Bratteli-Vershik Systems Associated to Positive Integer Polynomials

Sarah Bailey

University of North Carolina

History of Bratteli-Vershik Systems

- Bratteli first introduced diagrams in 1972 as a tool to study approximately finite C^*-algebras.

History of Bratteli-Vershik Systems

- Bratteli first introduced diagrams in 1972 as a tool to study approximately finite C^*-algebras.
- In 1976 Elliott introduced the idea of a dimension group as an isomorphism invariant of C^*-algebras and of their associated Bratteli diagrams.

History of Bratteli-Vershik Systems

- Bratteli first introduced diagrams in 1972 as a tool to study approximately finite C^*-algebras.
- In 1976 Elliott introduced the idea of a dimension group as an isomorphism invariant of C^*-algebras and of their associated Bratteli diagrams.
- Then in 1985, Vershik associated a dynamical system to the diagrams. These systems became known as Bratteli-Vershik systems.

History of Bratteli-Vershik Systems

- Bratteli first introduced diagrams in 1972 as a tool to study approximately finite C^*-algebras.
- In 1976 Elliott introduced the idea of a dimension group as an isomorphism invariant of C^*-algebras and of their associated Bratteli diagrams.
- Then in 1985, Vershik associated a dynamical system to the diagrams. These systems became known as Bratteli-Vershik systems.
- In 1992, Herman, Putnam, and Skau showed that every Cantor minimal system is isomorphic to a Bratteli-Vershik system generated by a essentially simple diagram, and made connections between the C^*-algebra theory and the dynamics associated with the diagram.

Bratteli Diagrams

Definition 1: A *Bratteli diagram* is an infinite directed graph with the following properties:

Bratteli Diagrams

Definition 1: A *Bratteli diagram* is an infinite directed graph with the following properties:

- The vertices and edges can be partitioned into finite sets: $\mathcal{V} = V_0 \cup V_1 \cup V_2$... and $\mathcal{E} = E_1 \cup E_2 \cup E_3 \ldots$

Bratteli Diagrams

Definition 1: A *Bratteli diagram* is an infinite directed graph with the following properties:

- The vertices and edges can be partitioned into finite sets: $\mathcal{V} = V_0 \cup V_1 \cup V_2 ...$ and $\mathcal{E} = E_1 \cup E_2 \cup E_3 \dots$
- $V_0 = \{v_0\}$

Bratteli Diagrams

Definition 1: A *Bratteli diagram* is an infinite directed graph with the following properties:

- The vertices and edges can be partitioned into finite sets: $\mathcal{V} = V_0 \cup V_1 \cup V_2...$ and $\mathcal{E} = E_1 \cup E_2 \cup E_3 \ldots$
- $V_0 = \{v_0\}$
- The edges in E_n connect the vertices in V_{n-1} to the vertices in V_n.

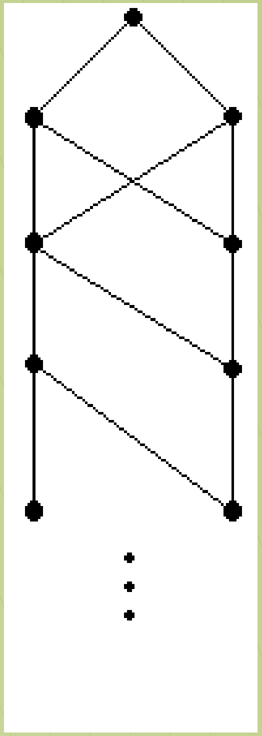
Bratteli Diagrams

Definition 1: A *Bratteli diagram* is an infinite directed graph with the following properties:

- The vertices and edges can be partitioned into finite sets: $\mathcal{V} = V_0 \cup V_1 \cup V_2 ...$ and $\mathcal{E} = E_1 \cup E_2 \cup E_3 \ldots$
- $V_0 = \{v_0\}$
- The edges in E_n connect the vertices in V_{n-1} to the vertices in V_n.
- All vertices in $\mathcal{V}$ have at least one edge leaving from it, and all vertices except v_0 have at least one edge coming into it.

Example

Example of $(\mathcal{V}, \mathcal{E})$:



$$|V_1| = |V_2| = |V_3| = |V_4| = 2$$

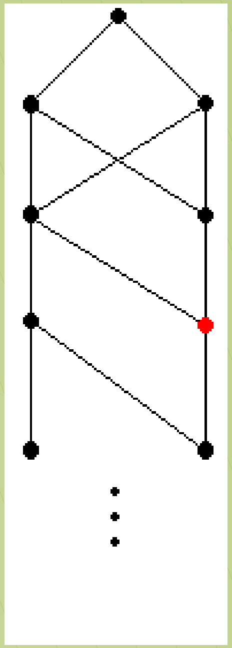
$|E_1| = 2$, $|E_2| = 4$, and $|E_3| = |E_4| = 3$.

Notation

For a specific vertex $v \in \mathcal{V}$, denote v by (k, l) whenever v is in V_k and is $l + 1$ in from the right.

Notation

For a specific vertex $v \in \mathcal{V}$, denote v by (k, l) whenever v is in V_k and is $l + 1$ in from the right.



The red vertex is denoted $(3, 0)$.

Bratteli diagram associated to $p(x)$

Let $a_0, a_1, ..., a_n \in \mathbb{N}$ and $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$.

Bratteli diagram associated to $p(x)$

Let $a_0, a_1, ..., a_n \in \mathbb{N}$ and $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$.

Define the *Bratteli diagram associated to* $p(x)$, $(\mathcal{V}, \mathcal{E})_{p(x)}$, to be the Bratteli diagram such that:

Bratteli diagram associated to $p(x)$

Let $a_0, a_1, ..., a_n \in \mathbb{N}$ and $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$.

Define the *Bratteli diagram associated to* $p(x)$, $(\mathcal{V}, \mathcal{E})_{p(x)}$, to be the Bratteli diagram such that:

- $|V_0| = 1$ and $|V_k| = |V_{k-1}| + n$ for all $k > 0$.

Bratteli diagram associated to $p(x)$

Let $a_0, a_1, ..., a_n \in \mathbb{N}$ and $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$.

Define the *Bratteli diagram associated to* $p(x)$, $(\mathcal{V}, \mathcal{E})_{p(x)}$, to be the Bratteli diagram such that:

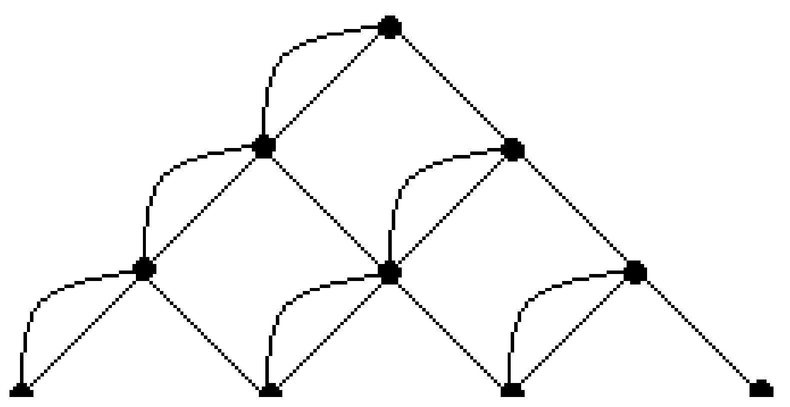
- $|V_0| = 1$ and $|V_k| = |V_{k-1}| + n$ for all $k > 0$.
- The number of edges from (k, l) to $(k+1, l+j)$ is a_j.

