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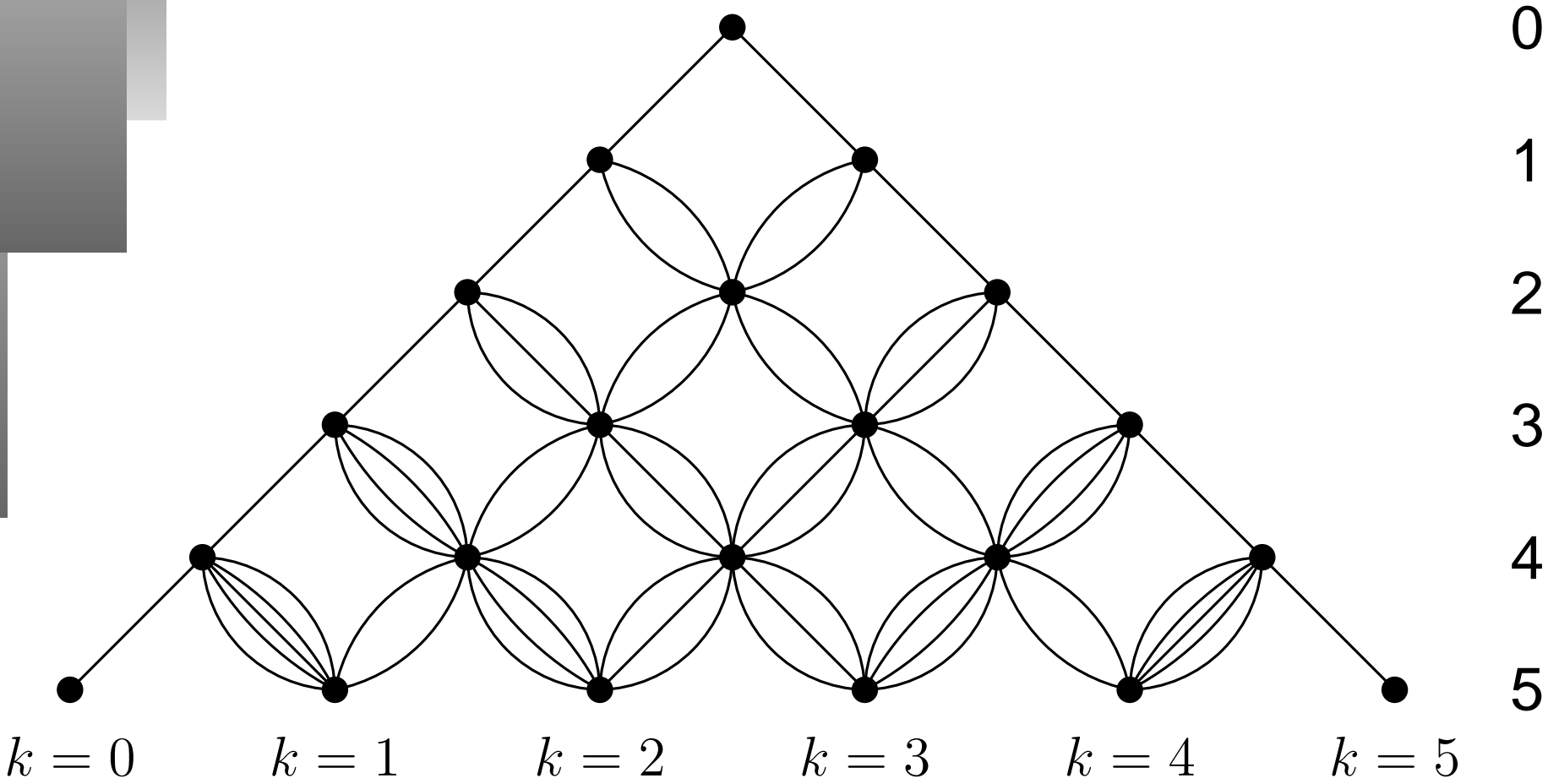
# **The Symmetric Measure of the Adic Transformation on the Euler Graph**

Sarah Bailey

University of North Carolina

# Euler Diagram

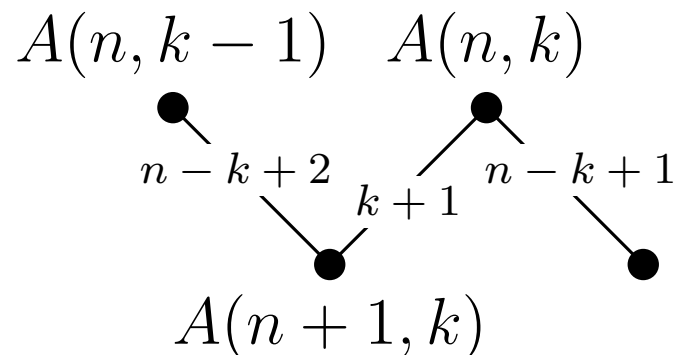
Level



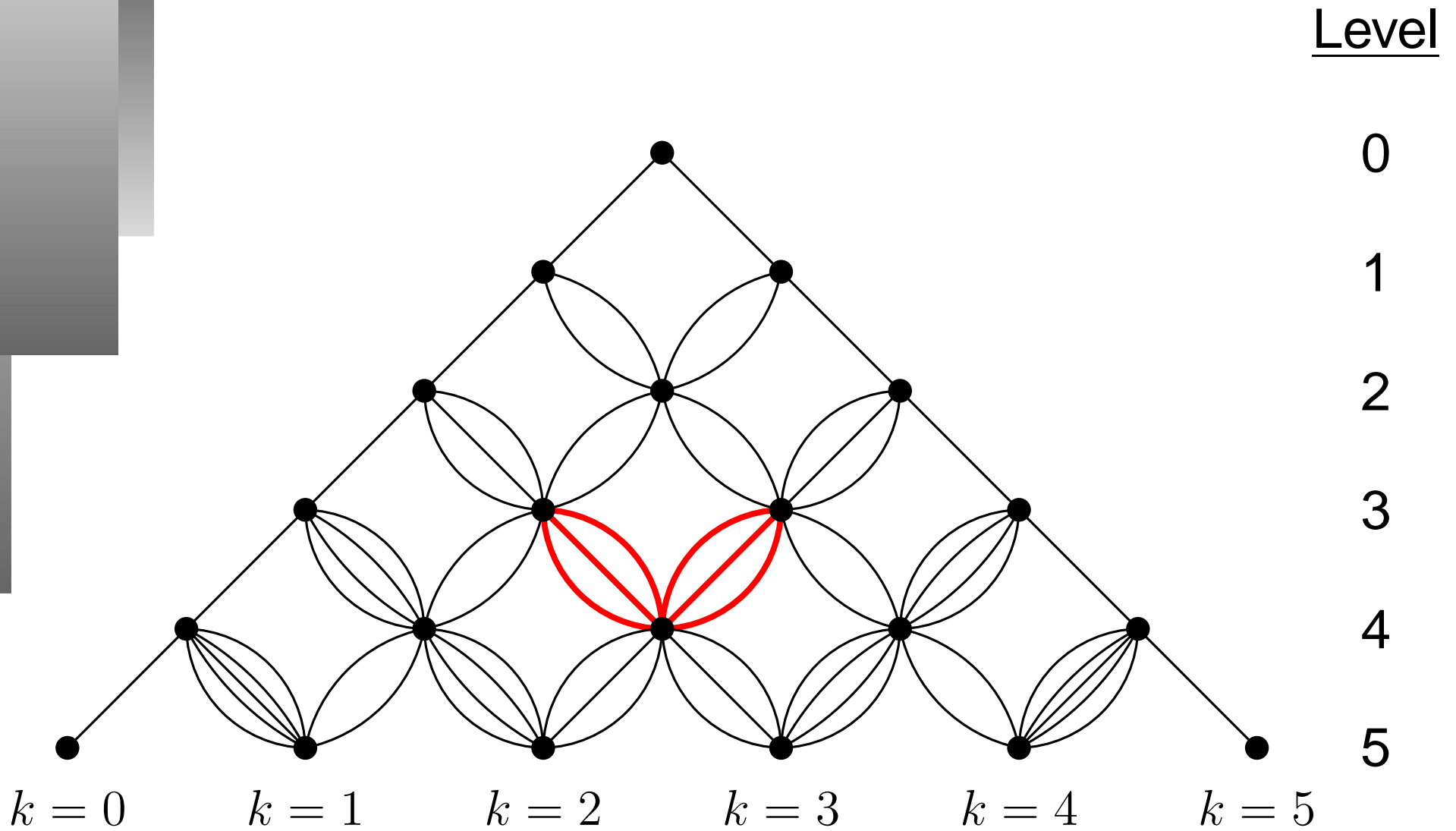
# Euler Diagram

- The number of paths into any vertex  $(n, k)$  is the Eulerian number  $A(n, k)$
- From any vertex  $(n, k)$  there are  $k + 1$  edges leaving to the left, and  $n - k + 1$  edges leaving to the right.
- This gives rise to the recursion:

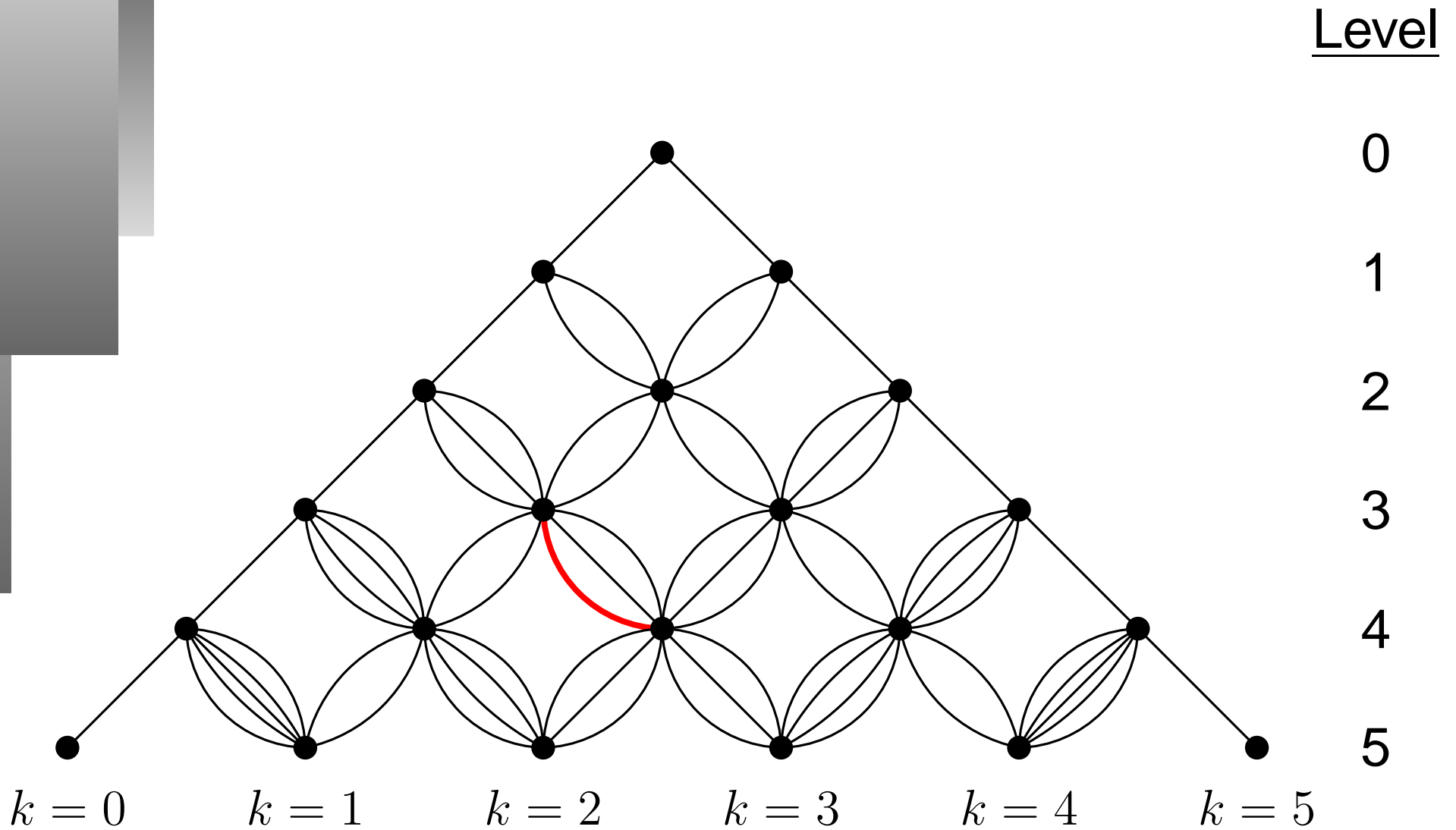
$$A(n + 1, k) = (n - k + 2)A(n, k - 1) + (k + 1)A(n, k).$$



