Uniqueness of the Symmetric Measure

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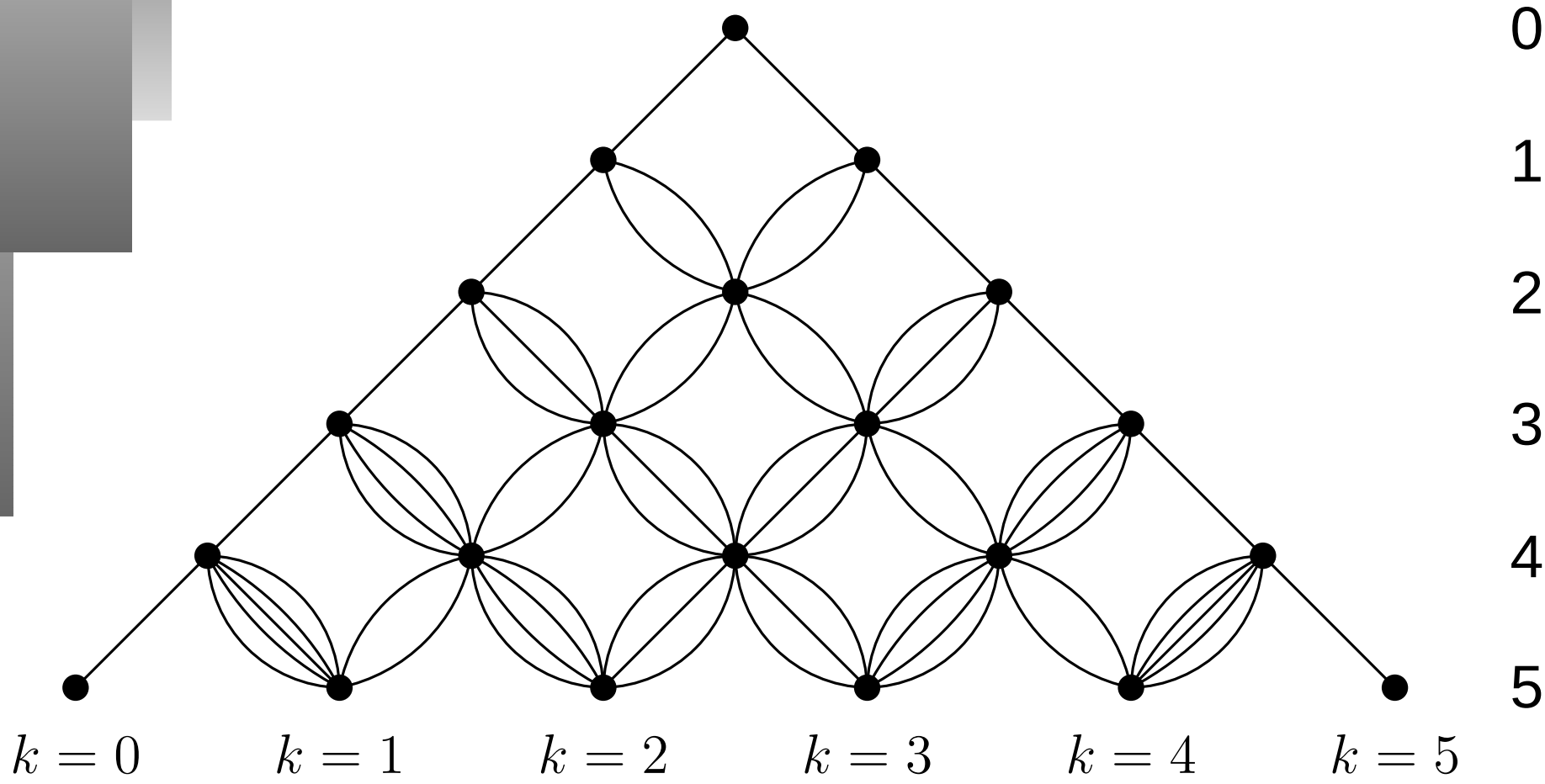
University of North Carolina

Euler Diagram



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- From any vertex (n, k) there are $k + 1$ edges leaving to the left, and $n - k + 1$ edges leaving to the right.

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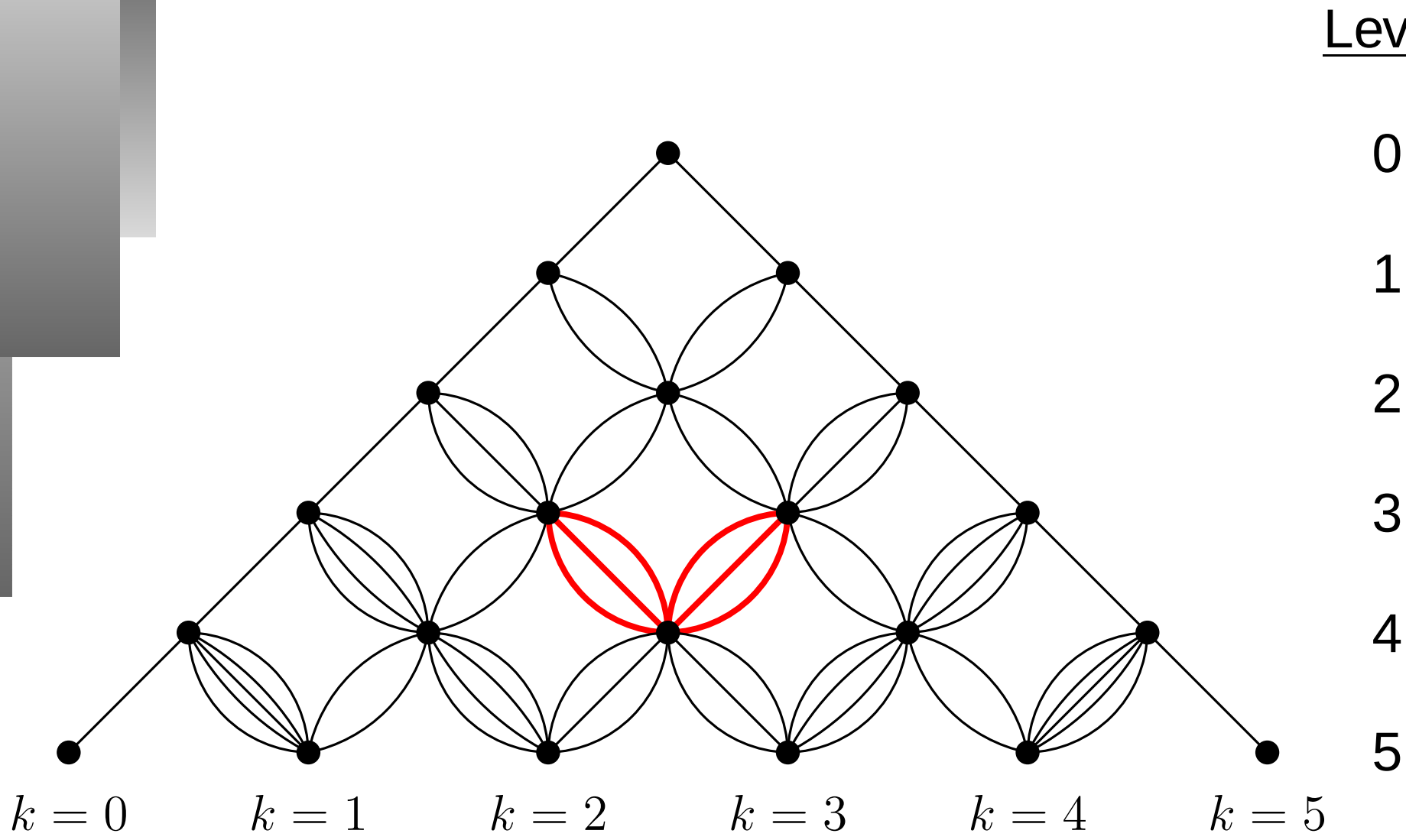
- X is a compact metric space with metric given by: For $x, y \in X$, $d(x,y) = 2^{-i+1}$ where $i = \inf\{j | x_j \neq y_j\}$.
- A cylinder set $C = [c_1 c_2 \ldots c_n]$ is a clopen set such that $x \in C$ implies $x = c_1 \ldots c_n x_{n+1} \ldots$. These sets generate the topology.