Bratteli diagram associated to $p(x)$

Let $a_0, a_1, ..., a_n \in \mathbb{N}$ and $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$.

Define the *Bratteli diagram associated to* $p(x)$, $(\mathcal{V}, \mathcal{E})_{p(x)}$, to be the Bratteli diagram such that:

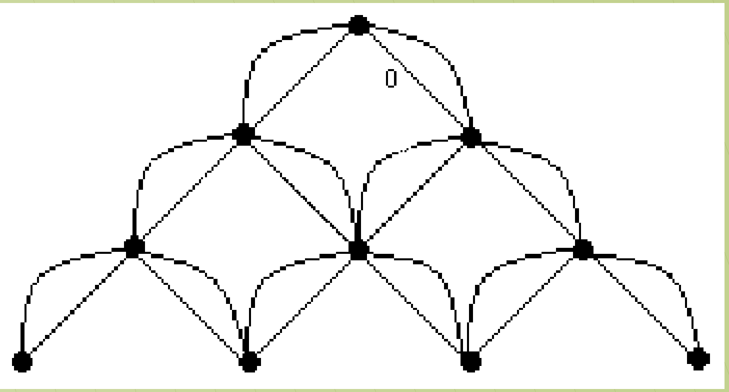
- $|V_0| = 1$ and $|V_k| = |V_{k-1}| + n$ for all $k > 0$.
- The number of edges from (k, l) to $(k+1, l+j)$ is a_j.



$(V, E)_{2x+1}$

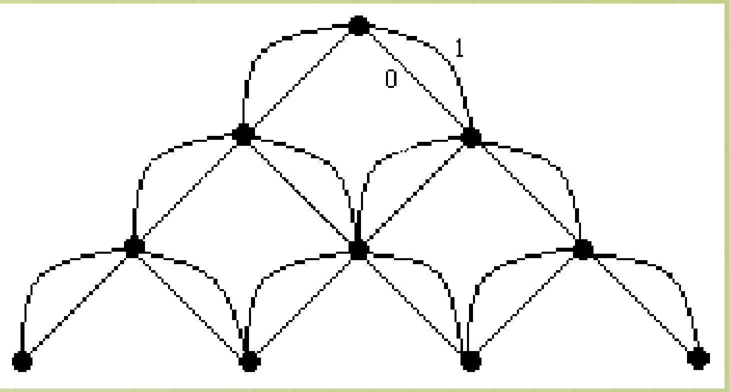
Labelling

- Beginning with the leftmost edge connecting $(0, 0)$ with $(1, 0)$ label it 0.



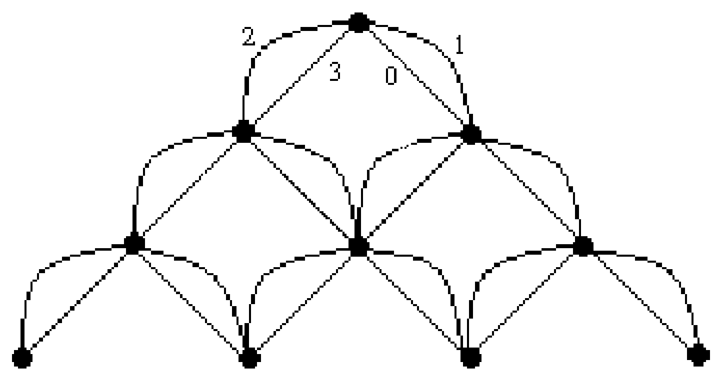
Labelling

- Beginning with the leftmost edge connecting $(0, 0)$ with $(1, 0)$ label it 0.
- Continue labelling the edges connecting $(0, 0)$ to $(1, 0)$ in numerical order until all a_0 have been labelled from 0 to $a_0 - 1$.



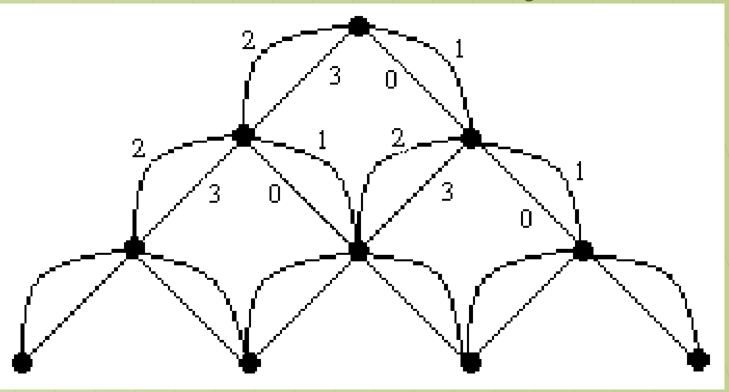
Labelling

- Beginning with the leftmost edge connecting $(0,0)$ with $(1,0)$ label it 0.
- Continue labelling the edges connecting $(0,0)$ to $(1,0)$ in numerical order until all a_0 have been labelled from 0 to $a_0 - 1$.
- Label left to right in numerical order beginning with a_0 the edges connecting $(0,0)$ with $(1,1)$.



Labelling

- Beginning with the leftmost edge connecting $(0,0)$ with $(1,0)$ label it 0.
- Continue labelling the edges connecting $(0,0)$ to $(1,0)$ in numerical order until all a_0 have been labelled from 0 to $a_0 - 1$.
- Label left to right in numerical order beginning with a_0 the edges connecting $(0,0)$ with $(1,1)$.
- In this same fashion label the edges connecting (k,l) to $(k+1, l+j)$ with labels $a_0 + ... + a_{j-1}$ through $a_0 + ... + a_j - 1$



Partial Order

- Using this labelling, we will define a partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$.

Partial Order

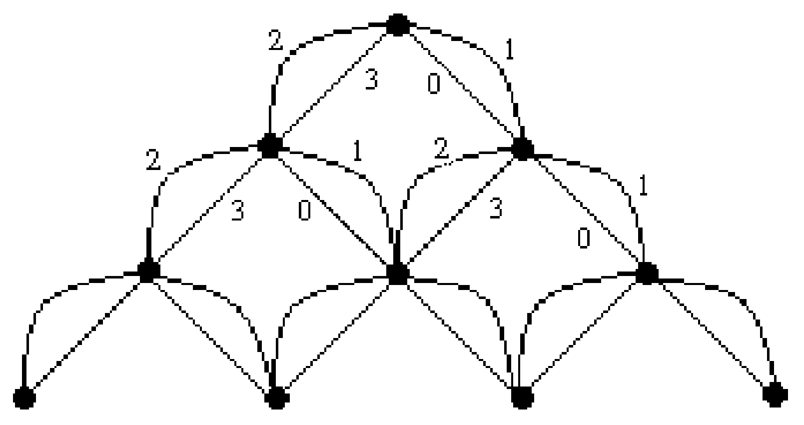
- Using this labelling, we will define a partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$.
- Two edges, e_0, e_1 are said to be *comparable* if they have the same range. Then $e_0 < e_1$ if the label corresponding to e_0 is less than the label corresponding to e_1.