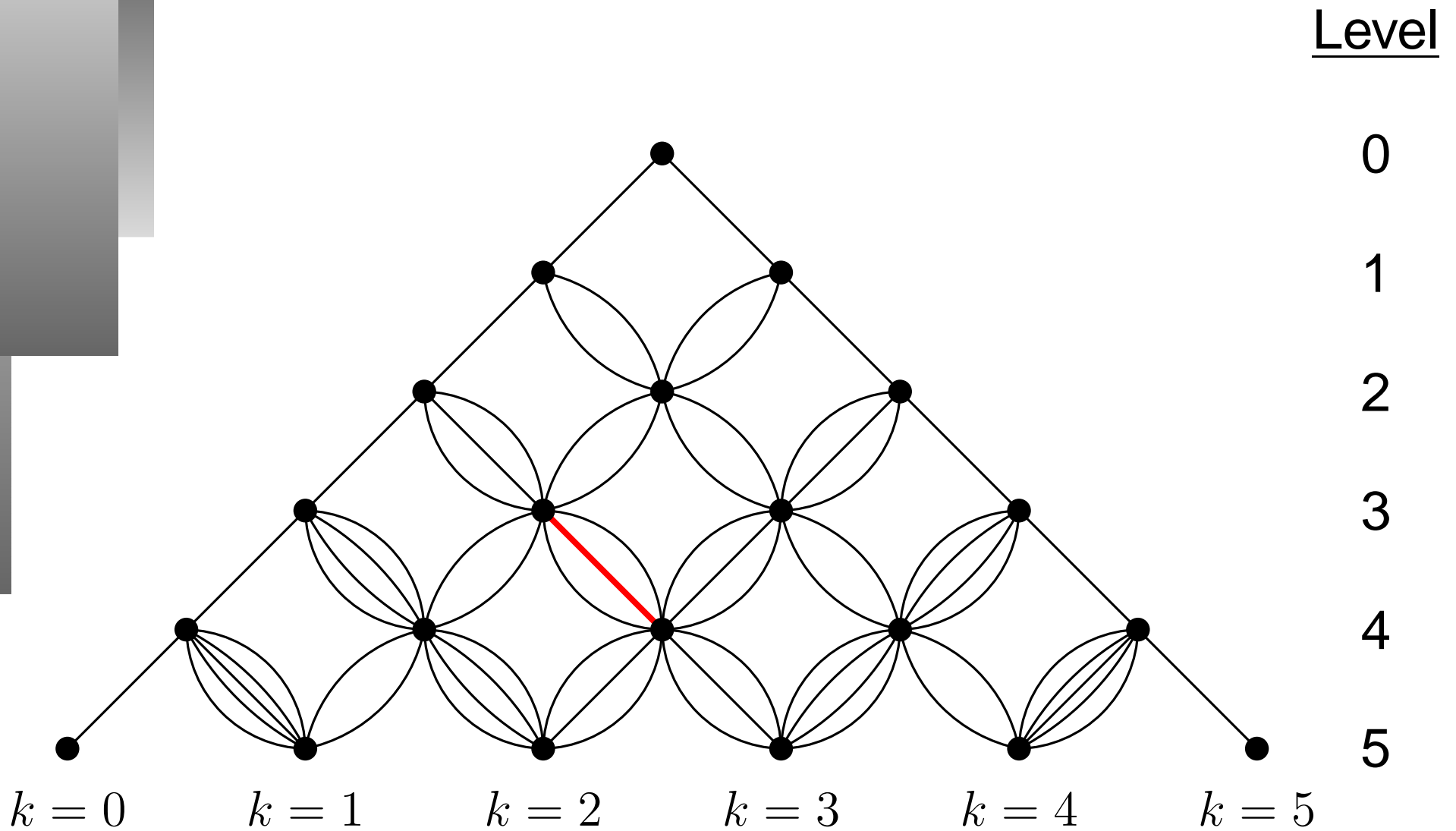
# Edge Ordering



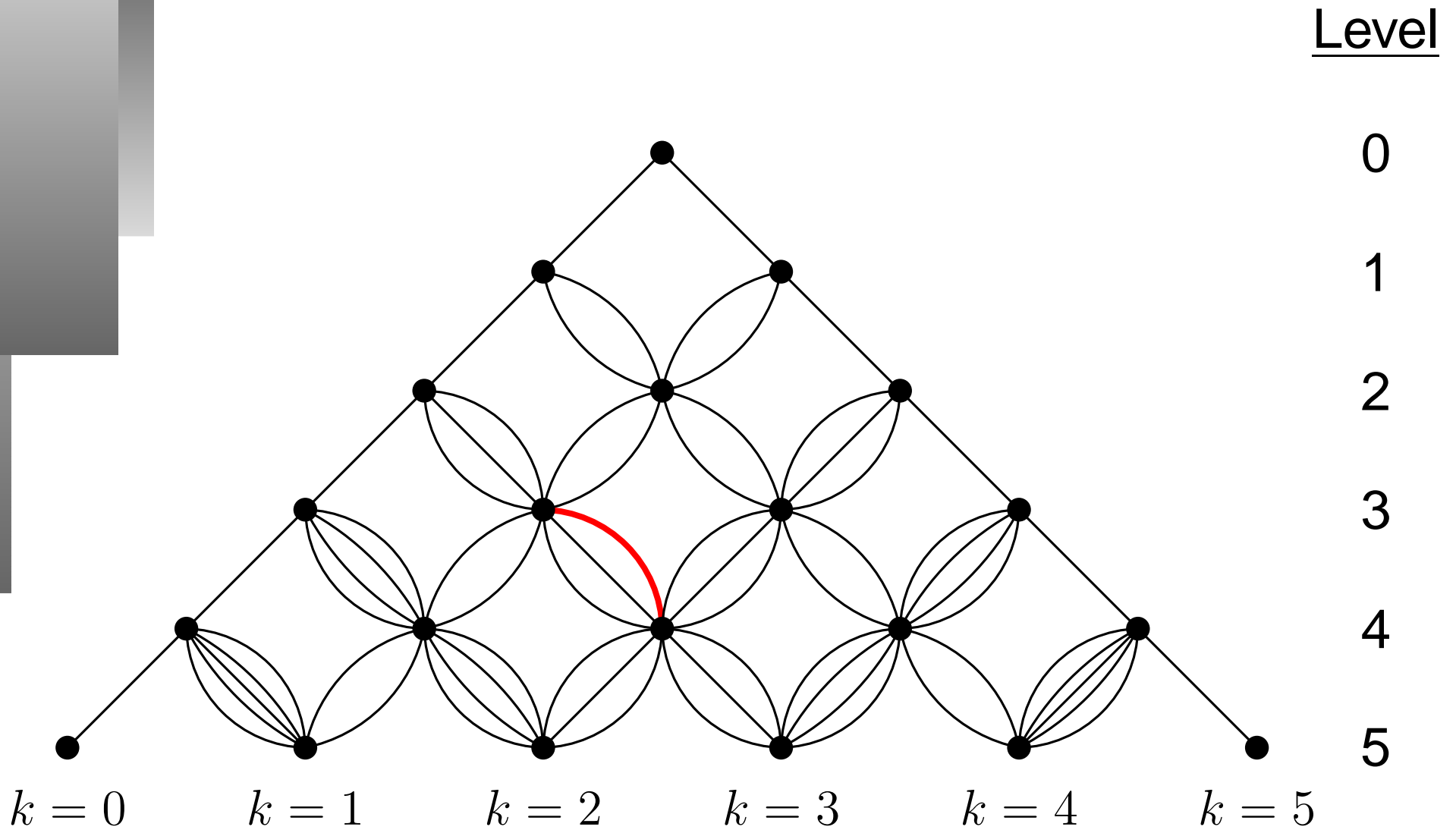
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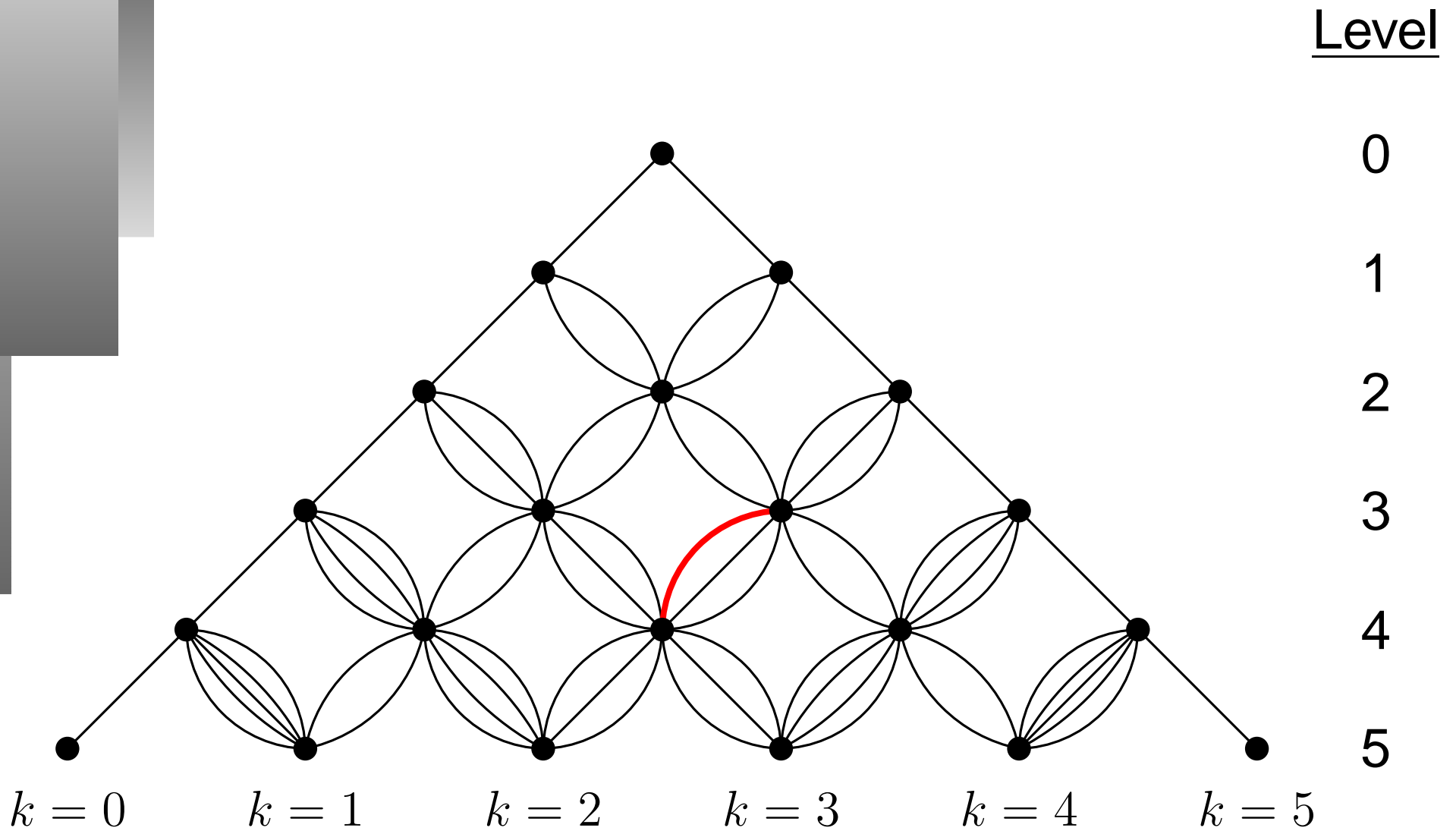
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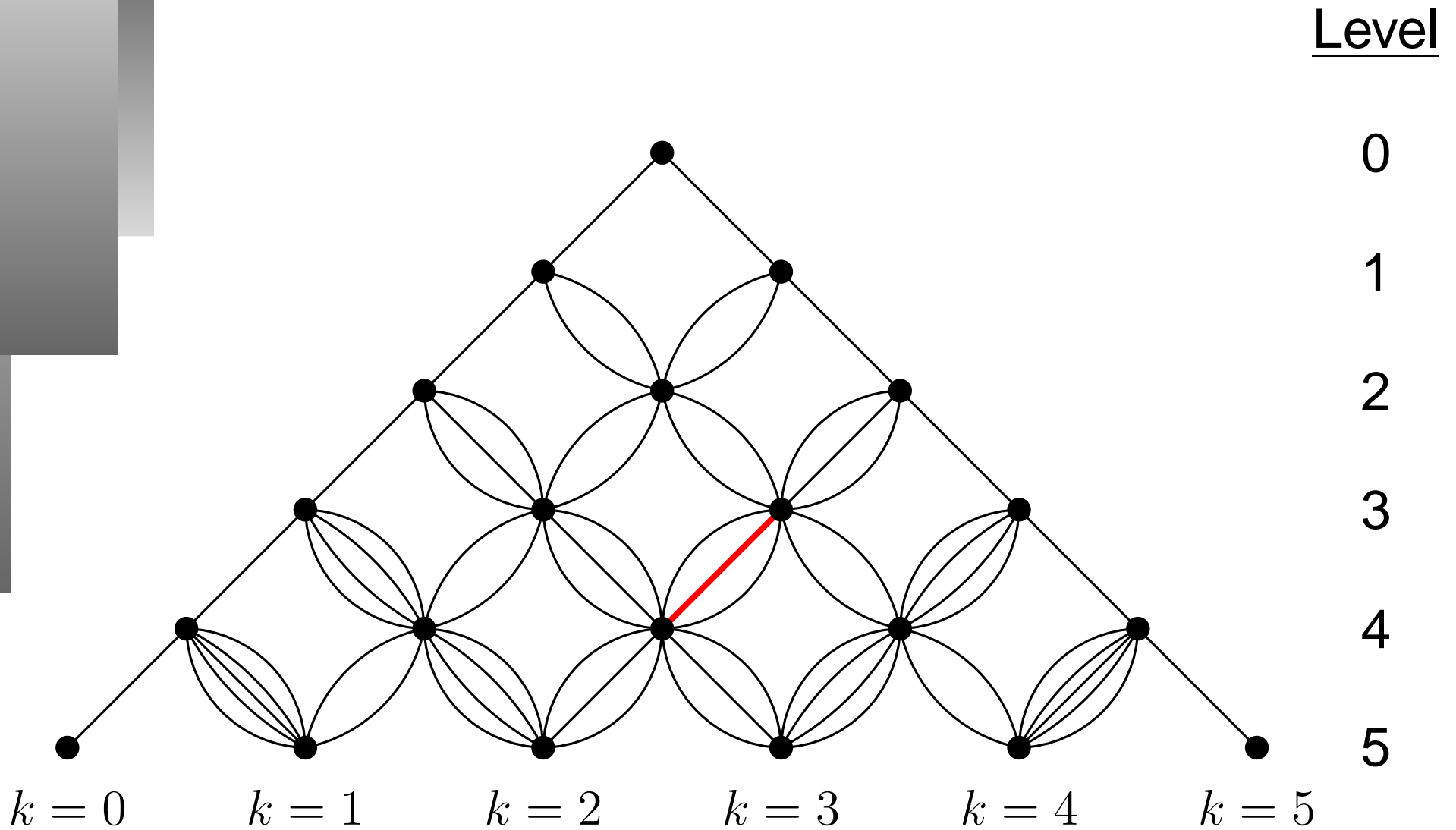
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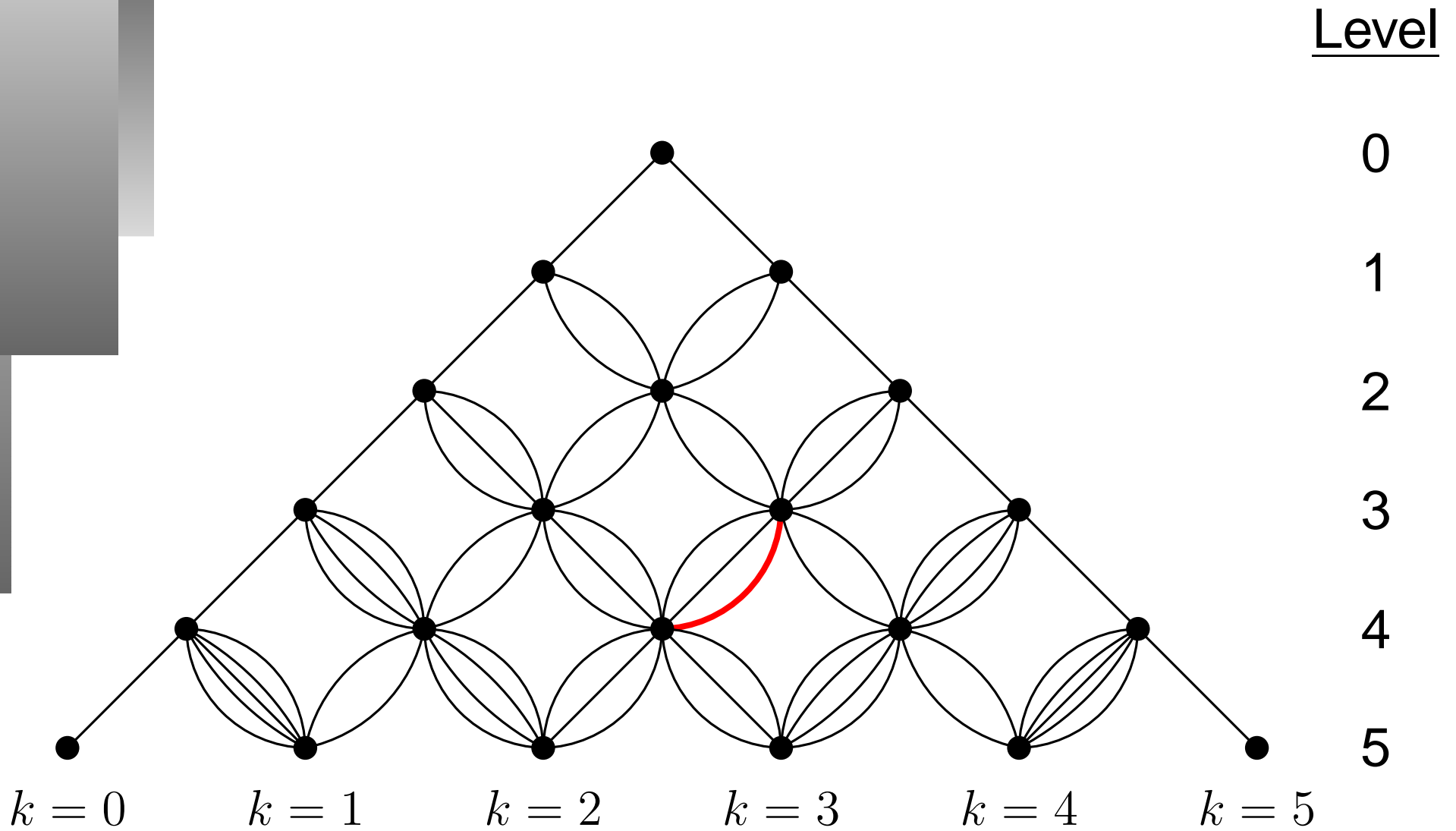
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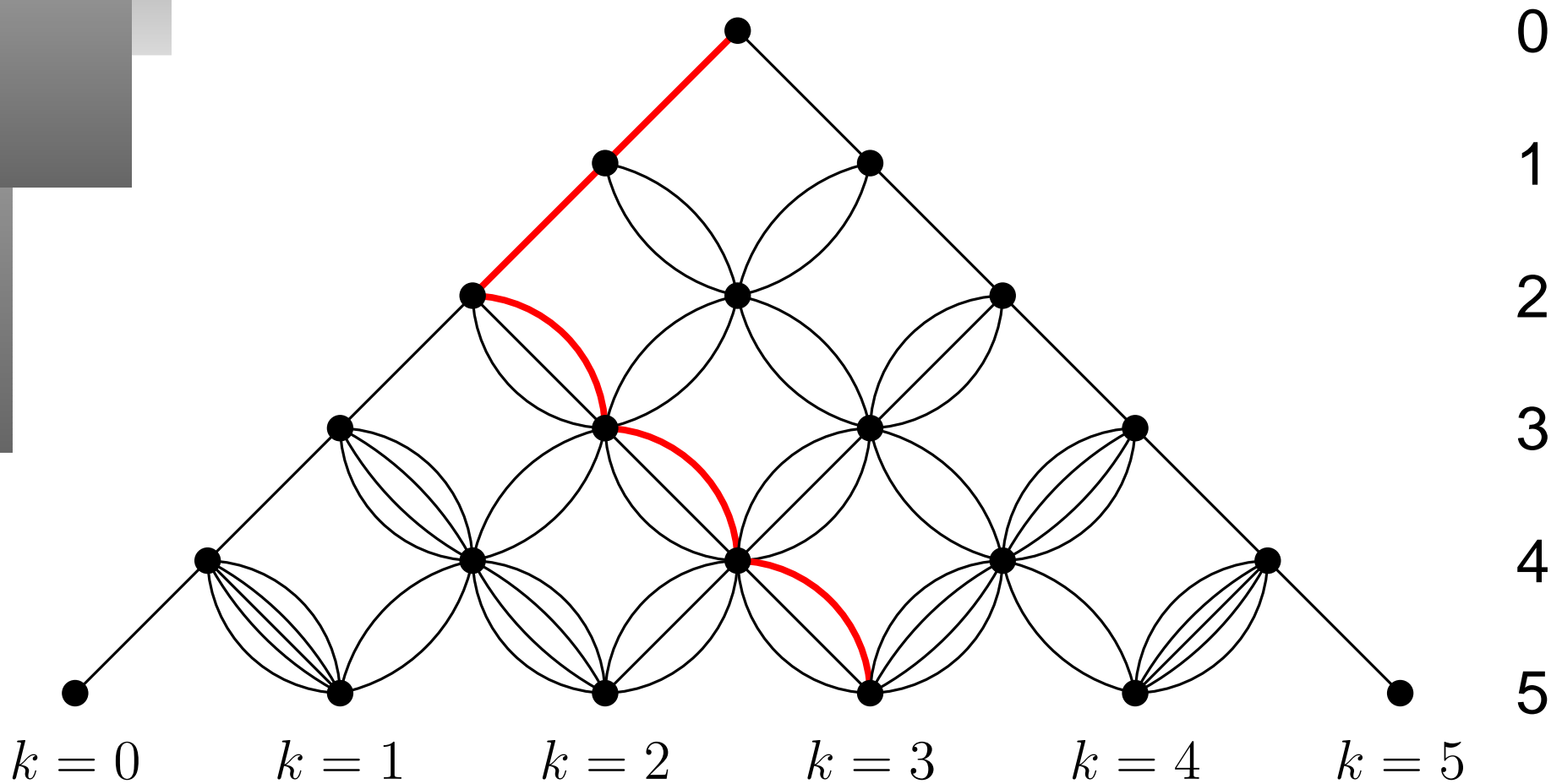