Edge Ordering



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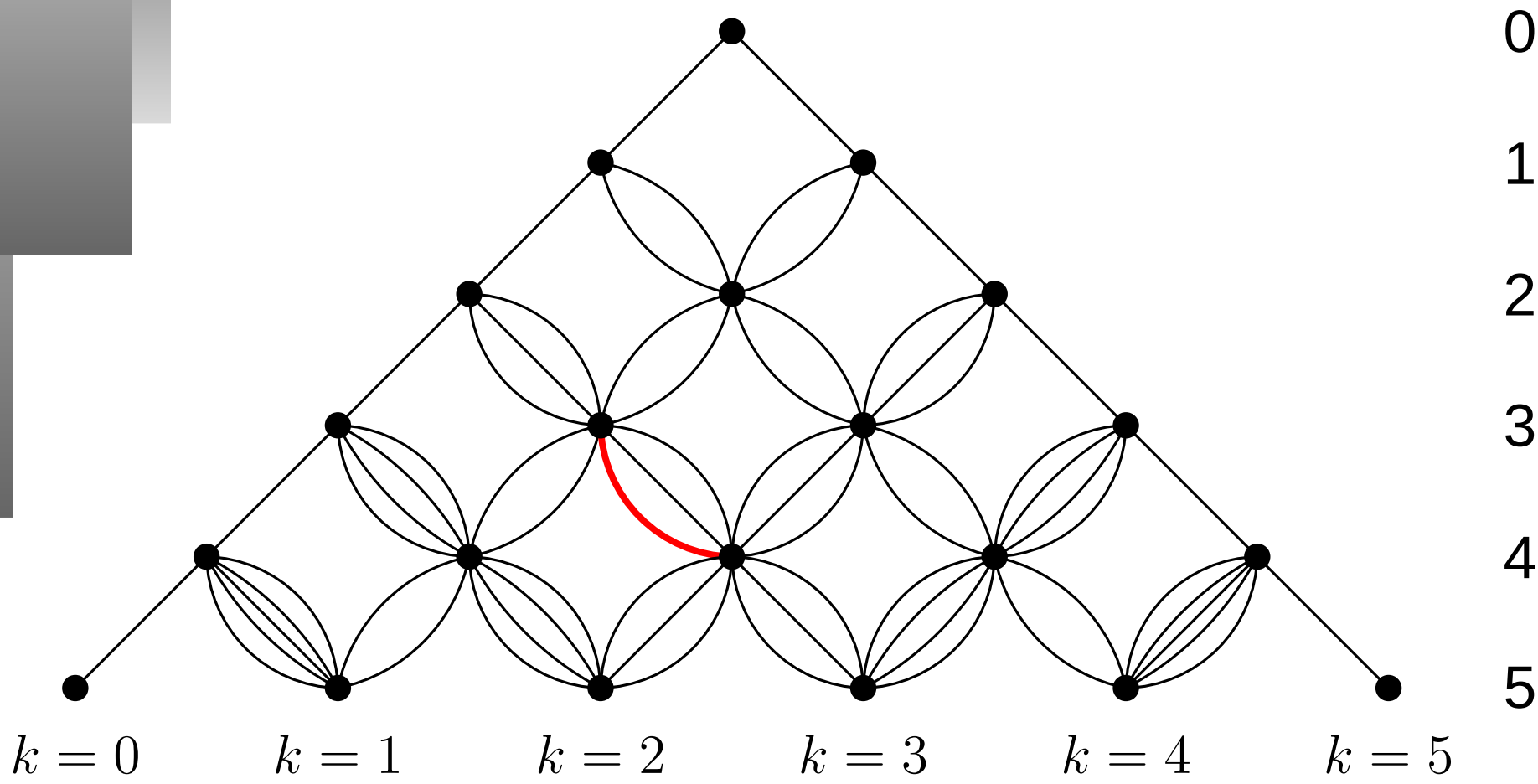

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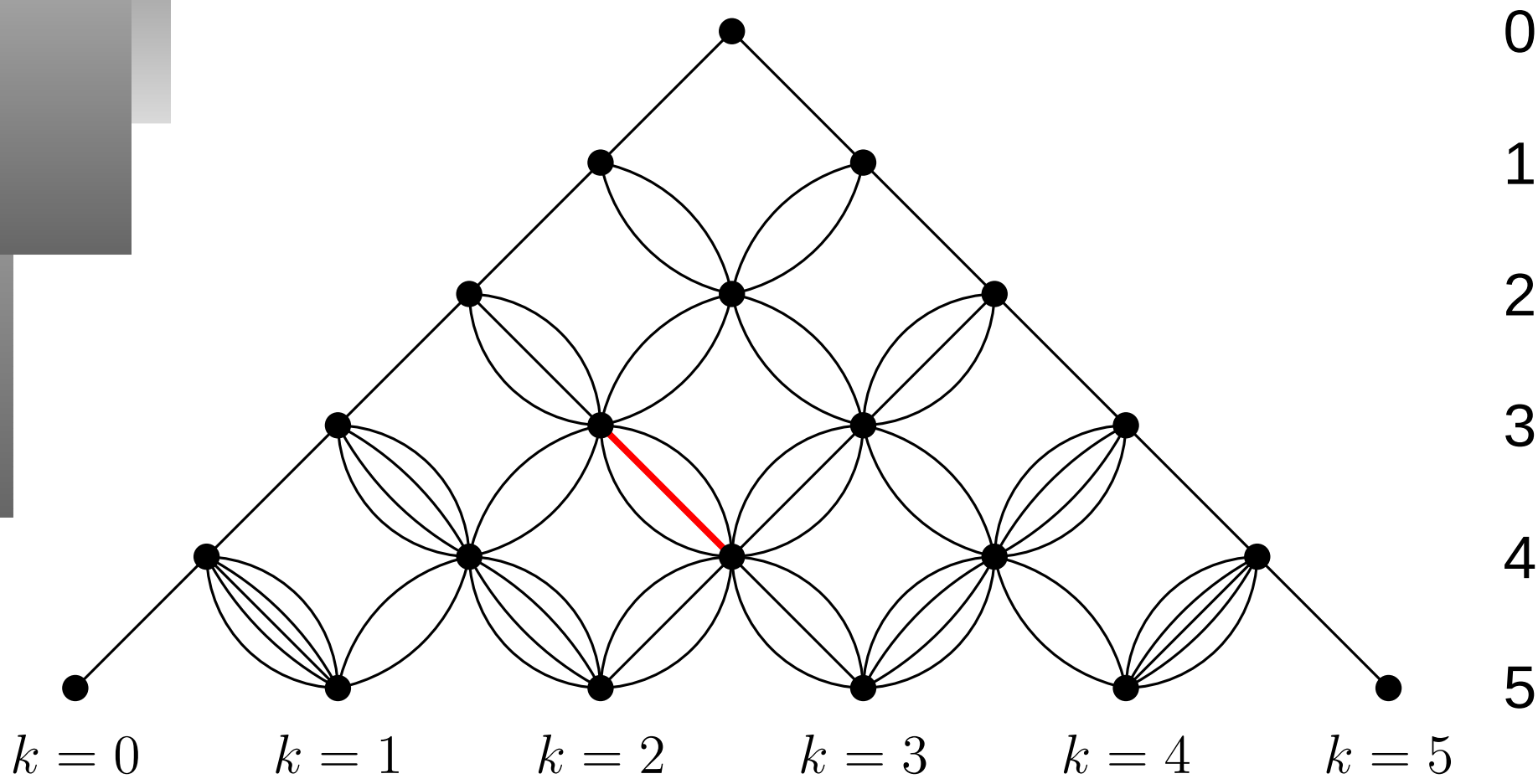