Partial Order

- Using this labelling, we will define a partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$.
- Two edges, e_0, e_1 are said to be *comparable* if they have the same range. Then $e_0 < e_1$ if the label corresponding to e_0 is less than the label corresponding to e_1.
- The labelling has been constructed so that edges with the same range increase as you look up from left to right.

Partial Order

- Using this labelling, we will define a partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$.
- Two edges, e_0, e_1 are said to be *comparable* if they have the same range. Then $e_0 < e_1$ if the label corresponding to e_0 is less than the label corresponding to e_1.
- The labelling has been constructed so that edges with the same range increase as you look up from left to right.



The edges with range (2,1) increase from the left to right.

Path Space

- For any $(\mathcal{V},\mathcal{E})_{p(x)}$, there is an associated path space $X_{p(x)}$ which consists of infinite edge paths on $(\mathcal{V},\mathcal{E})_{p(x)}$. If $x \in X_{p(x)}$ define $(x)_k$ to be the edge label in the k^{th} position.

$$x = 1023210.... \text{ then } (x)_0 = 1,\ (x)_1 = 0, (x)_2 = 2\ldots$$

Path Space

- For any $(\mathcal{V},\mathcal{E})_{p(x)}$, there is an associated path space $X_{p(x)}$ which consists of infinite edge paths on $(\mathcal{V},\mathcal{E})_{p(x)}$. If $x \in X_{p(x)}$ define $(x)_k$ to be the edge label in the k^{th} position.

$$x = 1023210.... \text{ then } (x)_0 = 1,\ (x)_1 = 0, (x)_2 = 2\ldots$$

- This is a compact Hausdorf space. A metric is given by

$$d(x,y) = \{\frac{1}{2^i} |\ i \text{ is the first place } (x)_i \neq (y)_i\}.$$

Path Space

- For any $(\mathcal{V},\mathcal{E})_{p(x)}$, there is an associated path space $X_{p(x)}$ which consists of infinite edge paths on $(\mathcal{V},\mathcal{E})_{p(x)}$. If $x \in X_{p(x)}$ define $(x)_k$ to be the edge label in the k^{th} position.

$$x = 1023210.... \text{ then } (x)_0 = 1,\ (x)_1 = 0, (x)_2 = 2\dots$$

- This is a compact Hausdorf space. A metric is given by

$$d(x,y) = \{\frac{1}{2^i}|\ i \text{ is the first place } (x)_i \neq (y)_i\}.$$

- By $(k)^\infty$ we will mean the infinite edge path consisting of all k's,

$$kkkkkkkkkk....$$

Partial order on $X_{p(x)}$

- The partial order on the edges of $(\mathcal{V},\mathcal{E})_{p(x)}$ can be extended to the edge paths in $X_{p(x)}$.

Partial order on $X_{p(x)}$

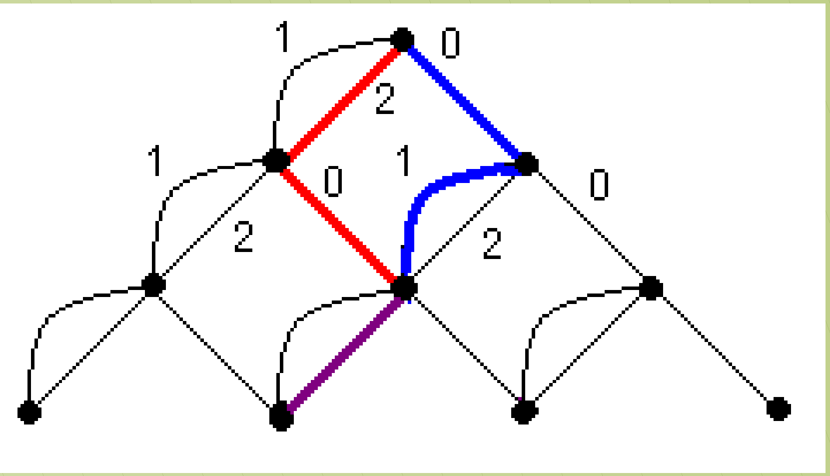
- The partial order on the edges of $(\mathcal{V},\mathcal{E})_{p(x)}$ can be extended to the edge paths in $X_{p(x)}$.
- Two paths x and y are comparable if they both pass through some vertex, $(k,m) \neq (0,0)$, $(x)_k \neq (y)_k$, and $(x)_j = (y)_j$ for all $j > k$.

Partial order on $X_{p(x)}$

- The partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$ can be extended to the edge paths in $X_{p(x)}$.
- Two paths x and y are comparable if they both pass through some vertex, $(k, m) \neq (0, 0)$, $(x)_k \neq (y)_k$, and $(x)_j = (y)_j$ for all $j > k$.
- $x < y$ if at the first occurrence of the above condition, $(x)_k < (y)_k$.

Partial order on $X_{p(x)}$

- The partial order on the edges of $(\mathcal{V}, \mathcal{E})_{p(x)}$ can be extended to the edge paths in $X_{p(x)}$.
- Two paths x and y are comparable if they both pass through some vertex, $(k, m) \neq (0, 0)$, $(x)_k \neq (y)_k$, and $(x)_j = (y)_j$ for all $j > k$.
- $x < y$ if at the first occurrence of the above condition, $(x)_k < (y)_k$.



Adic Transformation

- Paths in $X_{p(x)}$ are maximal if there are no greater paths.

Adic Transformation

- Paths in $X_{p(x)}$ are maximal if there are no greater paths.
- Define the adic transformation $T_{p(x)}$ on $(\mathcal{V},\mathcal{E})_{p(x)}$ by the following:

$$T_{p(x)} : X_{p(x)} \to X_{p(x)}$$

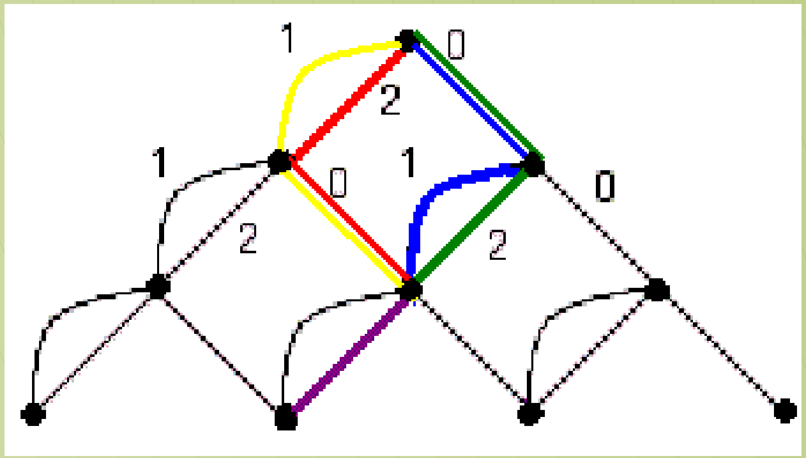
$$T_{p(x)}(x) = \begin{cases} \text{the smallest } y > x & x \notin X_{p(x)}^{max} \\ (0)^\infty & x = (a_0 - 1)^\infty \\ (a_0 + \ldots + a_{n-1})^\infty & x = (a_0 + \ldots + a_n - 1)^\infty \\ x & \text{otherwise} \end{cases}$$

Adic Transformation

- Paths in $X_{p(x)}$ are maximal if there are no greater paths.
- Define the adic transformation $T_{p(x)}$ on $(\mathcal{V},\mathcal{E})_{p(x)}$ by the following:

$$T_{p(x)} : X_{p(x)} \to X_{p(x)}$$

$$T_{p(x)}(x) = \begin{cases} \text{the smallest } y > x & x \notin X_{p(x)}^{max} \\ (0)^\infty & x = (a_0 - 1)^\infty \\ (a_0 + \ldots + a_{n-1})^\infty & x = (a_0 + \ldots + a_n - 1)^\infty \\ x & \text{otherwise} \end{cases}$$



Invariant Measures

Invariant Measures

- $(X_{p(x)}, T_{p(x)})$ is a topological dynamical system, so the next question to ask is about the $T_{p(x)}$-invariant ergodic measures.