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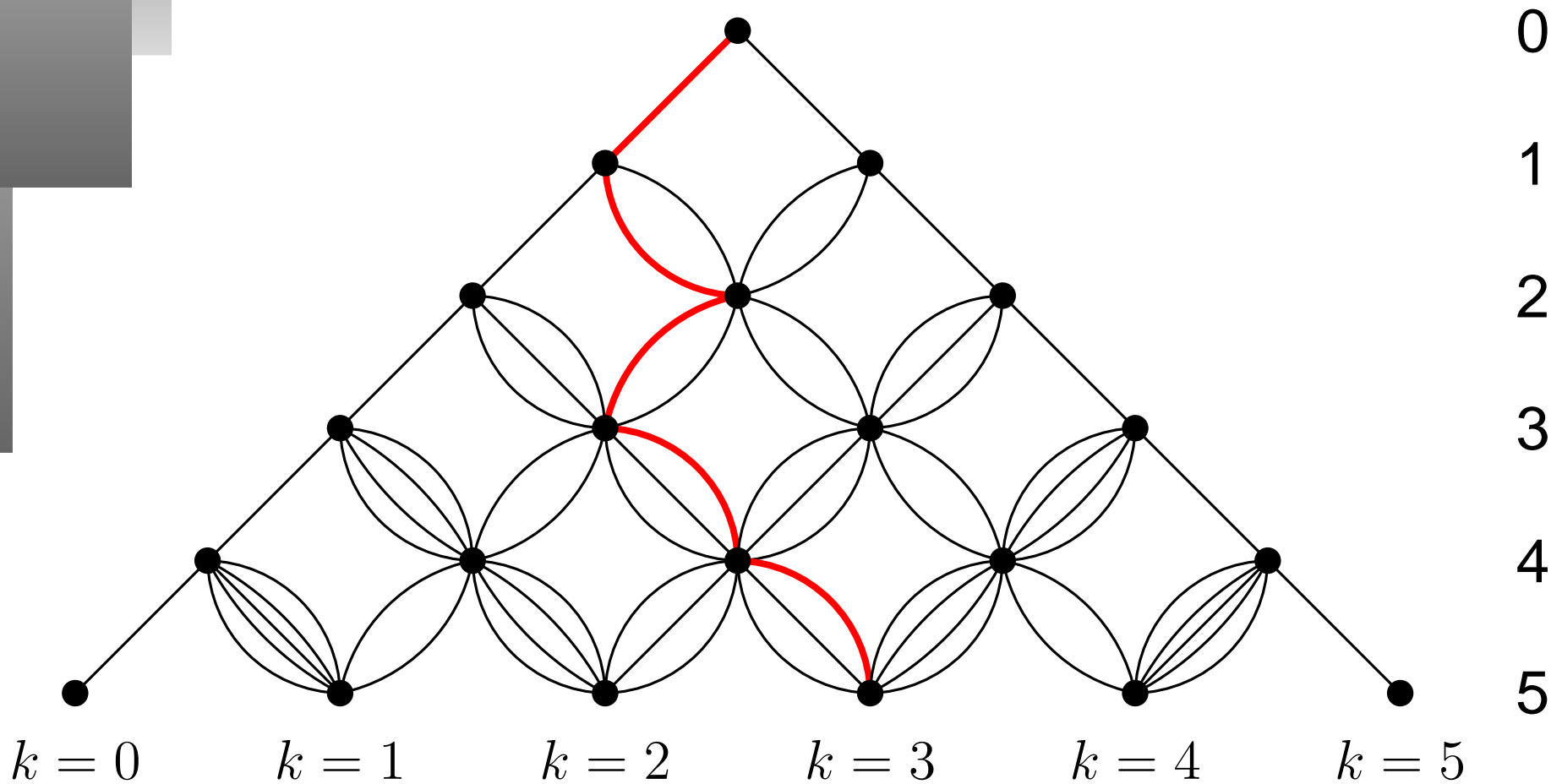
# Euler Adic