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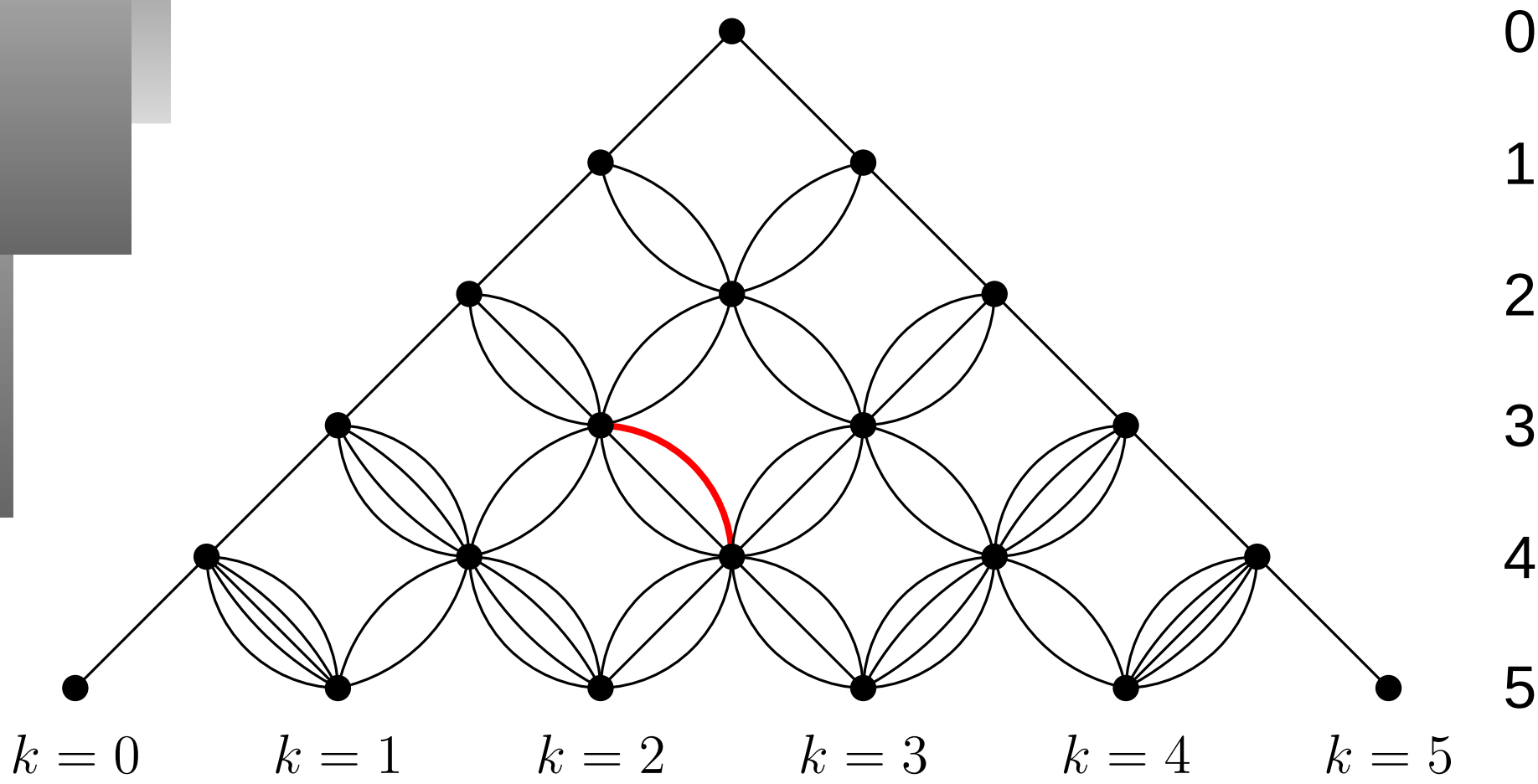

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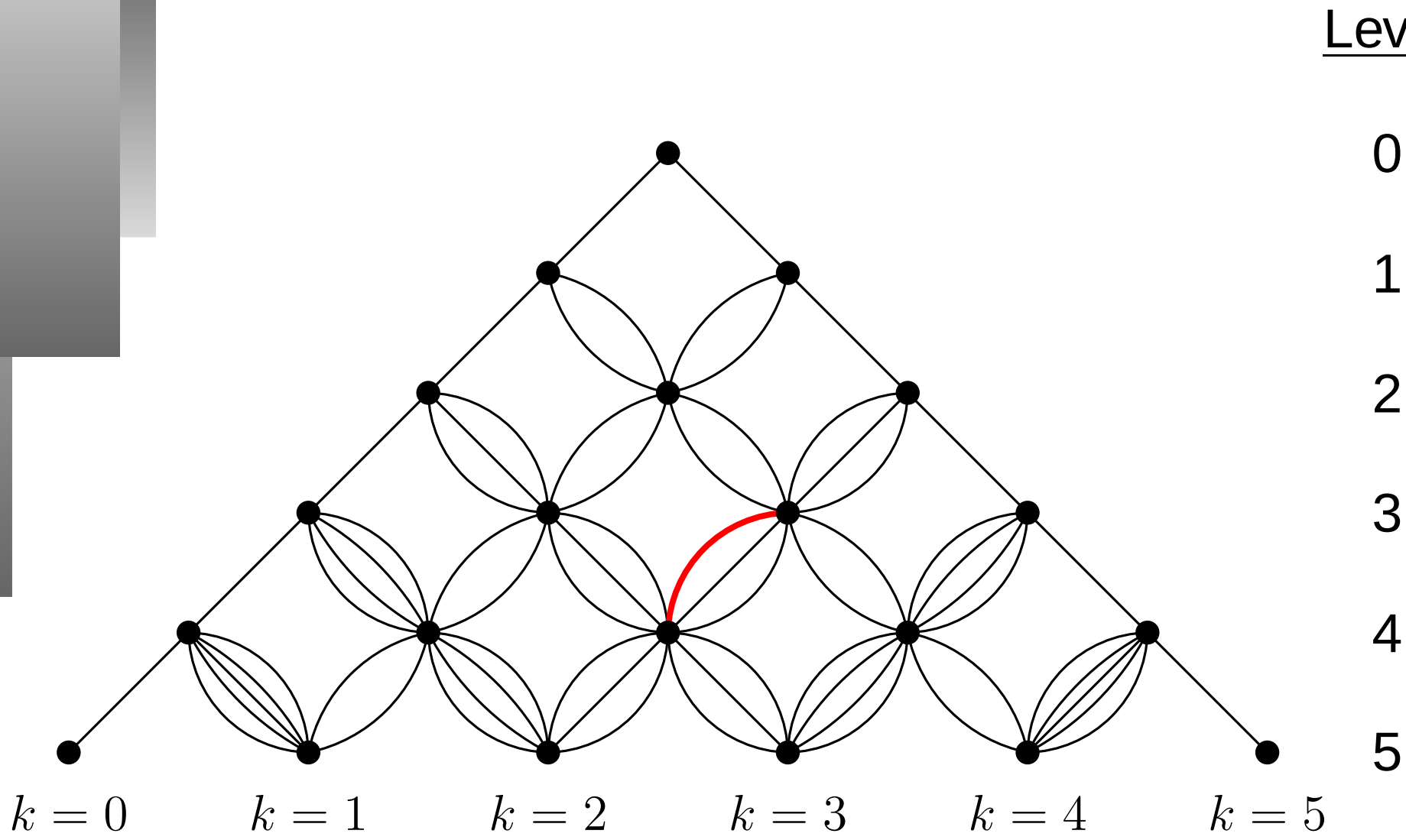