Invariant Measures

- $(X_{p(x)}, T_{p(x)})$ is a topological dynamical system, so the next question to ask is about the $T_{p(x)}$-invariant ergodic measures.
- Vershik showed in the case (X_{x+1}, T_{x+1}), the invariant ergodic measures are the Bernoulli measures $\mathcal{B}(p, 1-p)$.

Invariant Measures

- $(X_{p(x)}, T_{p(x)})$ is a topological dynamical system, so the next question to ask is about the $T_{p(x)}$-invariant ergodic measures.
- Vershik showed in the case (X_{x+1}, T_{x+1}), the invariant ergodic measures are the Bernoulli measures $\mathcal{B}(p, 1-p)$.
- Méla showed in the case that all the coefficients of $p(x)$ are one, that the invariant ergodic measures are the Bernoulli measure $\mathcal{B}(0, ..., 0, 1)$ and the one-parameter family $\mathcal{B}(q, t_q, \frac{t_q^2}{q}, \frac{t_q^3}{q^2}, .., \frac{t_q^n}{q^{n-1}})$, where t_q is the unique solution in [0,1] to the equation:

$$q^n - q^{n-1} + q^{n-1}t + q^{n-2}t^2 + ... + qt^{n-1} + t^n = 0.$$

Theorem 1

Theorem 1

Theorem 1. *The $T_{p(x)}$-invariant, ergodic probability measures for $(X_{p(x)}, T_{p(x)})$ are the one-parameter family of Bernoulli measures*

$$\mathcal{B}\left(\underbrace{q,\dots,q}_{a_0\ times},\underbrace{t_q,\dots,t_q}_{a_1\ times},\underbrace{\frac{t_q^2}{q},\dots,\frac{t_q^2}{q}}_{a_2\ times},\dots,\underbrace{\frac{t_q^n}{q^{n-1}},\dots,\frac{t_q^n}{q^{n-1}}}_{a_n\ times}\right),$$

where $q \in (0, \frac{1}{a_0})$, and t_q is the unique solution in $[0,1]$ to the equation:

$$a_0 q^n + a_1 q^{n-1} t + \dots + a_n t^n - q^{n-1} = 0,$$

as well as $\mathcal{B}\left(\underbrace{\frac{1}{a_0},\dots,\frac{1}{a_0}}_{a_0\ times},0,\dots,0\right)$*, and* $\mathcal{B}\left(0,\dots,0,\underbrace{\frac{1}{a_n},\dots,\frac{1}{a_n}}_{a_n\ times}\right)$.

Skip Proof

Sketch of Proof

Proof:

- If C^k is the cylinder set C extended by k, and μ is a fully-supported ergodic measure, then

$$\frac{\mu(C^k)}{\mu(C)} = \frac{\mu(C^{jk})}{\mu(C^j)}$$

Sketch of Proof

Proof:

- If C^k is the cylinder set C extended by k, and μ is a fully-supported ergodic measure, then

$$\frac{\mu(C^k)}{\mu(C)} = \frac{\mu(C^{jk})}{\mu(C^j)}$$

- This implies $\frac{\mu(C^k)}{\mu(C)}$ is independent of C, hence all fully-supported ergodic $T_{p(x)}-$invariant measures are Bernoulli.

Sketch of Proof

Proof:

- If C^k is the cylinder set C extended by k, and μ is a fully-supported ergodic measure, then

$$\frac{\mu(C^k)}{\mu(C)} = \frac{\mu(C^{jk})}{\mu(C^j)}$$

- This implies $\frac{\mu(C^k)}{\mu(C)}$ is independent of C, hence all fully-supported ergodic $T_{p(x)}-$invariant measures are Bernoulli.
- From here on out we assume the measures are Bernoulli.

Continuation of Proof

- Assuming invariant Bernoulli measures, we define independent, identically distributed random variables $X_k(x) = (x)_k$.

Continuation of Proof

- Assuming invariant Bernoulli measures, we define independent, identically distributed random variables $X_k(x) = (x)_k$.
- Then the Hewitt-Savage zero-one law gives that the σ-algebra generated by the sets of infinite paths in $X_{p(x)}$ with the first k edges as permutations of one another is trivial.

Continuation of Proof

- Assuming invariant Bernoulli measures, we define independent, identically distributed random variables $X_k(x) = (x)_k$.
- Then the Hewitt-Savage zero-one law gives that the σ-algebra generated by the sets of infinite paths in $X_{p(x)}$ with the first k edges as permutations of one another is trivial.
- The σ-algebra generated by the $T_{p(x)}$ invariant sets is a coarser σ-algebra, hence also trivial.

Continuation of Proof

- Assuming invariant Bernoulli measures, we define independent, identically distributed random variables $X_k(x) = (x)_k$.
- Then the Hewitt-Savage zero-one law gives that the σ-algebra generated by the sets of infinite paths in $X_{p(x)}$ with the first k edges as permutations of one another is trivial.
- The σ-algebra generated by the $T_{p(x)}$ invariant sets is a coarser σ-algebra, hence also trivial.
- So the invariant Bernoulli measures are ergodic. It only remains to determine what these measures are.

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.
- In particular, edges that connect the same two vertices have the same probability.

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.
- In particular, edges that connect the same two vertices have the same probability.
- Then the entire probability vector is determined solely by $\mu([0]) = q$ and $\mu([a_0]) = t$.

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.
- In particular, edges that connect the same two vertices have the same probability.
- Then the entire probability vector is determined solely by $\mu([0]) = q$ and $\mu([a_0]) = t$.
- Since the probabilities sum to 1 we have:

$$a_0 q^n + a_1 q^{n-1} t + \ldots + a_n t^n - q^{n-1} = 0.$$

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.
- In particular, edges that connect the same two vertices have the same probability.
- Then the entire probability vector is determined solely by $\mu([0]) = q$ and $\mu([a_0]) = t$.
- Since the probabilities sum to 1 we have:

$$a_0 q^n + a_1 q^{n-1} t + \ldots + a_n t^n - q^{n-1} = 0.$$

- Then the IVT, and the first derivative test tells us that there is a unique solution, t_q, in $[0, 1]$.