$T : X \rightarrow X$  with  $T(x)$  being the next "largest" path. Level



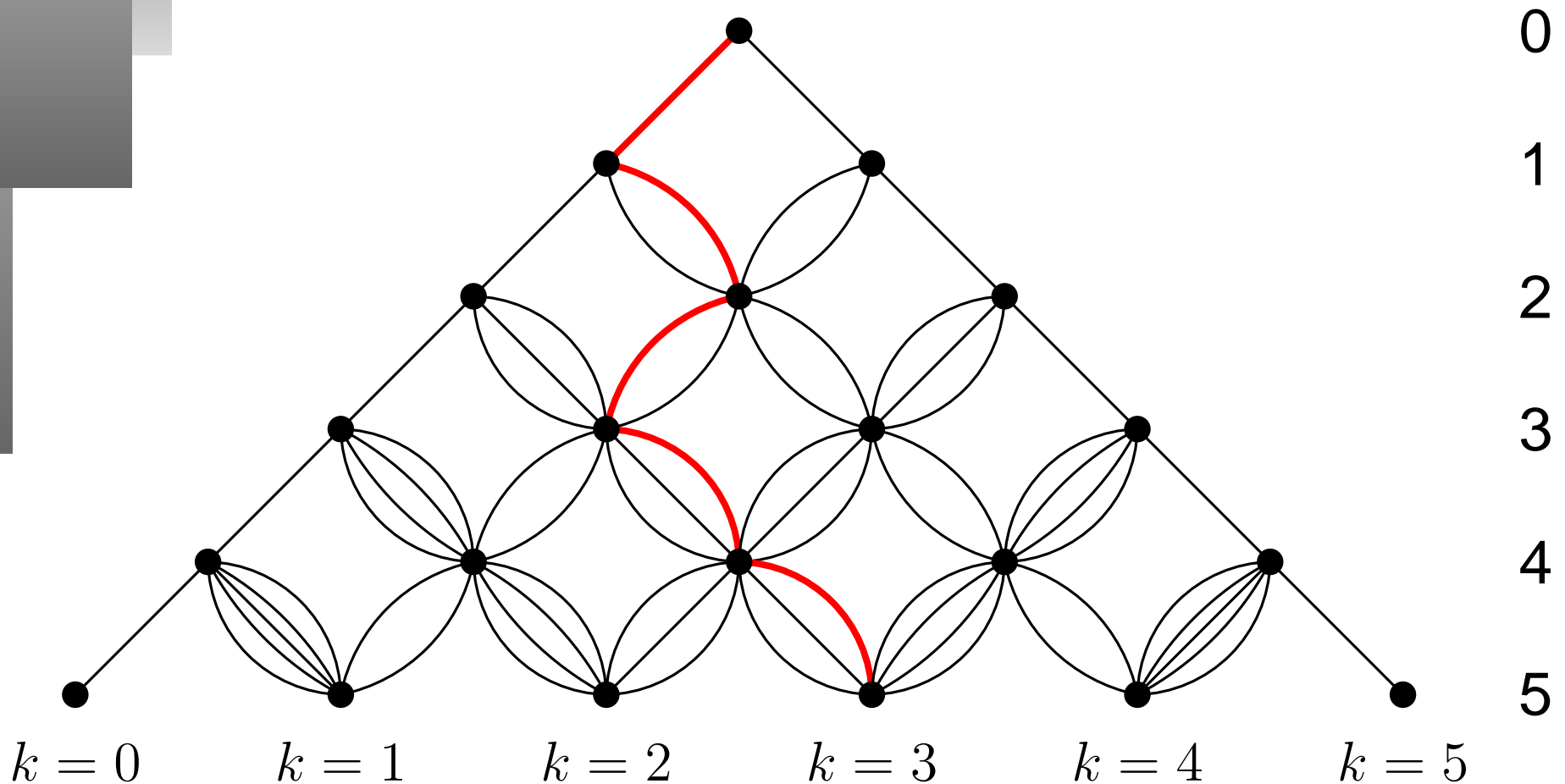
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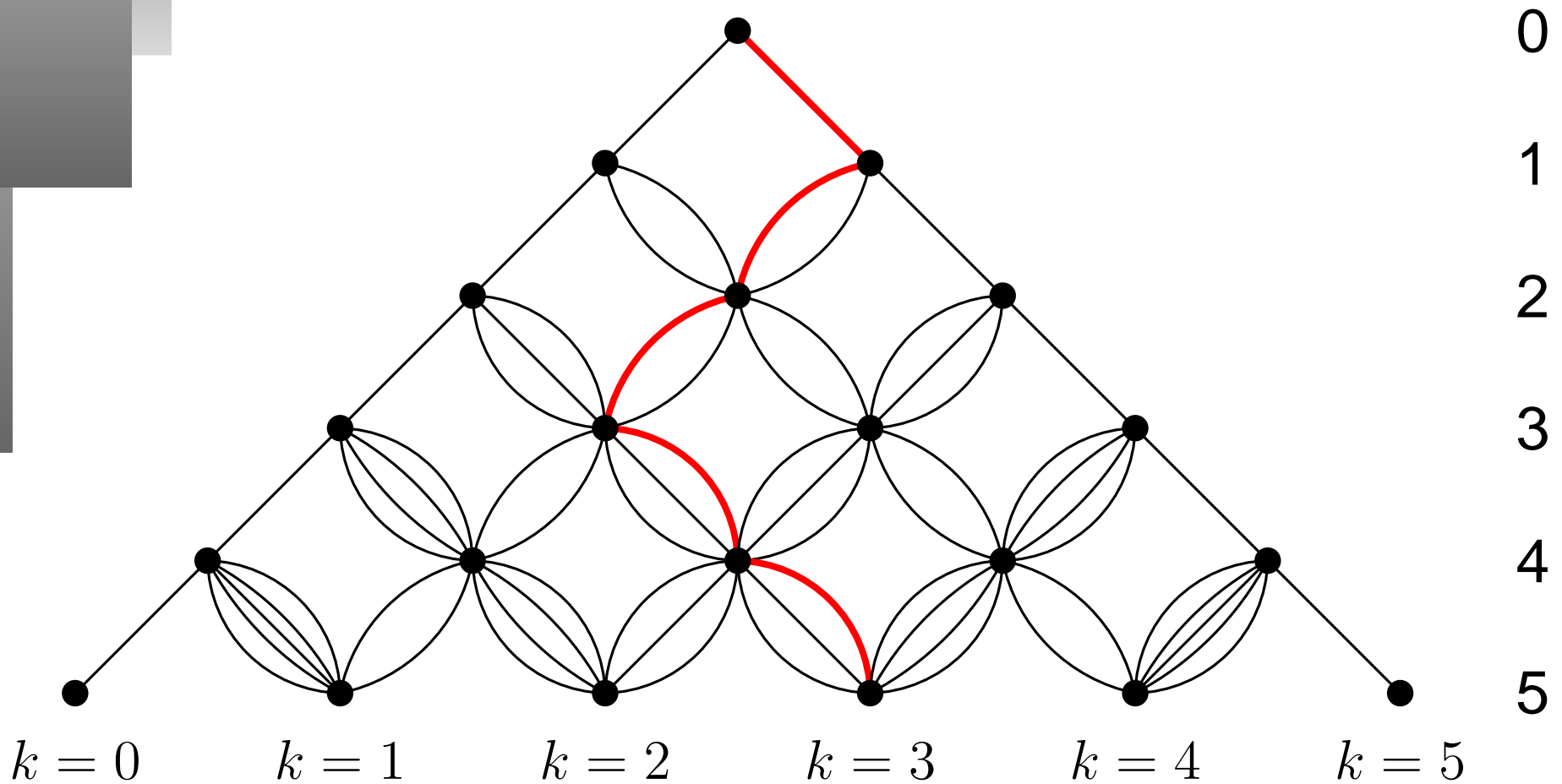
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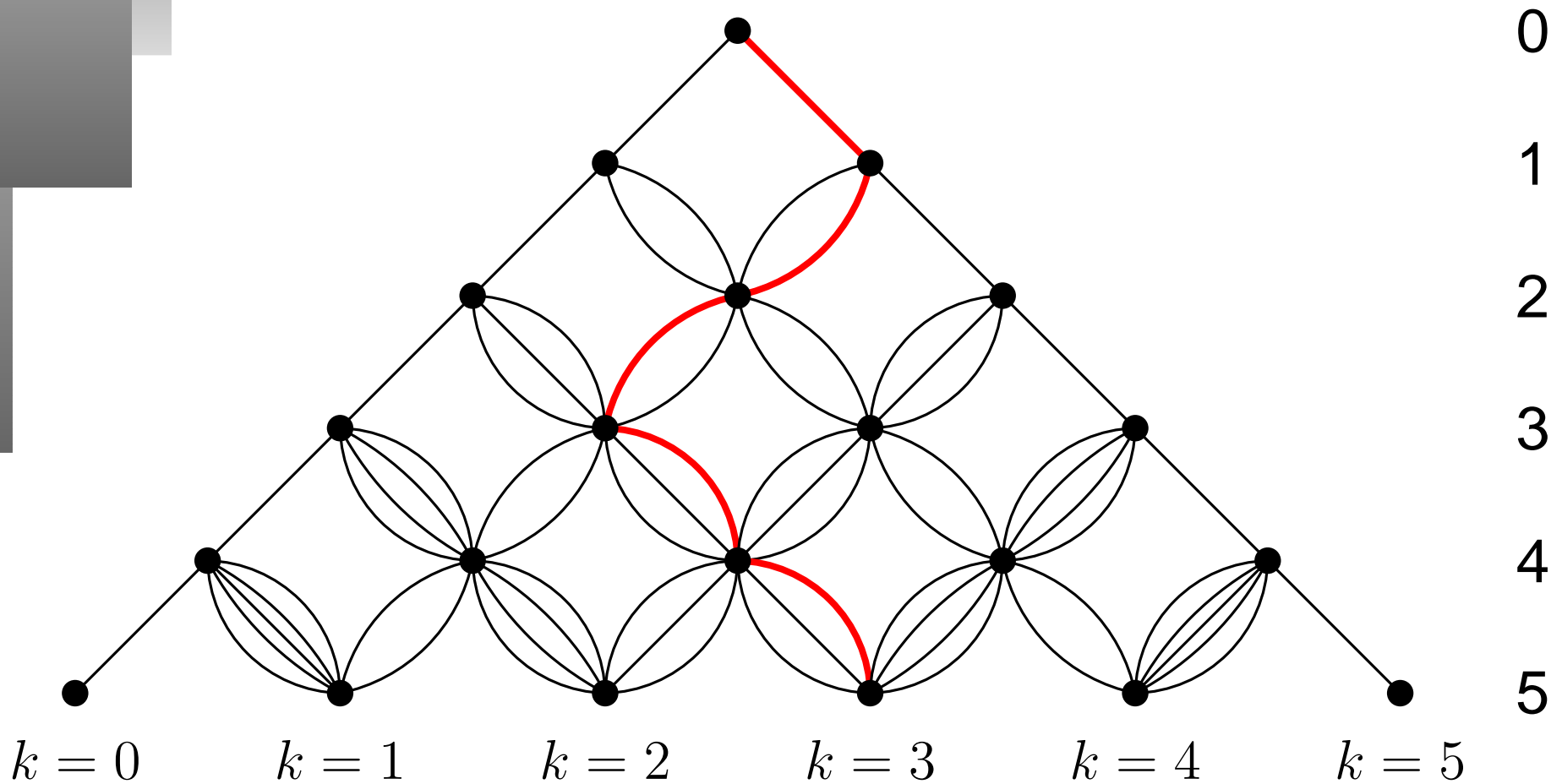
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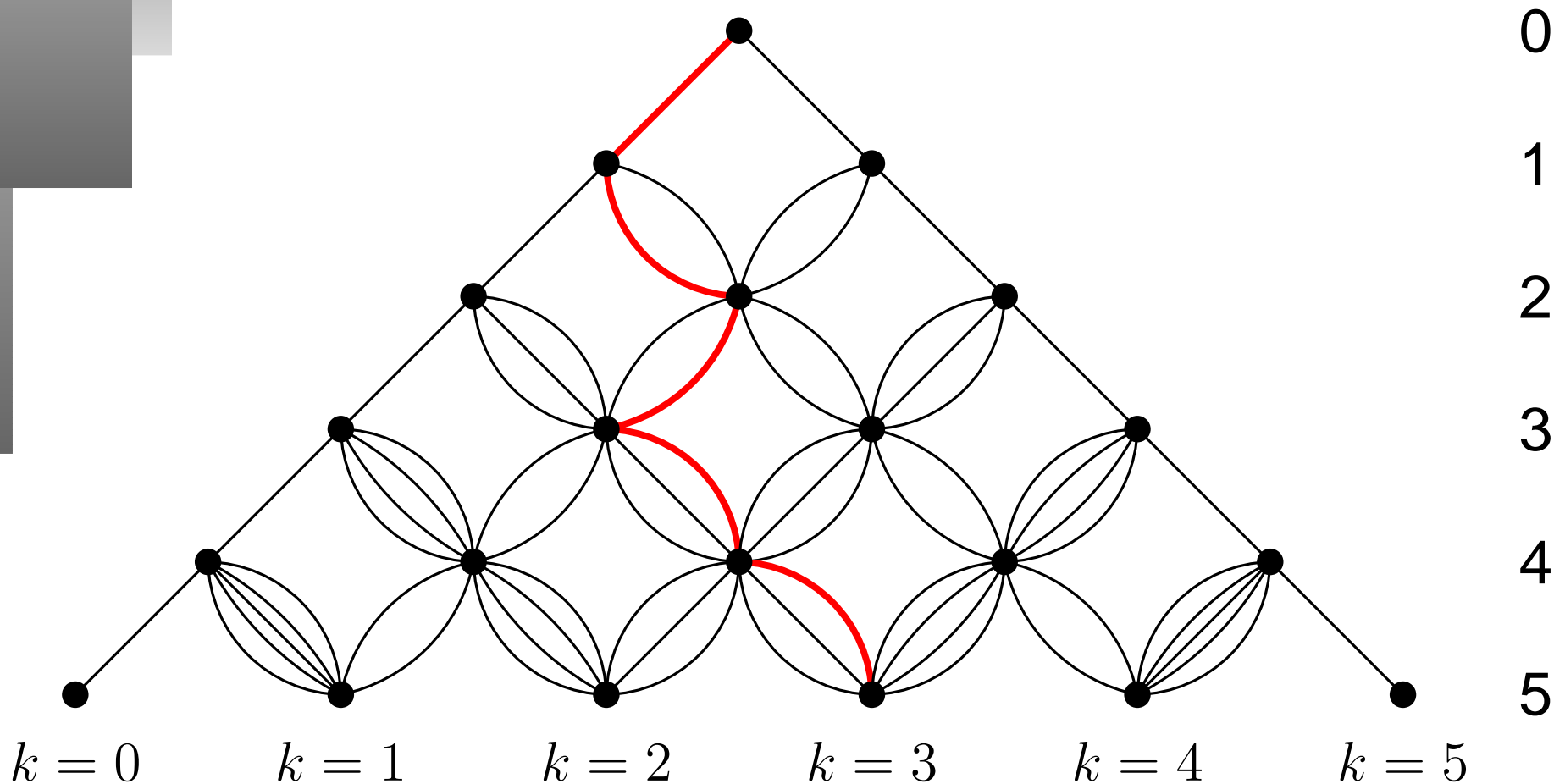