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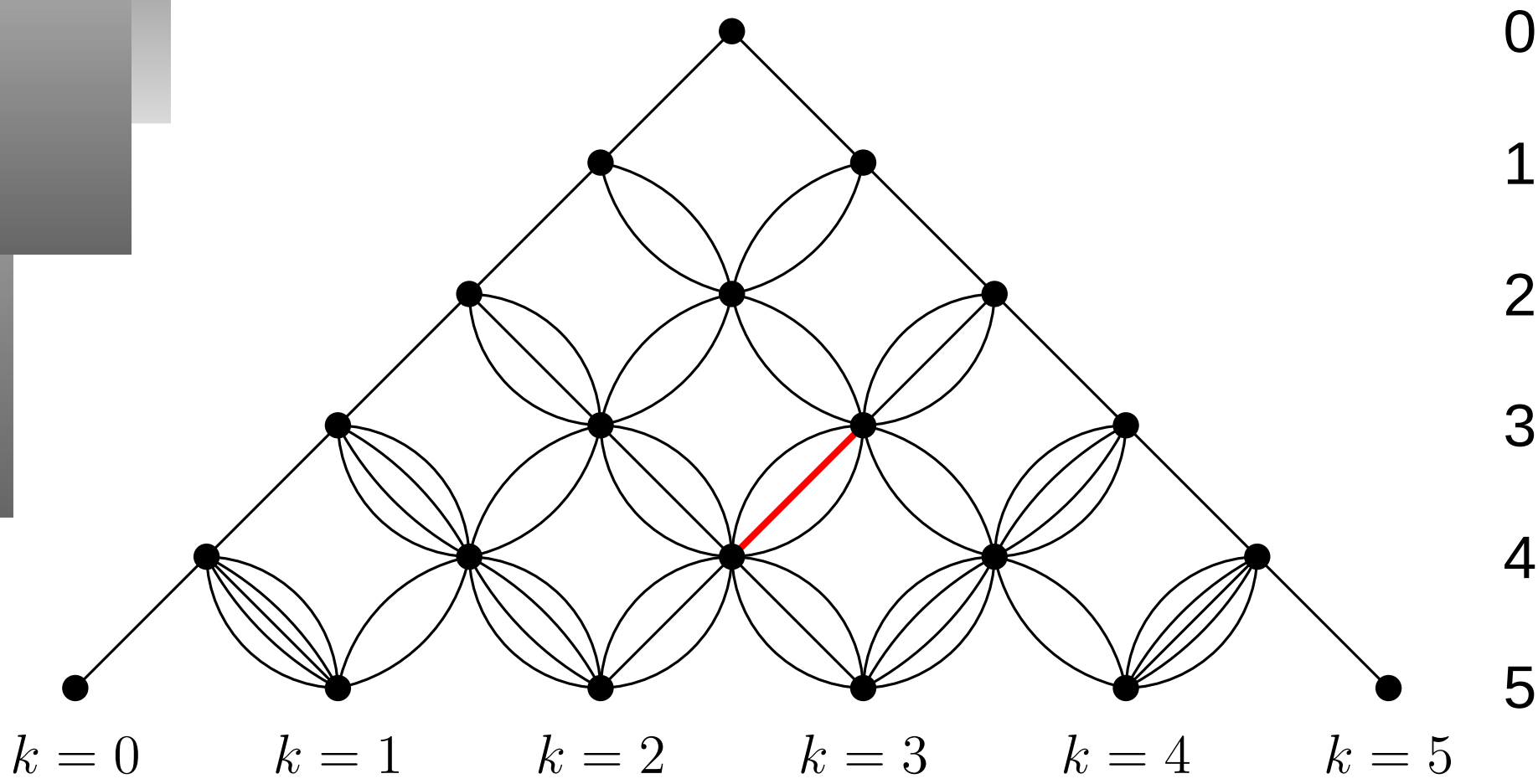

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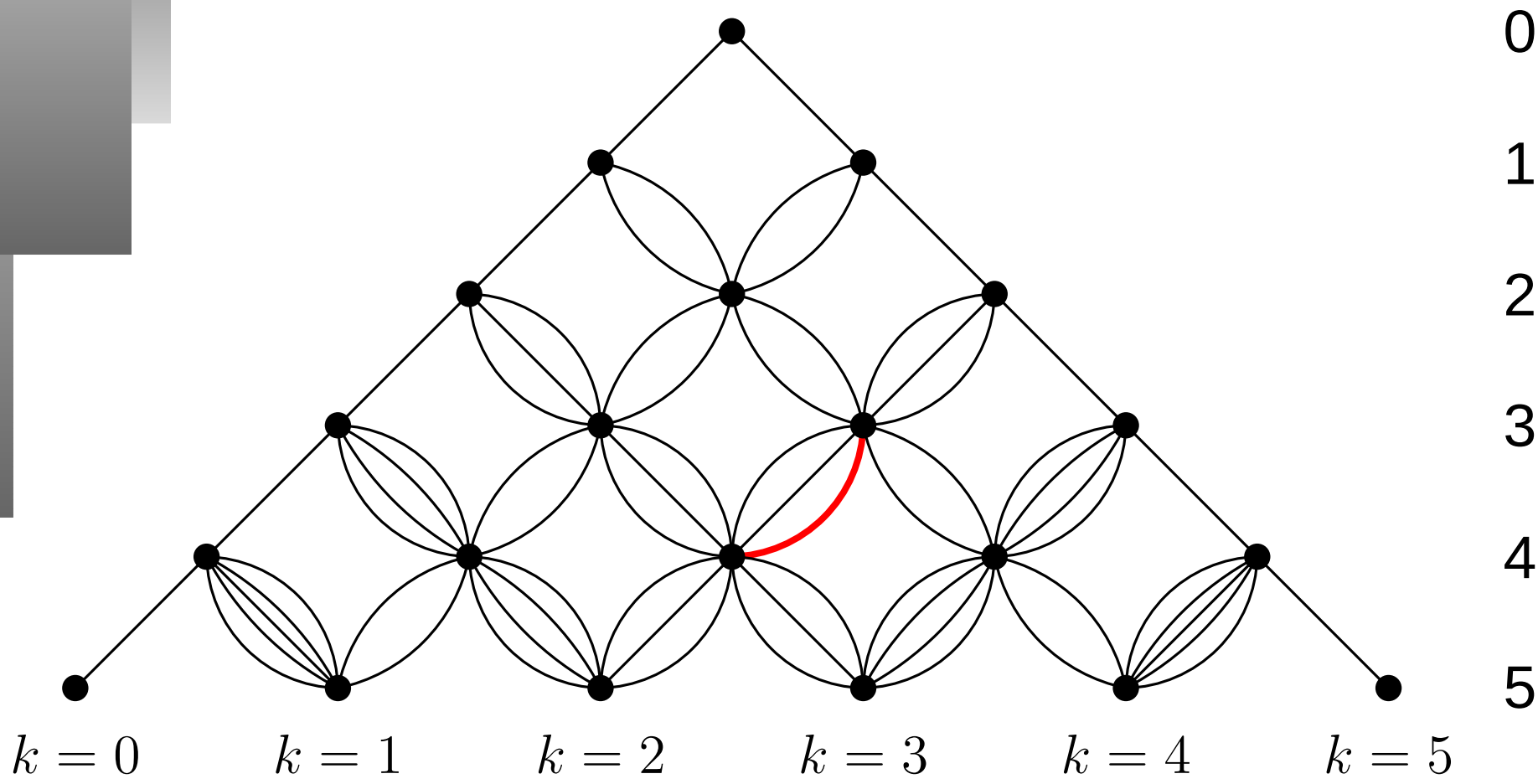
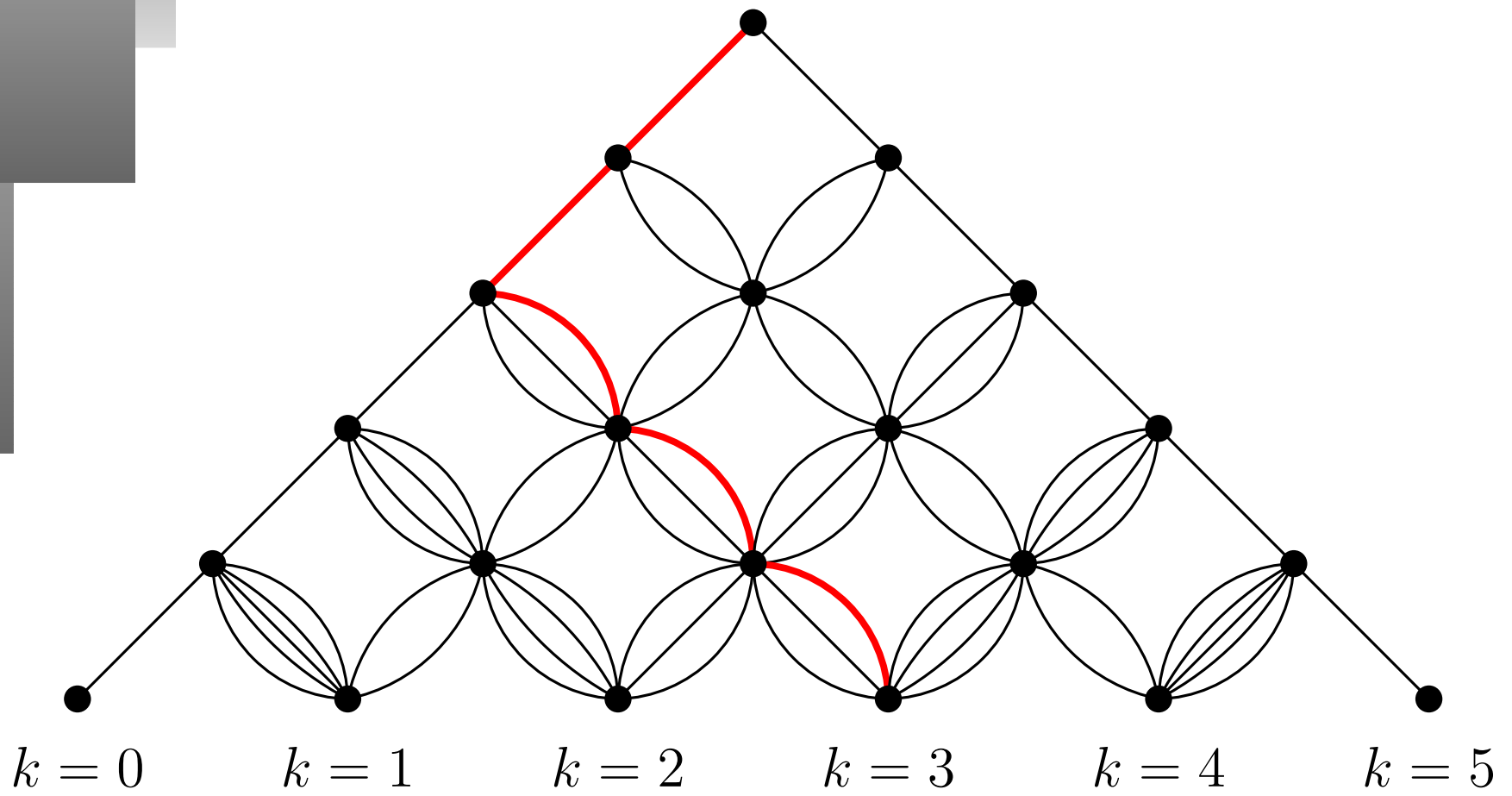