Continuation of Proof

- Any measure is $T_{p(x)}$-invariant if and only if all cylinders with the same terminal vertex have the same measure.
- In particular, edges that connect the same two vertices have the same probability.
- Then the entire probability vector is determined solely by $\mu([0]) = q$ and $\mu([a_0]) = t$.
- Since the probabilities sum to 1 we have:

$$a_0q^n + a_1q^{n-1}t + ... + a_nt^n - q^{n-1} = 0.$$

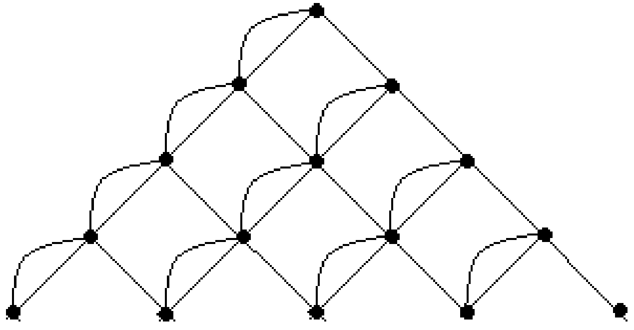
- Then the IVT, and the first derivative test tells us that there is a unique solution, t_q, in $[0, 1]$.
- The fully-supported, $T_{p(x)}$-invariant, ergodic measures are as stated in the theorem.

Conclusion of Proof

- It remains to determine the $T_{p(x)}$-invariant ergodic measures that are not fully-supported.

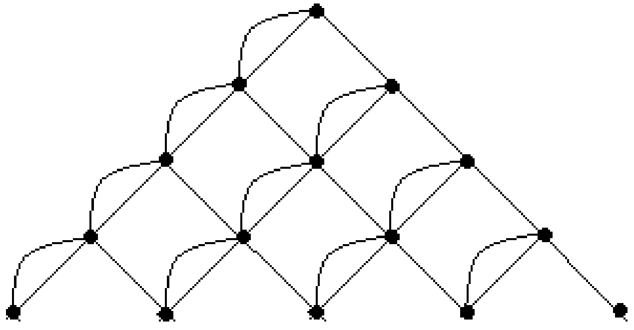
Conclusion of Proof

- It remains to determine the $T_{p(x)}$-invariant ergodic measures that are not fully-supported.
- The only closed invariant sets are the infinite rectangles defined by $(0, 0)$ and a eventually diagonal path.



Conclusion of Proof

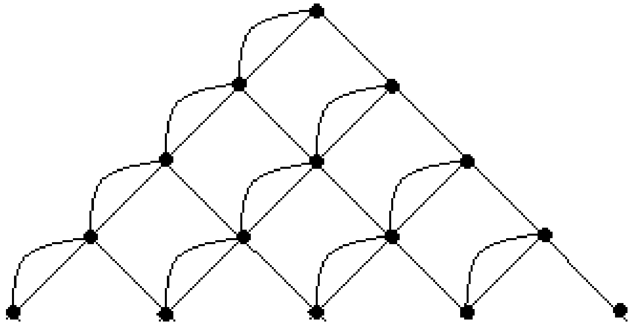
- It remains to determine the $T_{p(x)}$-invariant ergodic measures that are not fully-supported.
- The only closed invariant sets are the infinite rectangles defined by $(0,0)$ and a eventually diagonal path.



- Consider $C = [j]$ (where j is not on the far edges), and by the ergodic theorem, $\mu(C) = \lim\limits_{m\to\infty} \dfrac{1}{m} \displaystyle\sum_{k=0}^{m-1} \chi_c \circ T^k = 0$.

Conclusion of Proof

- It remains to determine the $T_{p(x)}$-invariant ergodic measures that are not fully-supported.
- The only closed invariant sets are the infinite rectangles defined by $(0,0)$ and a eventually diagonal path.



- Consider $C = [j]$ (where j is not on the far edges), and by the ergodic theorem, $\mu(C) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \chi_c \circ T^k = 0$.
- Repeat this process until all support is on the far edges.◇

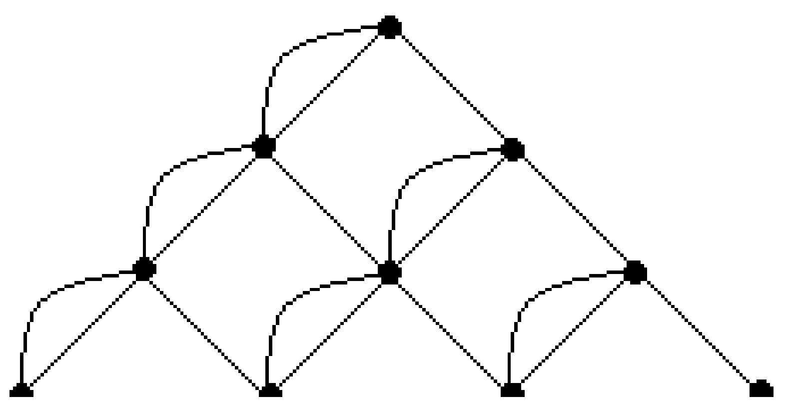
Associated Dimension Groups

Incidence Matrices

For a Bratteli Diagram $(\mathcal{V}, \mathcal{E})$, the incidence matrices ϕ_k are the $|V_{k-1}| \times |V_k|$ matrices such that $(\phi_k)_{ij}$ is the number of edges connecting $(k-1, i)$ with (k, j).

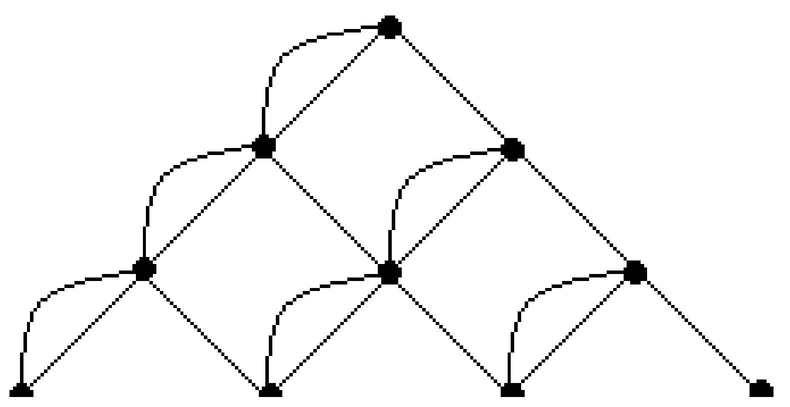
Incidence Matrices

For a Bratteli Diagram $(\mathcal{V}, \mathcal{E})$, the incidence matrices ϕ_k are the $|V_{k-1}| \times |V_k|$ matrices such that $(\phi_k)_{ij}$ is the number of edges connecting $(k-1, i)$ with (k, j).



Incidence Matrices

For a Bratteli Diagram $(\mathcal{V}, \mathcal{E})$, the incidence matrices ϕ_k are the $|V_{k-1}| \times |V_k|$ matrices such that $(\phi_k)_{ij}$ is the number of edges connecting $(k-1, i)$ with (k, j).