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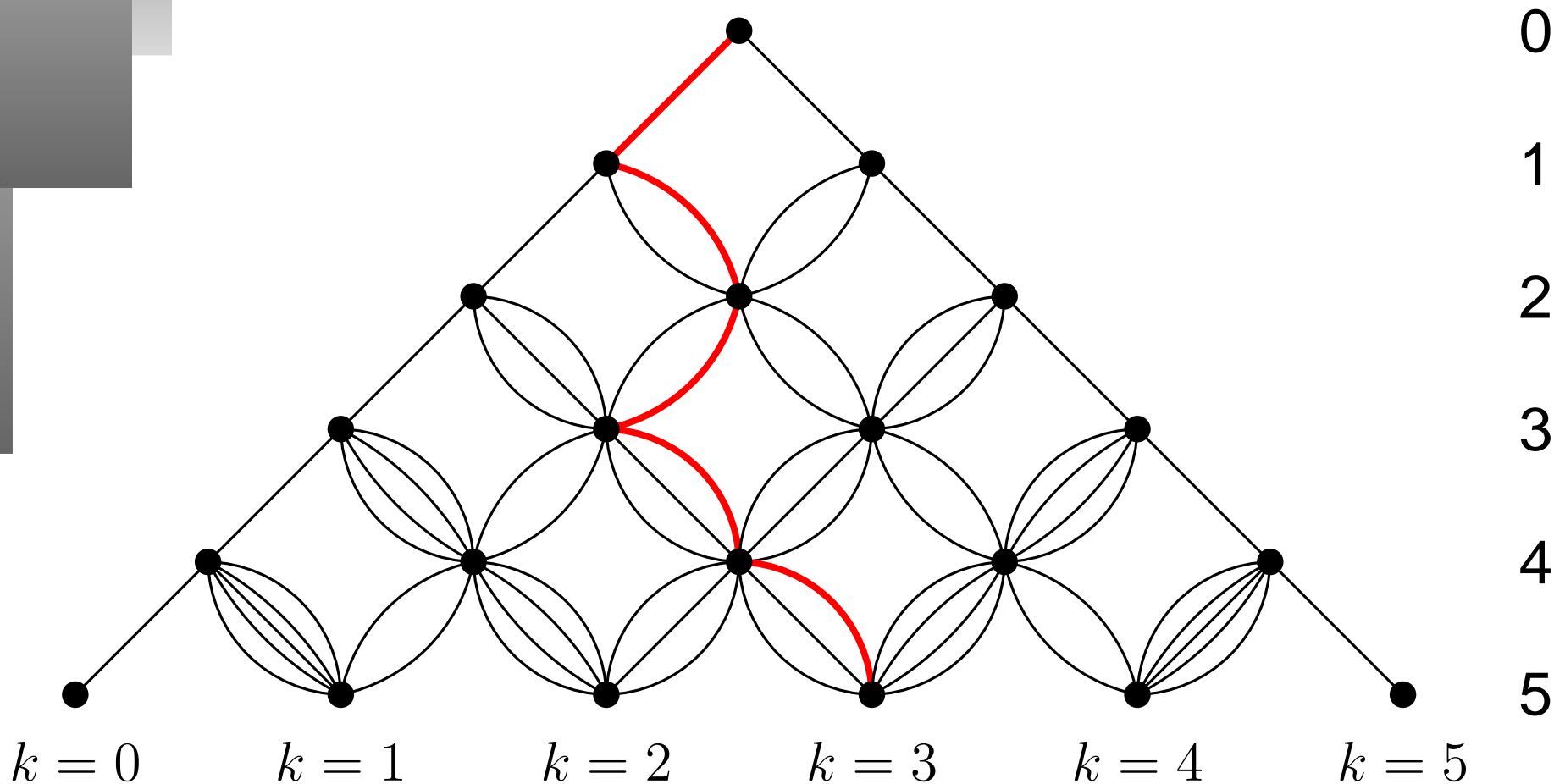
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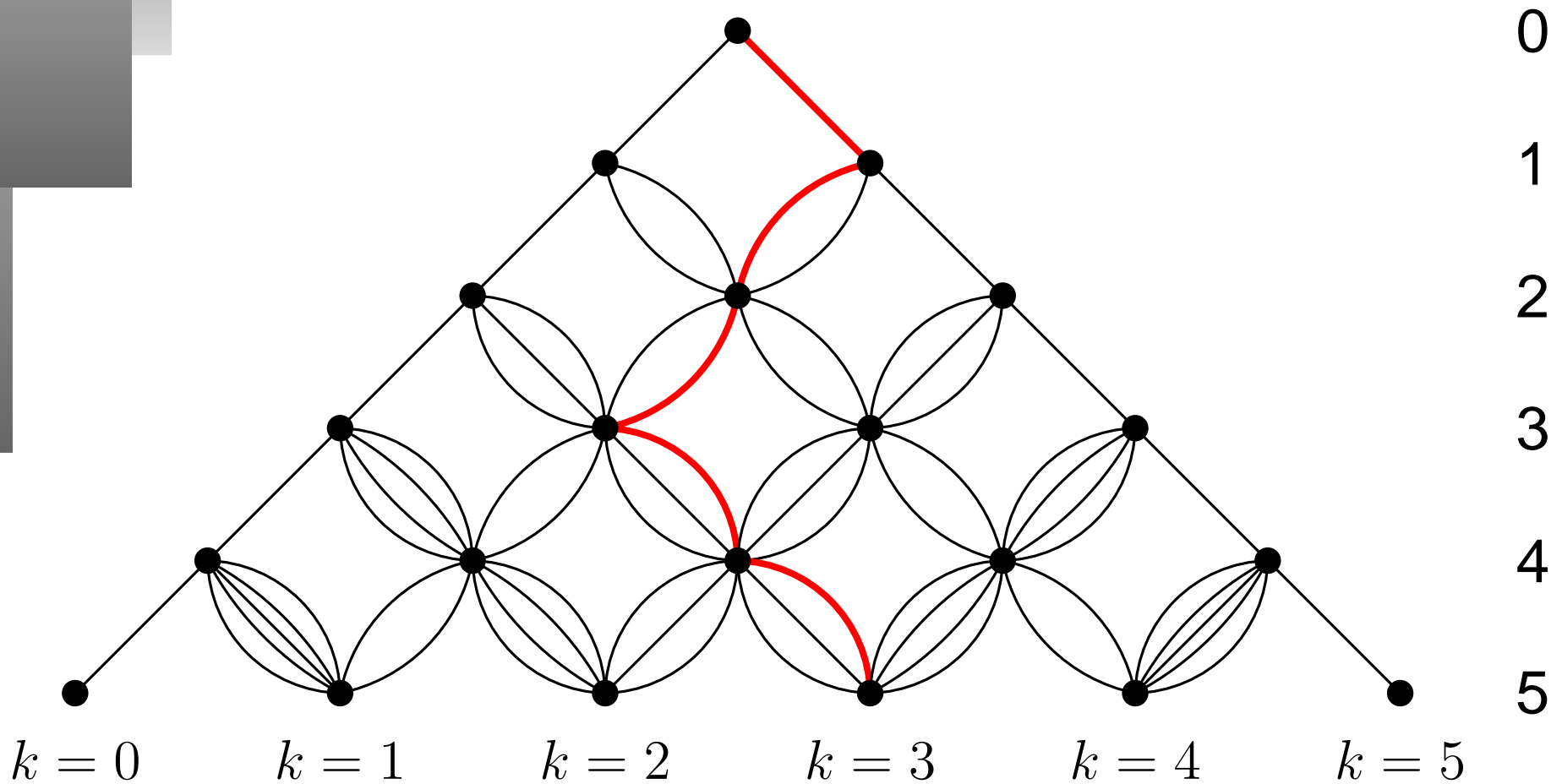
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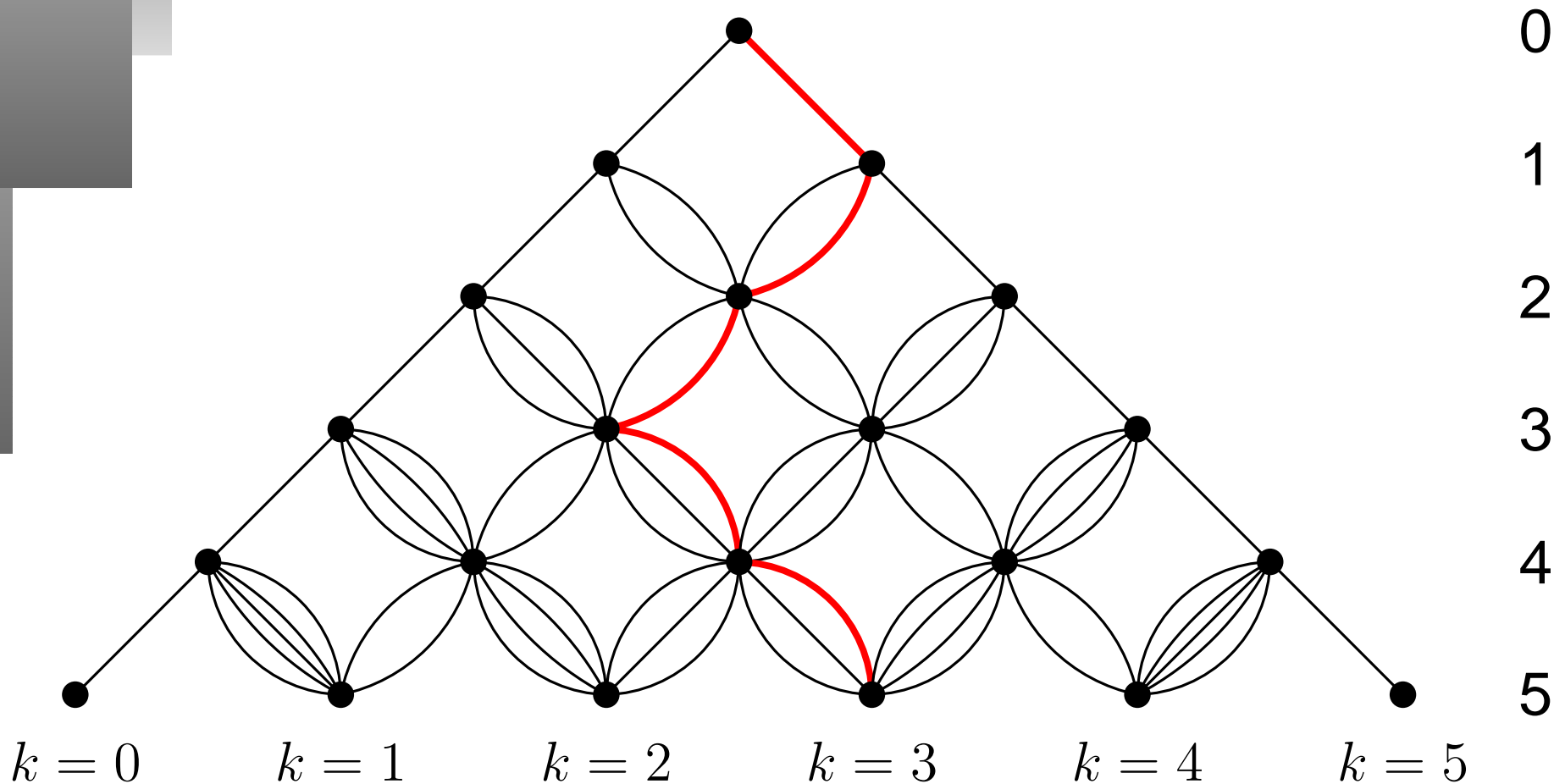
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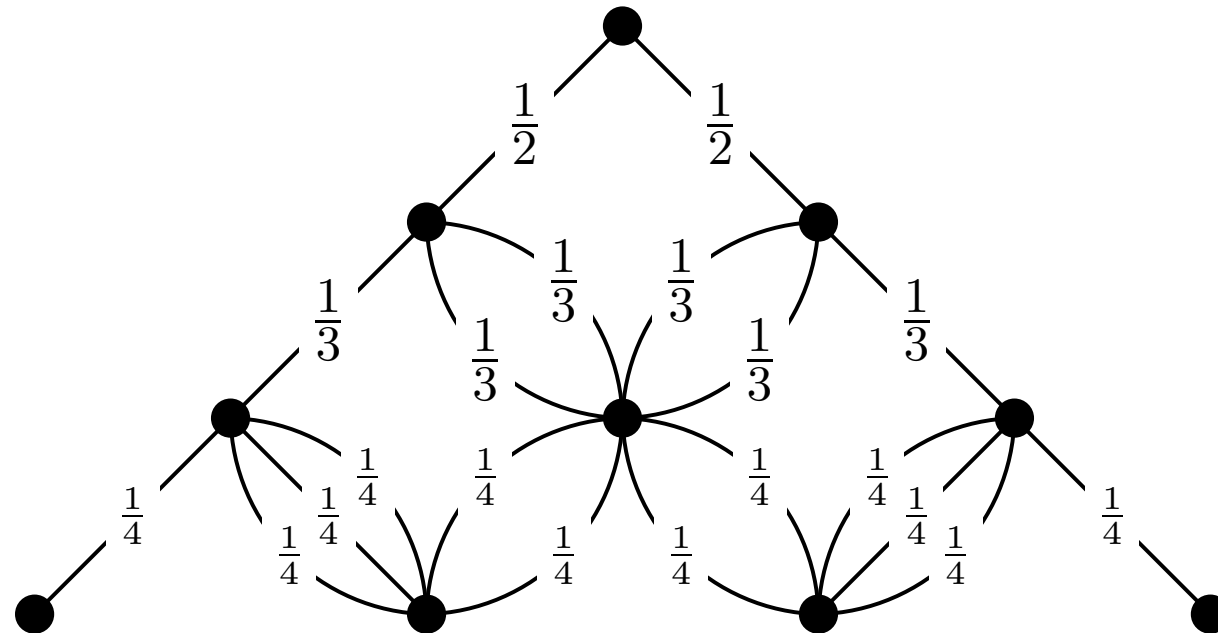