Euler Adic

$T : X \to X$ with $T(x)$ being the next "largest" path.



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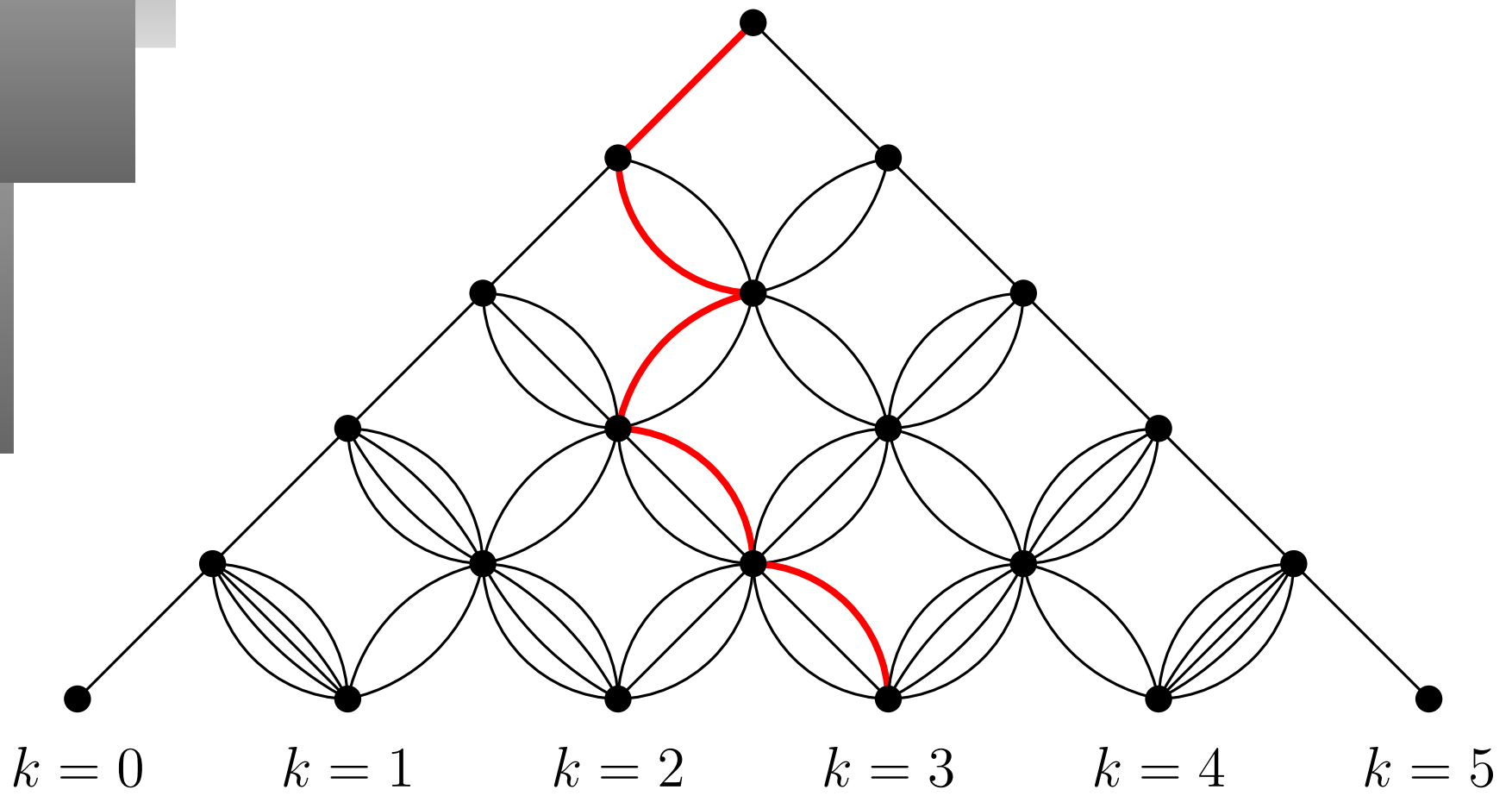

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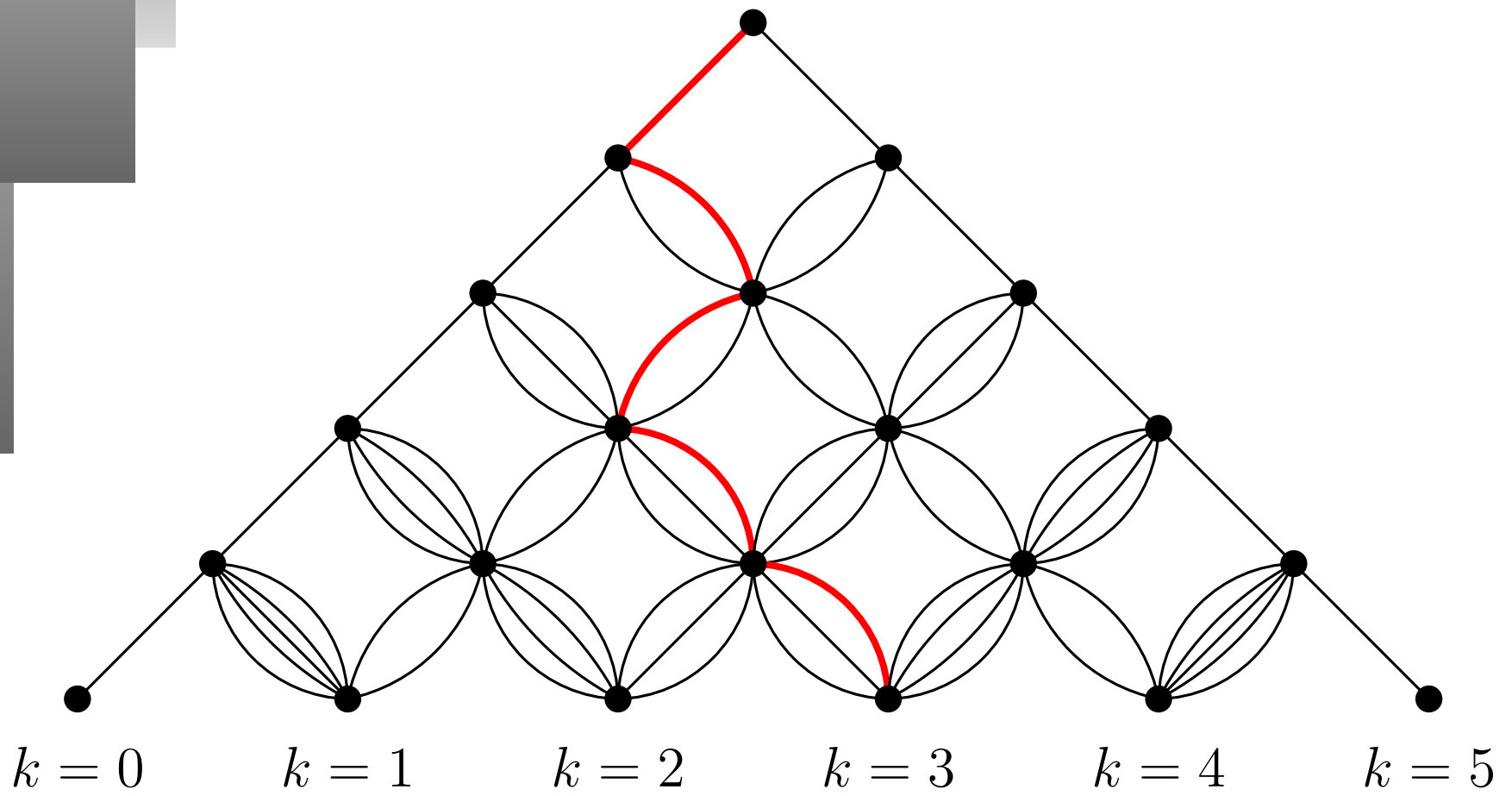