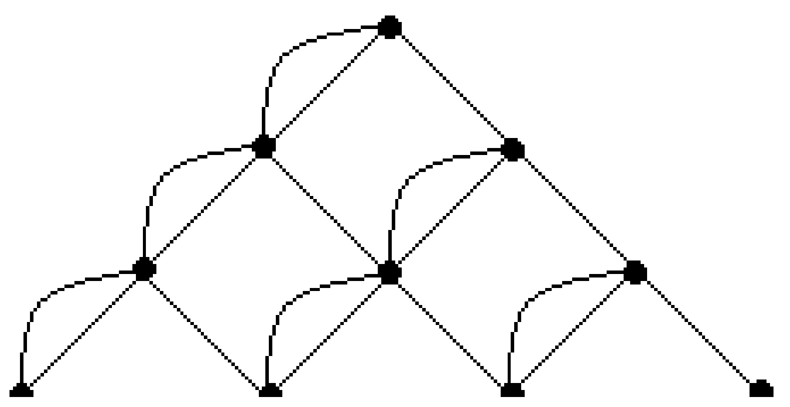


Here,

$$\phi_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Incidence Matrices

For a Bratteli Diagram $(\mathcal{V}, \mathcal{E})$, the incidence matrices ϕ_k are the $|V_{k-1}| \times |V_k|$ matrices such that $(\phi_k)_{ij}$ is the number of edges connecting $(k-1, i)$ with (k, j).



Here,

$$\phi_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:
  - $G_+ + G_+ = G_+$

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:
  - $G_+ + G_+ = G_+$
  - $G_+ - G_+ = G$

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:
    - $G_+ + G_+ = G_+$
    - $G_+ - G_+ = G$
    - $G_+ \cap -G_+ = \{0\}$

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:
  - $G_+ + G_+ = G_+$
  - $G_+ - G_+ = G$
  - $G_+ \cap -G_+ = \{0\}$
- The ordered group $\mathbb{Z}^n$ with the positive set $G_+ = \mathbb{Z}^n_+$ is called a *simplicial* group.

Ordered Group

- An ordered group is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:
  - $G_+ + G_+ = G_+$
  - $G_+ - G_+ = G$
  - $G_+ \cap -G_+ = \{0\}$
- The ordered group $\mathbb{Z}^n$ with the positive set $G_+ = \mathbb{Z}^n_+$ is called a *simplicial* group.
- A *dimension group* is an ordered group which is isomorphic to a direct limit of simplicial groups.

Associated Dimension Groups

For any Bratteli diagram, there is an associated dimension group, $K_0((V, E))$, namely the direct limit of the following:

Associated Dimension Groups

For any Bratteli diagram, there is an associated dimension group, $K_0((V, E))$, namely the direct limit of the following:

$$\mathbb{Z}^{|V_0|=1} \xrightarrow{\phi_1} \mathbb{Z}^{|V_1|} \xrightarrow{\phi_2} \mathbb{Z}^{|V_2|} \xrightarrow{\phi_3} \ldots$$

where ϕ_k is the indicidence matrix between levels $k-1$ and k of the Bratteli diagram.

Associated Dimension Groups

For any Bratteli diagram, there is an associated dimension group, $K_0((V,E))$, namely the direct limit of the following:

$$\mathbb{Z}^{|V_0|=1} \xrightarrow{\phi_1} \mathbb{Z}^{|V_1|} \xrightarrow{\phi_2} \mathbb{Z}^{|V_2|} \xrightarrow{\phi_3} \ldots$$

where ϕ_k is the indicidence matrix between levels $k-1$ and k of the Bratteli diagram.

The positive set consists of the equivalence classes for which there is a representative which is a vector with all positive entries.

$K^0((X,T))$

- Let X be a totally disconnected compact metric space, and let $C(X,\mathbb{Z})$ denote the continuous functions on X with values in $\mathbb{Z}$.

$K^0((X,T))$

- Let X be a totally disconnected compact metric space, and let $C(X,\mathbb{Z})$ denote the continuous functions on X with values in $\mathbb{Z}$.
- The coboundary operator, $\partial_T : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ is defined by $\partial_T(f) = f - f \circ T$, and $f - f \circ T$ is called a coboundary.

$K^0((X,T))$

- Let X be a totally disconnected compact metric space, and let $C(X,\mathbb{Z})$ denote the continuous functions on X with values in $\mathbb{Z}$.
- The coboundary operator, $\partial_T : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ is defined by $\partial_T(f) = f - f \circ T$, and $f - f \circ T$ is called a coboundary.
- Let $K^0(X,T) = C(X,\mathbb{Z})/\partial_T C(X,\mathbb{Z})$

$K^0((X,T))$

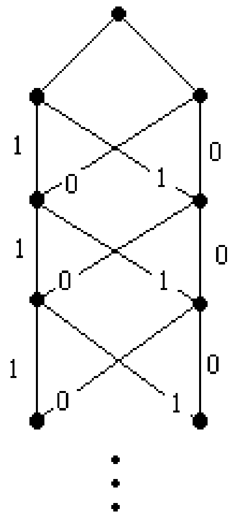
- Let X be a totally disconnected compact metric space, and let $C(X,\mathbb{Z})$ denote the continuous functions on X with values in $\mathbb{Z}$.
- The coboundary operator, $\partial_T : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ is defined by $\partial_T(f) = f - f \circ T$, and $f - f \circ T$ is called a coboundary.
- Let $K^0(X,T) = C(X,\mathbb{Z})/\partial_T C(X,\mathbb{Z})$
- Define the positive cone $K^0(X,T)^+ = \{[f] | f \in C(X,\mathbb{Z}^+)\}$.

Essentially Simple

- An ordered Bratteli diagram $(\mathcal{V}, \mathcal{E}, \geq)$ is essentially simple if X^{max} and X^{min} are both one point sets.

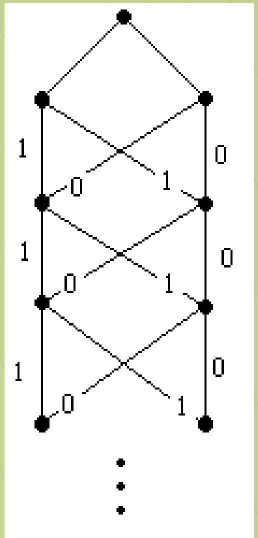
Essentially Simple

- An ordered Bratteli diagram $(\mathcal{V}, \mathcal{E}, \geq)$ is essentially simple if X^{max} and X^{min} are both one point sets.



Essentially Simple

- An ordered Bratteli diagram $(\mathcal{V}, \mathcal{E}, \geq)$ is essentially simple if X^{max} and X^{min} are both one point sets.



HPS If $(\mathcal{V}, \mathcal{E}, \geq)$ is an essentially simple ordered Bratteli diagram and (X, T) is the associated Bratteli Vershik system, is a natural order isomorphism:

- $K^0(X, \phi) \simeq K_0((V, E))$

$(\mathcal{V},\mathcal{E})_{p(x)}$

- $(\mathcal{V},\mathcal{E})_{p(x)}$ has countably many paths in $X_{p(x)}^{max}$ and $X_{p(x)}^{min}$, so not essentially simple. Can this result be extended?

$(\mathcal{V},\mathcal{E})_{p(x)}$

- $(\mathcal{V},\mathcal{E})_{p(x)}$ has countably many paths in $X_{p(x)}^{max}$ and $X_{p(x)}^{min}$, so not essentially simple. Can this result be extended?