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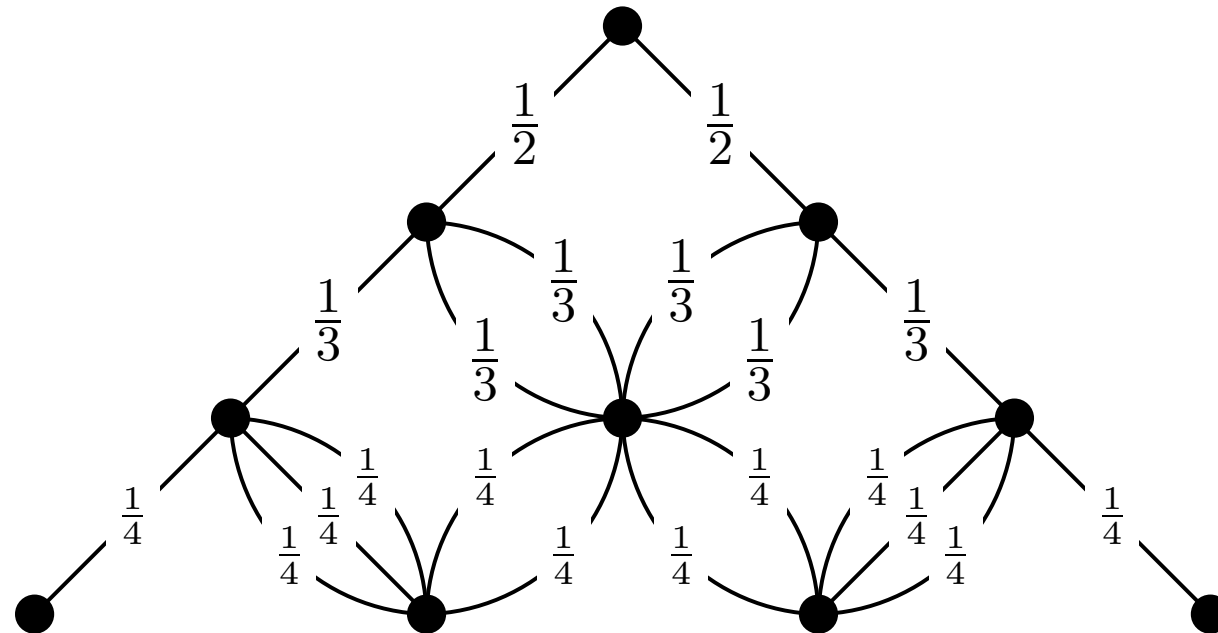