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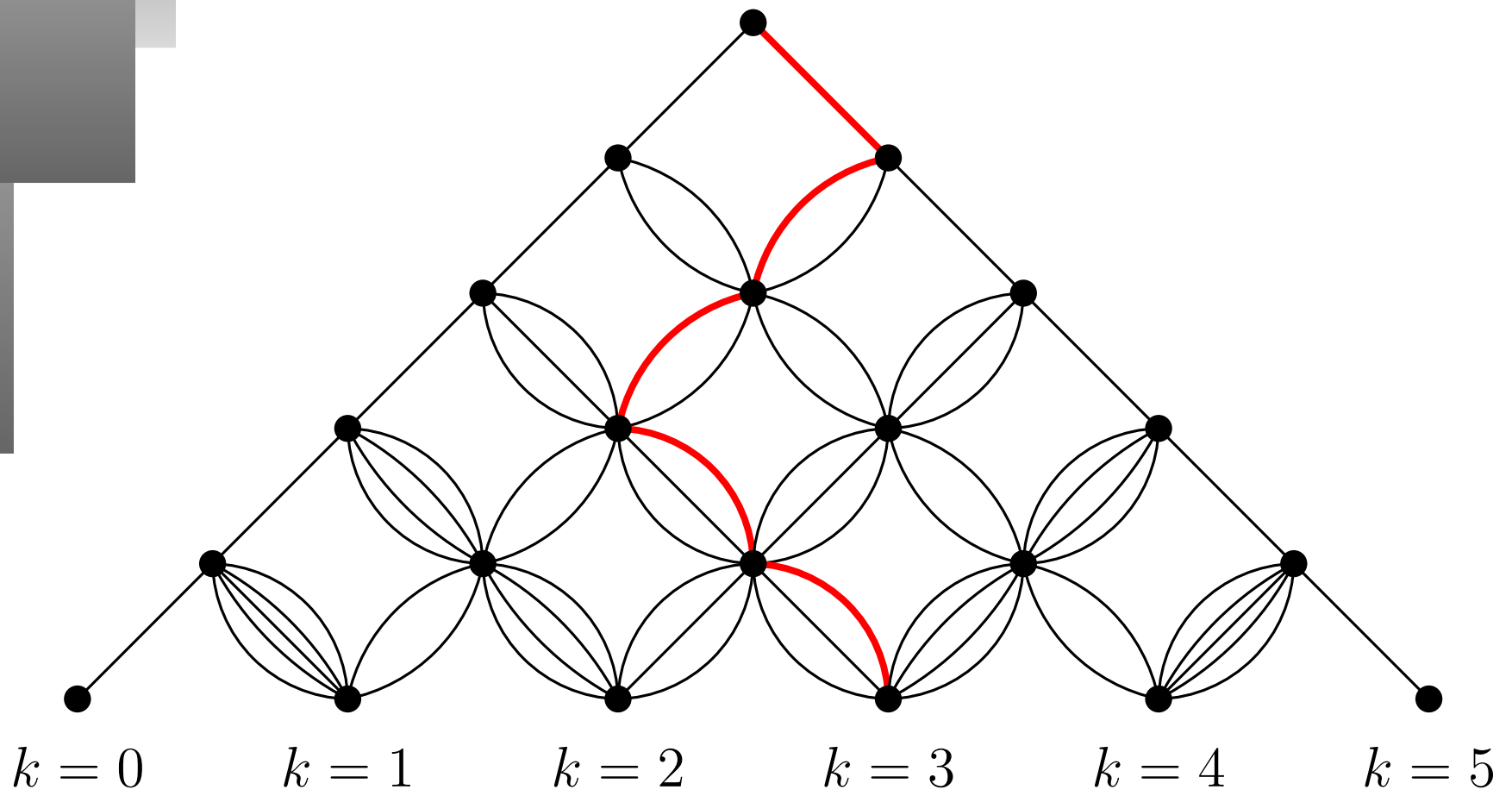

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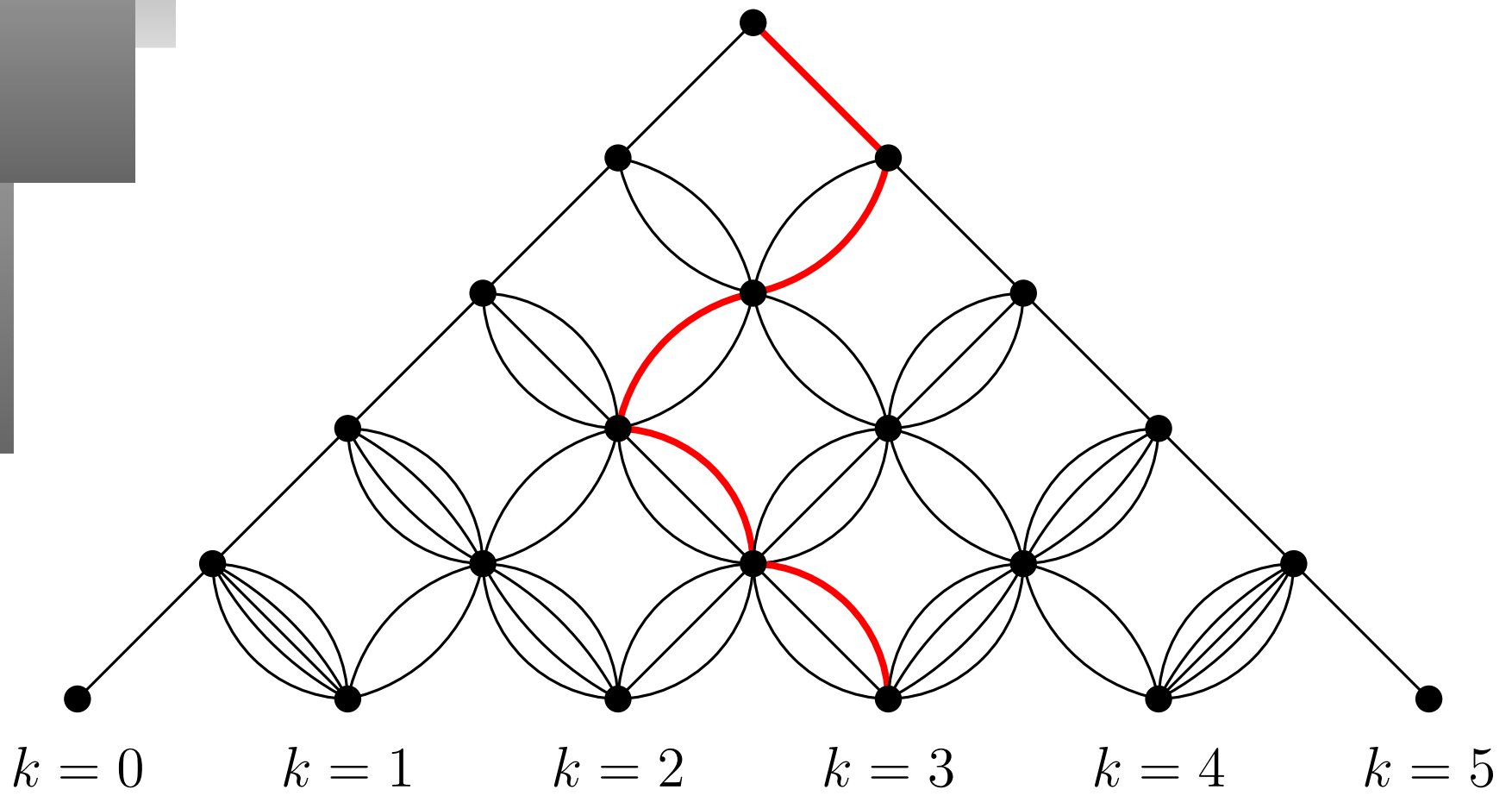

