and of the same length that

$$E_\mu(X_F|\mathcal{I})(x)=E_\mu(X_{F'}|\mathcal{I})(x)\ \text{ a.e. } d\mu.$$

Integrating gives $\mu(F)=\mu(F')$, so that $\mu=\eta$. $\square$

Theorem 4 *The symmetric measure η is ergodic and is the only fully supported invariant ergodic Borel probability measure for the Euler adic transformation.*

Proof. If we show that there is an ergodic measure μ which has $k_n(x)$ and $n-k_n(x)$ unbounded a.e., then it will follow from the Proposition that $\mu=\eta$, and hence η is ergodic and is the only fully-supported T-invariant ergodic measure on X.

If an ergodic measure has, say, $k_n(x)$ bounded on a set of positive measure, then $k_n(x)$ is bounded a.e., since each set $\{x|k_n(x)\le K\}$ is T-invariant. Let $\mathcal{E}_0=\emptyset$, and for each $K=1,2,\dots$ let $\mathcal{E}_K$ be the set of ergodic measures supported on either $\{x\in X|k_n(x)\le K \text{ for all } n\}$ or $\{x\in X|n-k_n(x)\le K \text{ for all } n\}$. If no ergodic measure has $k_n(x)$ and $n-k_n(x)$ unbounded a.e., then the set of ergodic measures is

$$\mathcal{E}=\bigcup_K \mathcal{E}_K.$$

Form the ergodic decomposition of η:

$$\eta=\int_{\mathcal{E}} e\,dP_\eta(e)=\sum_{K=1}^{\infty}\int_{\mathcal{E}_K\setminus\mathcal{E}_{K-1}} e\,dP_\eta(e).$$

If S is the set of paths $x \in X$ for which both $k_n(x)$ and $n - k_n(x)$ are unbounded, then, from the remark before Lemma 2, $\eta(S) = 1$; but, for each K, $e(S) = 0$ for all e in $\mathcal{E}_K$. Hence there is an ergodic measure for which $k_n(x)$ and $n - k_n(x)$ are unbounded a.e.. $\square$

Remark. The connection of this Theorem with the statements made in the Abstract and Introduction about random permutations can be seen as follows. As noted above, the space X of infinite paths in the Euler graph is in correspondence with the set $\mathcal{L}$ of linear orderings of $\mathbb{N}$. A cylinder set in X determined by fixing an initial path of length n corresponds, as explained above, to a permutation π_{n+1} in the group S_{n+1} of permutations of $\{1, 2, \ldots, n+1\}$, and thus to the set of all elements of $\mathcal{L}$ for which $1, 2, \ldots, n+1$ appear in the order specified by π_{n+1}. This family of clopen cylinder sets defines a compact metrizable topology and a Borel structure on $\mathcal{L}$. One way to speak about "random permutations" would be to give a Borel probability measure on $\mathcal{L}$. A Borel probability measure on X is T-invariant if and only if the corresponding measure on $\mathcal{L}$ assigns, for each n and $0 \leq k \leq n$, equal measure to all cylinders determined by permutations $\pi_{n+1} \in S_{n+1}$ which have the *same fixed number k of rises*. According to the Theorem, the only such measure is the one determined by the symmetric measure η on X, which assigns to each basic set $\mathcal{L}(\pi_{n+1})$, for all $\pi_{n+1} \in S_{n+1}$, *no matter the number of rises* in π_{n+1}, the *same* measure, $1/(n+1)!$.

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References

[1] S. Bailey, M. Keane, K. Petersen, and I. Salama. Ergodicity of the adic transformation on the Euler graph. to appear in Math. Proc. Camb. Phil. Soc.

[2] L. Carlitz, D.C. Kurtz, R. Scoville, and O.P. Stackelberg. Asymptotic properties of Eulerian numbers. *Zeit Wahr. verw. Geb.*, 23:47–54, 1972.

[3] Carl-Gustav Esseen. On the application of the theory of probability to two combinatorial problems involving permutations. In *Proceedings of the Seventh Conference on Probability Theory (Braşov, 1982).* VNU Sci. Press, Utrecht, 1985.

[4] James C. Fu and W.Y. Wendy Lou. Joint distribution of rises and falls. *Ann. Inst. Statist. Math.*, 52:415–425, 2000.

[5] James C. Fu, W.Y. Wendy Lou, and Yueh-Jir Wang. On the exact distributions of Eulerian and Simon Newcomb numbers associated with random permutations. *Stat. Prob. Letters*, 42:115–125, 1999.

[6] G. Oshanin and R. Voituriez. Random walk generated by random permutations of $\{1, 2, 3, \ldots, n+1\}$. *J. Phys. A: Math. Gen.*, 37:6221–6241, 2004.

[7] A.M. Vershik. Description of invariant measures for the actions of some infinite-dimensional groups. *Dokl. Akad. Nauk SSSR*, 218:749–752, 1974.

[8] A.M. Vershik and S.V. Kerov. Asymptotic theory of characters of the symmetric group. *Funkts. Anal. Prilozhen.*, 15:15–27, 1981.