# Symmetric Measure



- Gives each edge connecting level  $n$  to level  $n + 1$  weight  $\frac{1}{n + 2}$ .
- Gives each cylinder of length  $n$  measure  $\frac{1}{n + 1!}$

# Symmetric Measure



- Clearly  $T$ -invariant.
- In joint work with Keane, Petersen, and Salama the symmetric measure has been shown to be ergodic.

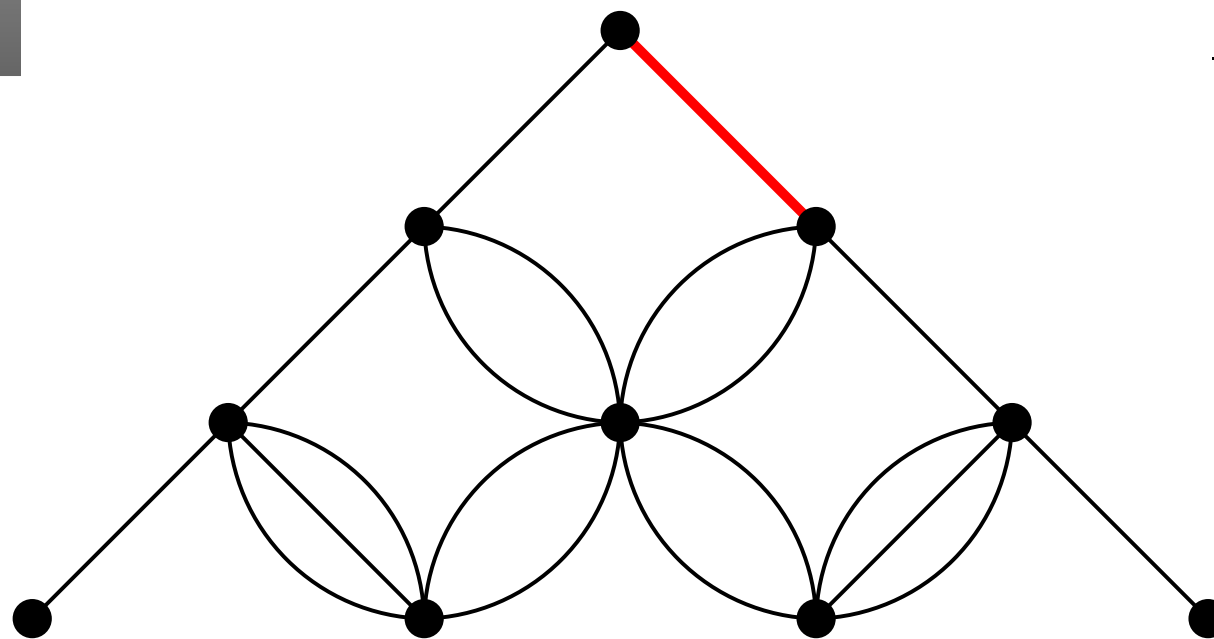
# *Cylinders and Permutations*

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- There is a bijective correspondence between cylinders of length  $n$  with  $k$  right turns and the set of Permutations of  $\{1, 2, \dots, n + 1\}$  with  $k$  rises.
- Hence the Eulerian number  $A(n, k)$  equals the number of permutations of the set  $\{1, 2, \dots, n + 1\}$  with  $k$  rises.

# Cylinders and Permutations

- The permutation 12 has one fall, and the corresponding cylinder has one right.

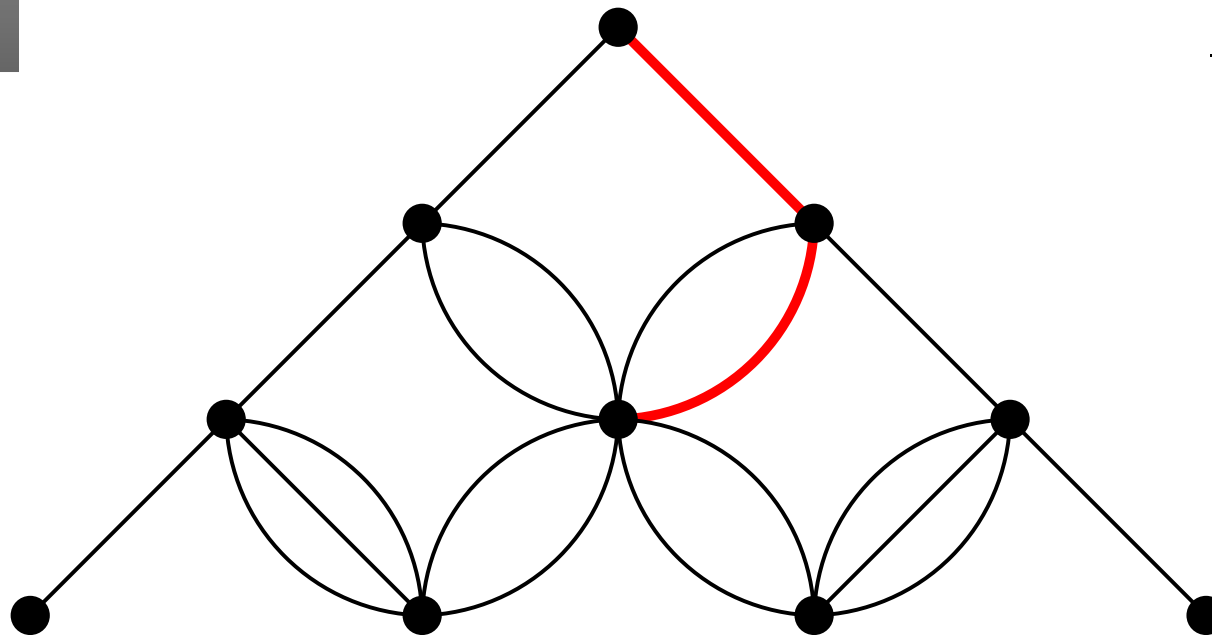


Permutation

12

# Cylinders and Permutations

- In order to extend the cylinder 12 by a left turn, insert a 3 in such a way to add a fall.

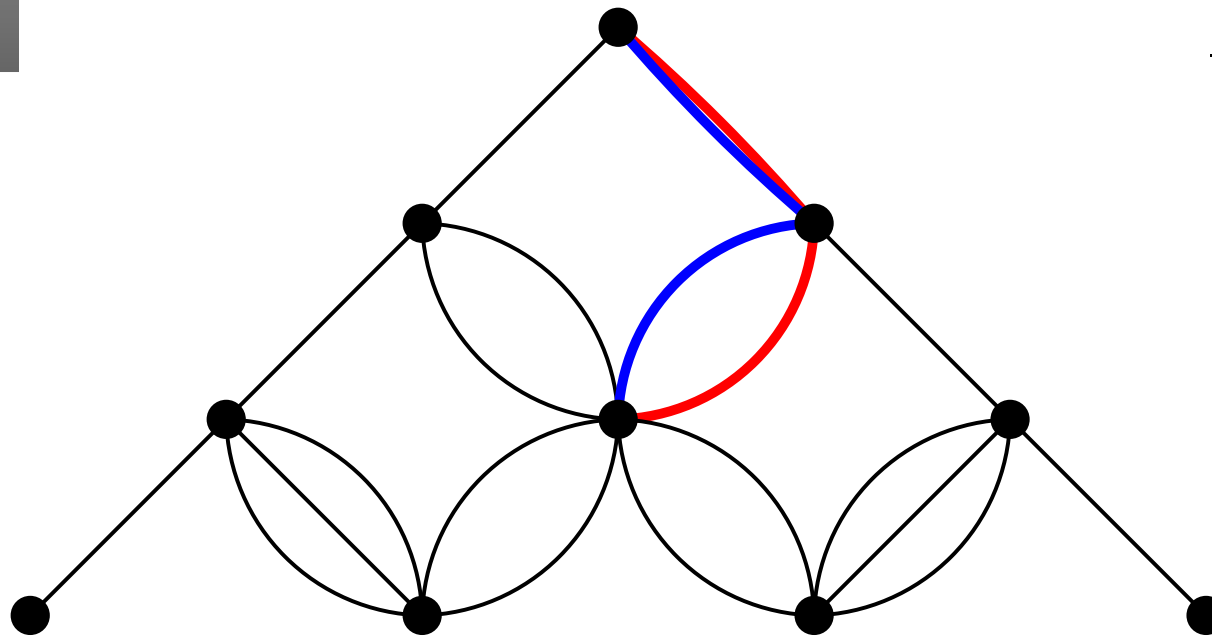


Permutation

312

# Cylinders and Permutations

- In order to extend the cylinder 12 by a left turn, insert a 3 in such a way to add a fall.