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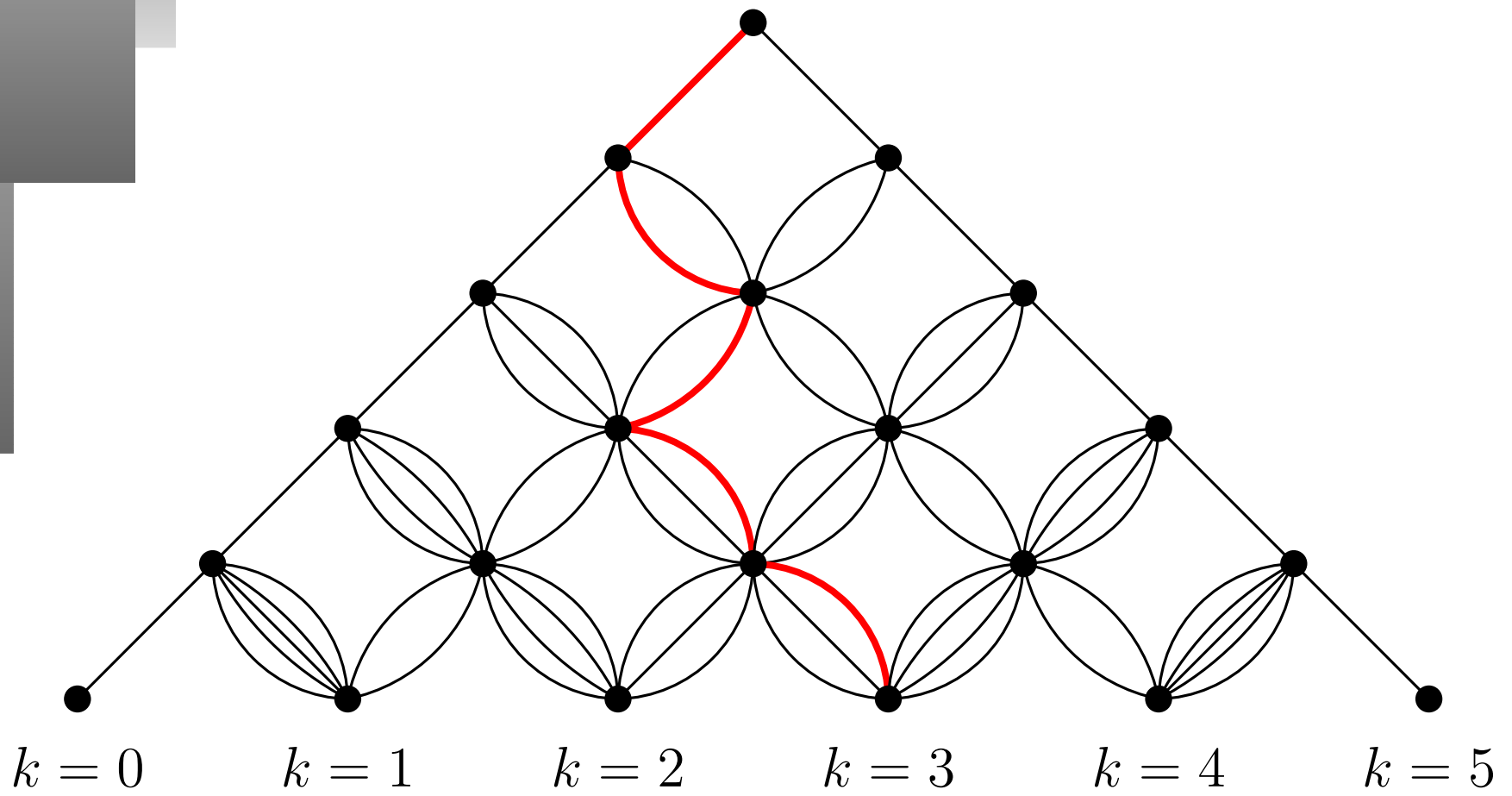

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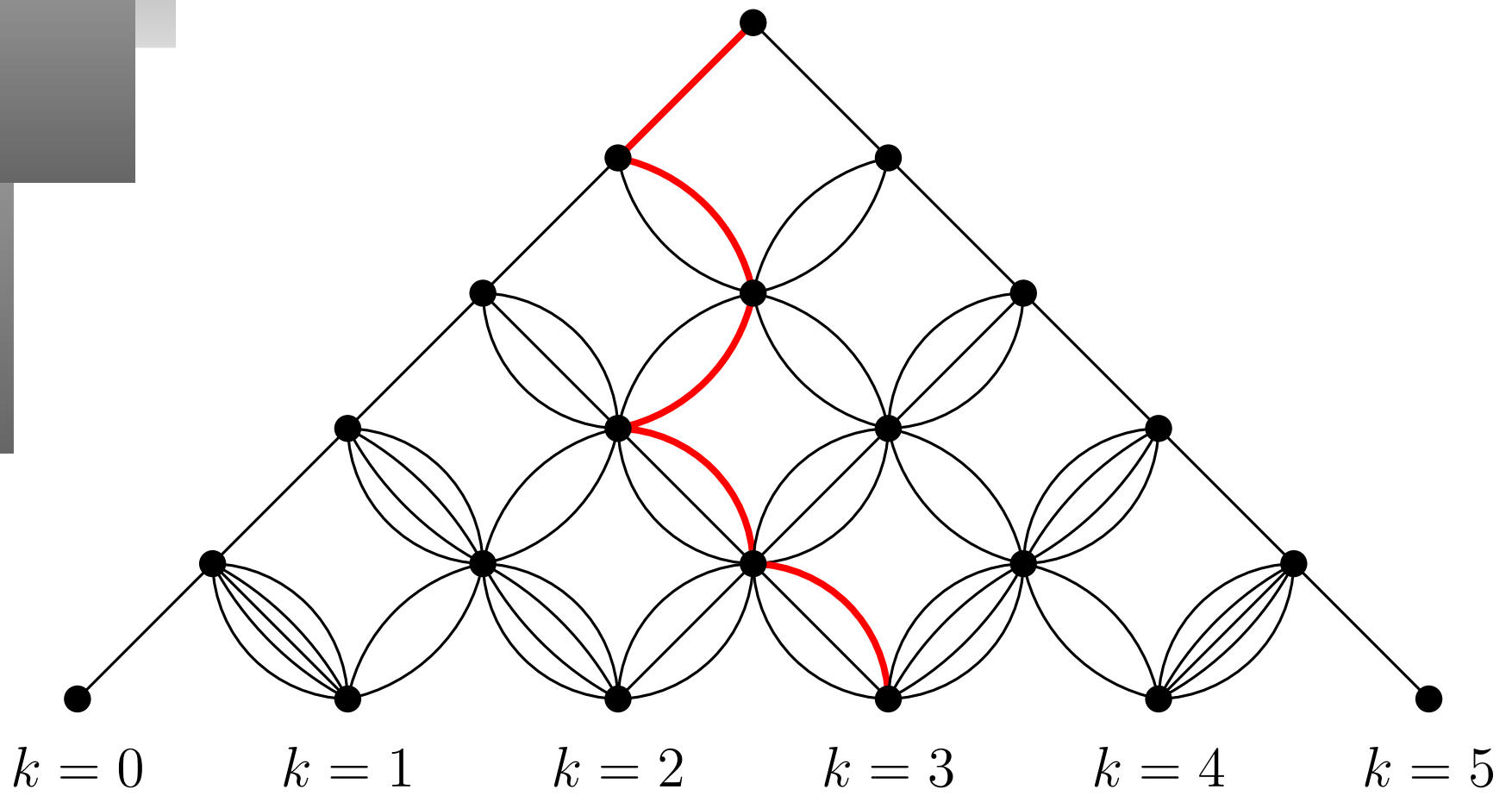