$(\mathcal{V},\mathcal{E})_{p(x)}$

- $(\mathcal{V},\mathcal{E})_{p(x)}$ has countably many paths in $X^{max}_{p(x)}$ and $X^{min}_{p(x)}$, so not essentially simple. Can this result be extended?

Theorem 2. *The dimension group associated to $(\mathcal{V},\mathcal{E})_{p(x)}$ is order isomorphic to the ordered group of rational functions of the form:*

$$\frac{r_{mn}(x)}{p(x)^m}$$

where $r_{mn}(x)$ is any polynomial with integer coefficients such that

$$\deg(r_{mn}(x)) \leq mn.$$

The positive set consists of the elements of G such that there is a k for which the numerator of

$$\frac{r_{mn}(x)(p(x))^k}{p(x)^{k+m}}$$

has all positive coefficients.

Skip Proof

Proof

- Recognize $v \in \mathbb{Z}^k$ as an integer polynomials of degree $k-1$, for example $[a\ b\ c] \sim ax^2 + bx + c$.

Proof

- Recognize $v \in \mathbb{Z}^k$ as an integer polynomials of degree $k-1$, for example $[a\ b\ c] \sim ax^2 + bx + c$.
- Then the incidence matrices acts as multiplication by $p(x)$ on these polynomials.

Proof

- Recognize $v \in \mathbb{Z}^k$ as an integer polynomials of degree $k-1$, for example $[a\ b\ c] \sim ax^2 + bx + c$.
- Then the incidence matrices acts as multiplication by $p(x)$ on these polynomials.
- Dividing through by $p(x)$ gives all polynomials in the same equivalence class the same "reduced" ratio of polynomials.

Proof

- Recognize $v \in \mathbb{Z}^k$ as an integer polynomials of degree $k-1$, for example $[a\ b\ c] \sim ax^2 + bx + c$.
- Then the incidence matrices acts as multiplication by $p(x)$ on these polynomials.
- Dividing through by $p(x)$ gives all polynomials in the same equivalence class the same "reduced" ratio of polynomials.
- Use these facts to show that this map from $\varinjlim \mathbb{Z}^i$ to G is an isomorphism. $\diamond$

Total Ergodicity

Total Ergodicity

- It has been shown that T_{x+1} is totally ergodic, i.e. T_{x+1}^{n} is ergodic for all $n \in \mathbb{N}$.

Total Ergodicity

- It has been shown that T_{x+1} is totally ergodic, i.e. T_{x+1}^n is ergodic for all $n \in \mathbb{N}$.
- This is not true for all $p(x)$. In particular, it is not true for (X_{2x+2}, T_{2x+2}).

Total Ergodicity

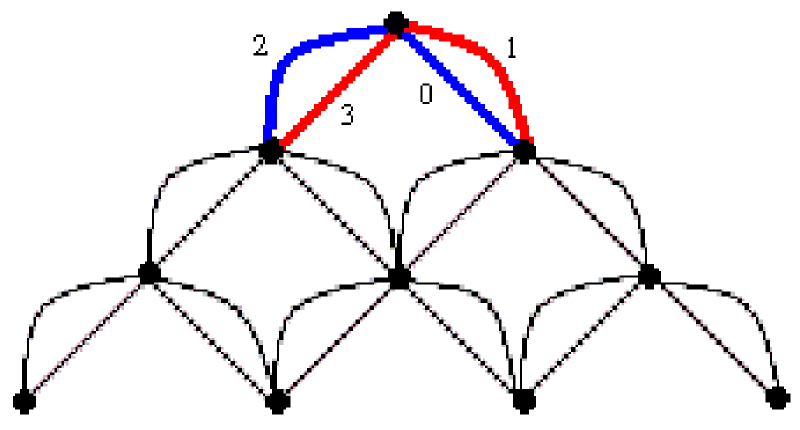
- It has been shown that T_{x+1} is totally ergodic, i.e. T_{x+1}^n is ergodic for all $n \in \mathbb{N}$.
- This is not true for all $p(x)$. In particular, it is not true for (X_{2x+2}, T_{2x+2}).
- $S = \{\mathcal{O}(x) | x \in X_{2x+2}^{max},\ x \neq (a_0 - 1)^\infty,\ x \neq (a_0 + ... + a_n - 1)^\infty\}$. Then S is a countable set consisting of paths that are eventually diagonal.

Total Ergodicity

- It has been shown that T_{x+1} is totally ergodic, i.e. T_{x+1}^n is ergodic for all $n \in \mathbb{N}$.
- This is not true for all $p(x)$. In particular, it is not true for (X_{2x+2}, T_{2x+2}).
- $S = \{\mathcal{O}(x) | x \in X_{2x+2}^{max},\ x \neq (a_0 - 1)^\infty,\ x \neq (a_0 + ... + a_n - 1)^\infty\}$. Then S is a countable set consisting of paths that are eventually diagonal.
- Then $E = X \setminus S$ is a set of full measure on which we can define eigenfunctions.

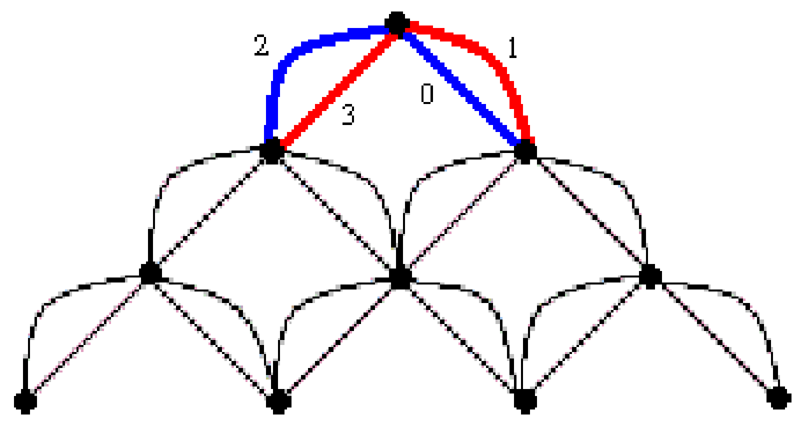
Invariant Sets

- Let A=$([0] \cup [2]) \setminus S$ and B=$([1] \cup [3]) \setminus S$.



Invariant Sets

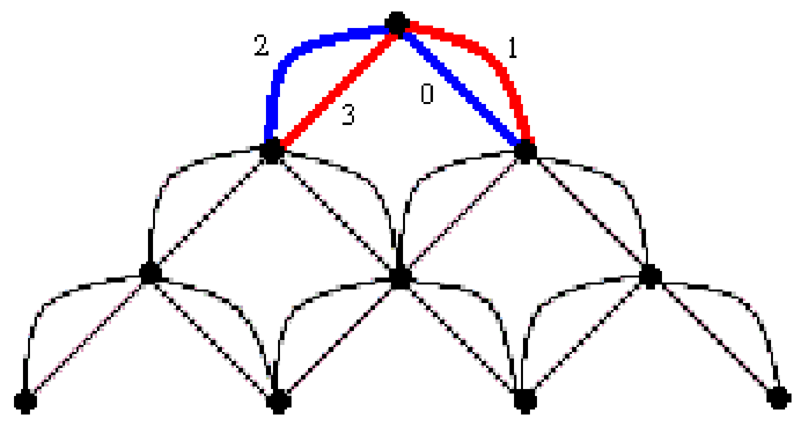
- Let A=$([0] \cup [2]) \setminus S$ and B=$([1] \cup [3]) \setminus S$.