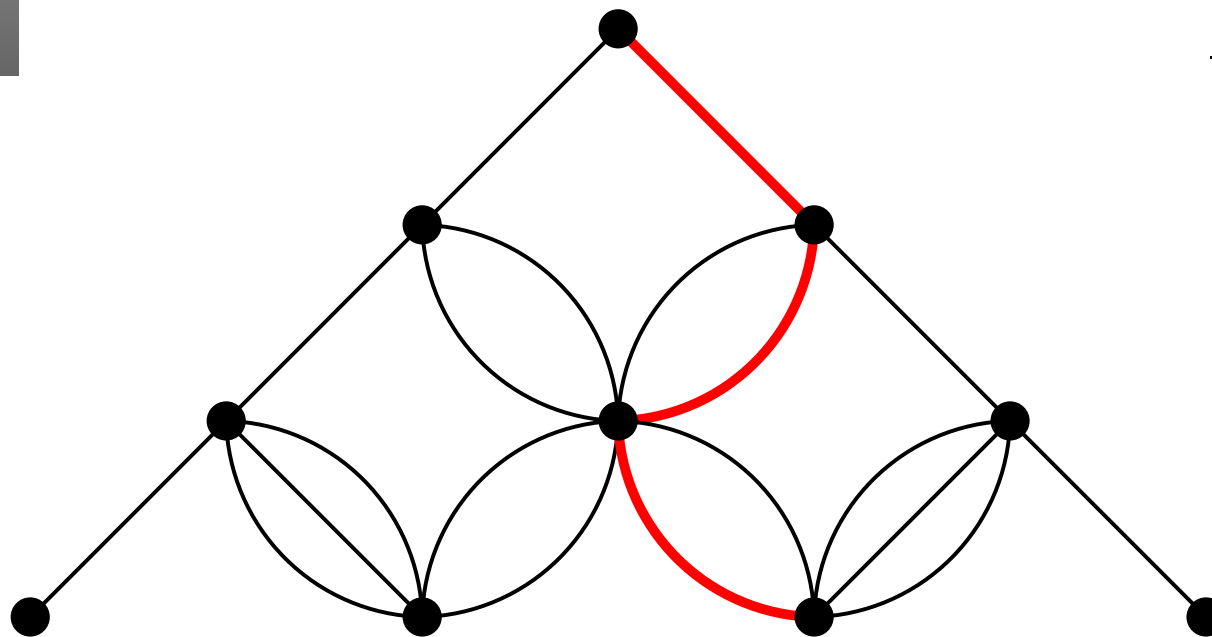
Permutation

312

or 132

# Cylinders and Permutations

- In order to extend the cylinder 312 by a right turn, insert a 4 in such a way to add a rise.

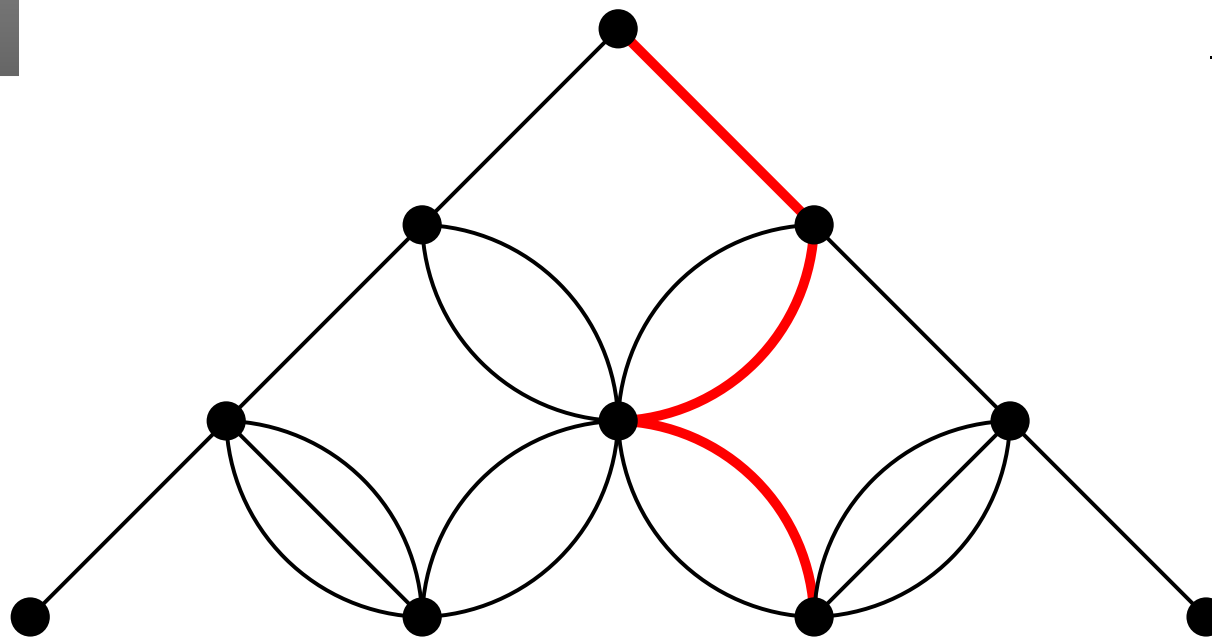


Permutation

3412

# Cylinders and Permutations

- In order to extend the cylinder 312 by a right turn, insert a 4 in such a way to add a rise.



Permutation

3412

or 3124

# Measure of Cylinder Sets

Let  $\mu$  be an invariant fully-supported ergodic probability measure for the Euler adic. Then for every cylinder set  $C$  and  $\mu$ -almost every  $x \in X$

$$\mu[C] = \lim_{n \rightarrow \infty} \frac{\dim(C, x_n)}{\dim(x_n)}$$

# Measure of Cylinder Sets

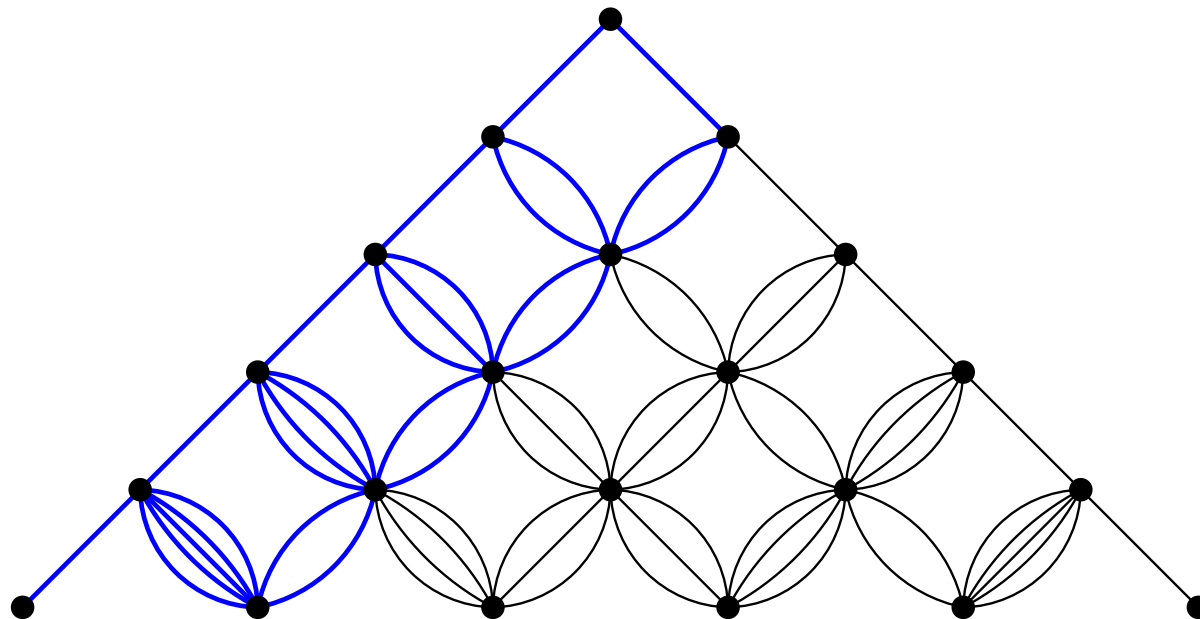
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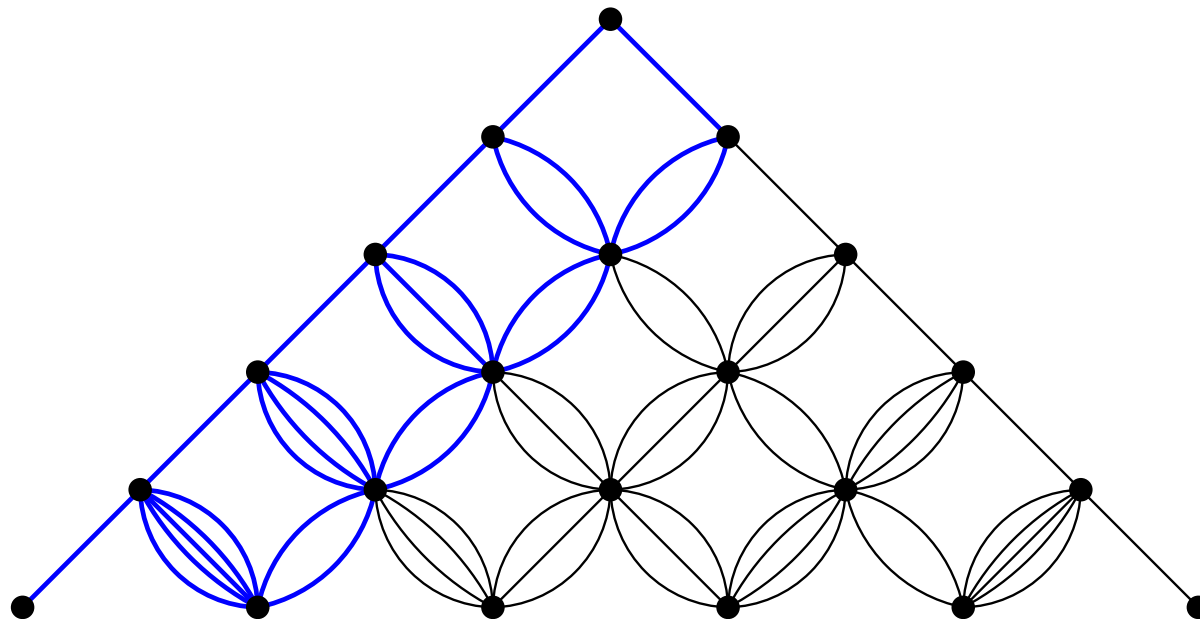
So, for two cylinders  $C_1, C_2$  and  $\mu$ -almost every  $x$

$$\frac{\mu[C_1]}{\mu[C_2]} = \lim_{n \rightarrow \infty} \frac{\dim(C_1, x_n)}{\dim(C_2, x_n)}$$

- The set of  $x$  with  $k < \infty$  rights is an invariant set.



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- The set of  $x$  with  $k < \infty$  rights is an invariant set.
- Likewise for the set of  $x$  with  $n - k < \infty$ .
- Hence  $\mu$ -almost every  $x$  has  $k$  and  $n - k$  unbounded.