- Then $T_{2x+2}(A) = B$ and $T_{2x+2}(B) = A$, hence $T^2_{2x+2}(A) = A$ and $T^2_{2x+2}(B) = B$.

Invariant Sets

- Let A=$([0]\cup[2])\setminus S$ and B=$([1]\cup[3])\setminus S$.



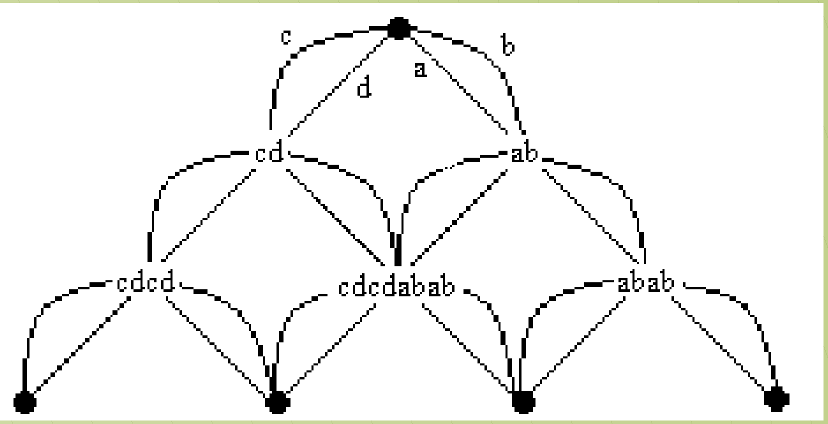
- Then $T_{2x+2}(A)=B$ and $T_{2x+2}(B)=A$, hence $T^2_{2x+2}(A)=A$ and $T^2_{2x+2}(B)=B$.
- Then

$$f(x)=\begin{cases}-1 & \text{if } x\in A\\ 1 & \text{if } x\in B\end{cases}$$

is an eigenfunction of T with eigenvalue -1, i.e. $f\circ T_{2x+2}=-f$

Coding

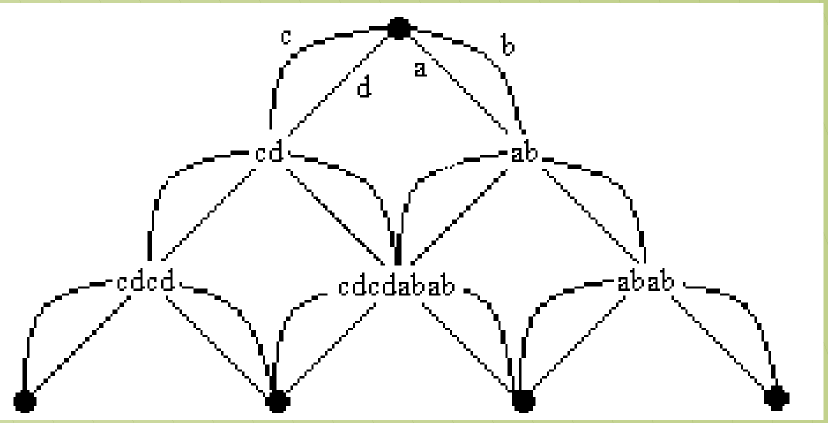
- Relabel the first edges connecting V_0 and V_1 in the fashion, $0 \to a$, $1 \to b$,...



Coding

- Relabel the first edges connecting V_0 and V_1 in the fashion, $0 \to a$, $1 \to b$,...
- Now for each vertex, (k,l) if there are j finite paths into (k,l), letting x be some path in $X_{p(x)}$ such that x follow the minimal path into (k,l). Now label (k,l) by:

$$(x)_0 (T_{p(x)} x)_0 ... (T_{p(x)}^{j-1} x)_0$$



Advantages

- Let $\sum$ be the subshift defined such that every block appears as a subblock of a vertex on the diagram.

Advantages

- Let $\sum$ be the subshift defined such that every block appears as a subblock of a vertex on the diagram.
- On a set of full measure, this will be an injective coding. It will collapse the odometer to 2 points, and get rid of the maximal orbits.

Advantages

- Let $\sum$ be the subshift defined such that every block appears as a subblock of a vertex on the diagram.
- On a set of full measure, this will be an injective coding. It will collapse the odometer to 2 points, and get rid of the maximal orbits.
- It has been shown that (X_{x+1}, T_{x+1}) is topologically weak mixing. Using this coding, we can see that (X_{2x+2}, T_{2x+2}) is not topologically weak mixing.

$[a] \cup [c] \to [b] \cup [d]$and vice versa

Advantages

- Let $\sum$ be the subshift defined such that every block appears as a subblock of a vertex on the diagram.
- On a set of full measure, this will be an injective coding. It will collapse the odometer to 2 points, and get rid of the maximal orbits.
- It has been shown that (X_{x+1}, T_{x+1}) is topologically weak mixing. Using this coding, we can see that (X_{2x+2}, T_{2x+2}) is not topologically weak mixing.

$$[a] \cup [c] \to [b] \cup [d] \text{and vice versa}$$

- It is also more natural to discuss $K^0(\sum, \sigma)$.

Summary of Questions

- What is $K^0(\sum, \sigma)$ and is it order isomorphic to $K_0((\mathcal{V}, \mathcal{E})_{p(x)})$?

Summary of Questions

- What is $K^0(\sum, \sigma)$ and is it order isomorphic to $K_0((\mathcal{V}, \mathcal{E})_{p(x)})$?
- What are some sufficient conditions on $p(x)$ to ensure eigenvalues? Which $p(x)$ give totally ergodic systems?

Summary of Questions

- What is $K^0(\sum, \sigma)$ and is it order isomorphic to $K_0((\mathcal{V}, \mathcal{E})_{p(x)})$?
- What are some sufficient conditions on $p(x)$ to ensure eigenvalues? Which $p(x)$ give totally ergodic systems?
- Which systems are topologically weakly mixing?

Summary of Questions

- What is $K^0(\sum, \sigma)$ and is it order isomorphic to $K_0((\mathcal{V}, \mathcal{E})_{p(x)})$?
- What are some sufficient conditions on $p(x)$ to ensure eigenvalues? Which $p(x)$ give totally ergodic systems?
- Which systems are topologically weakly mixing?

Bibliography

References

[1] P. Billingsley, *Probability and Measure*, Wiley, 1995.

[2] Mike Boyle, *Algebraic aspects of symbolic dynamics*, Lecture Notes.

[3] Ola Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Transactions of the American Mathematical Society **171** (1972), 195–234.

[4] F. Durand, B. Host, and C. Skau, *Substitutional dynamcial systems, Bratteli diagrams and dimension groups*, Ergodic Theory and Dynamcial Systems **19** (1999), 953–993.

[5] G.A. Elliott, *On the classification of inductive limits of sequences of semisimple finite dimensional algebras*, Journal of Algebra **38** (1976), 29–44.

[6] Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, International Journal of Mathematics **3** (1992), no. 6, 827–864.

[7] Xavier Méla, *Dynamical properties of the Pascal adic and related systems*, PhD dissertation, University of North Carolina, Chapel Hill, 2002.