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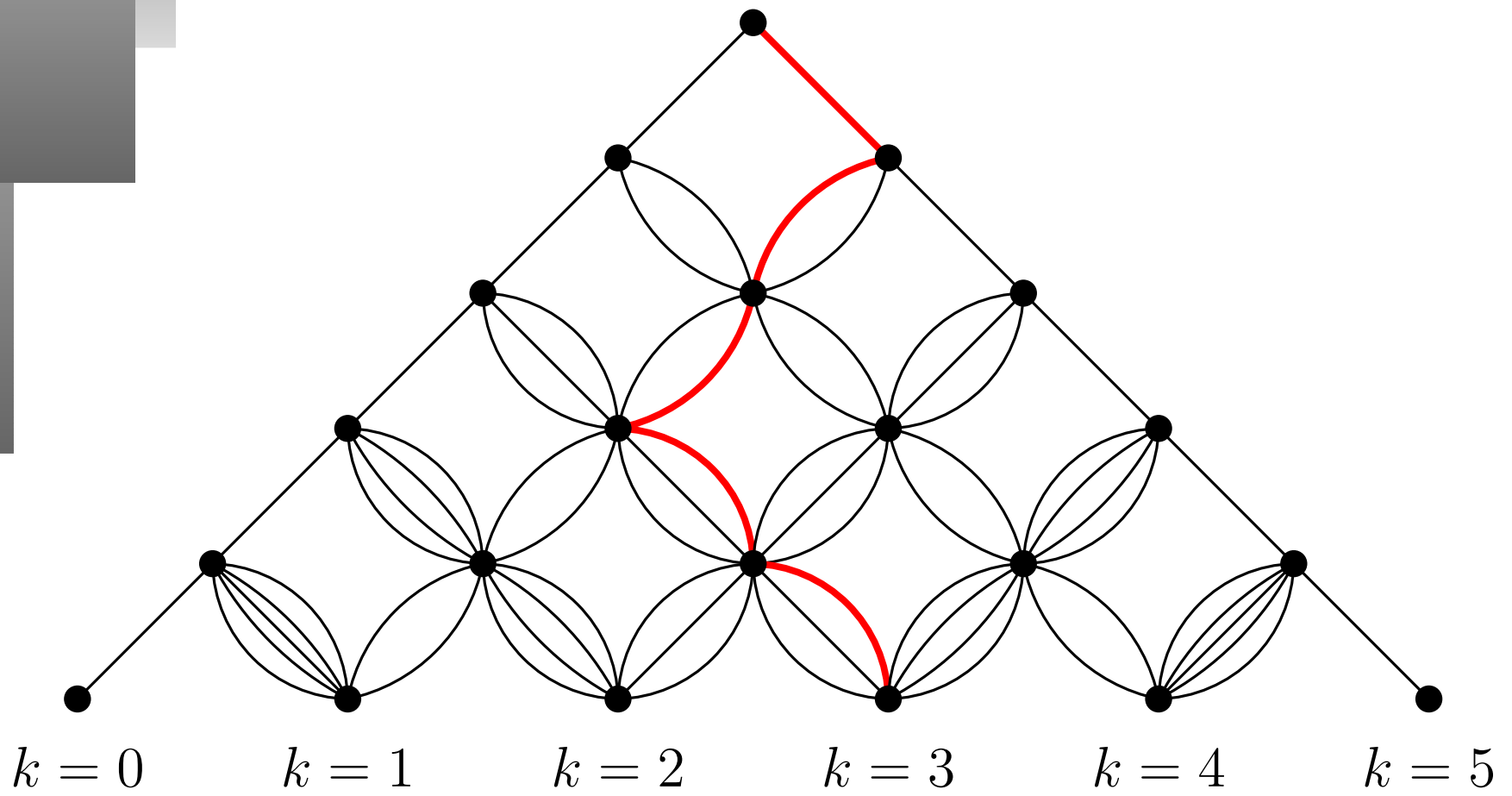

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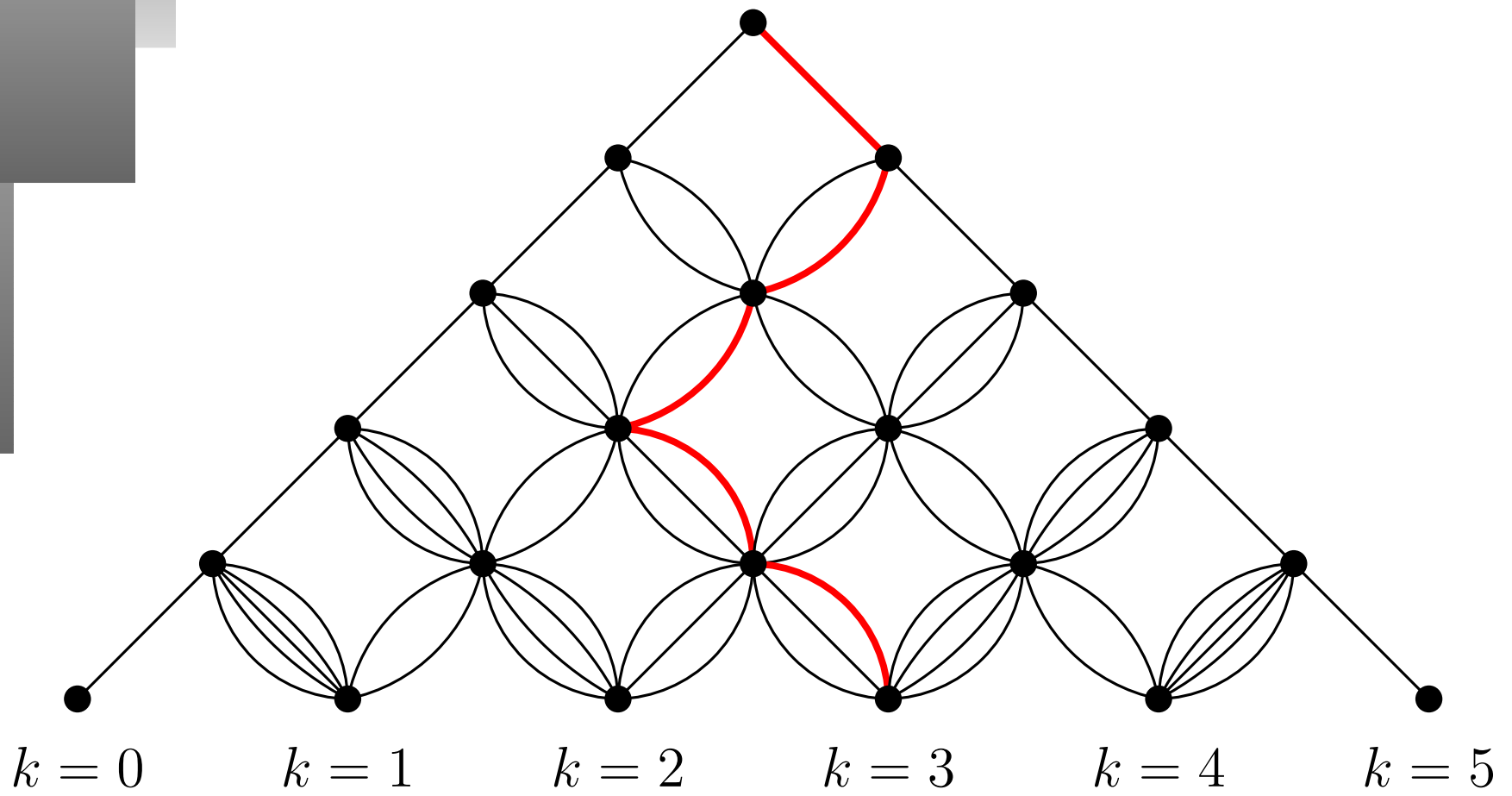

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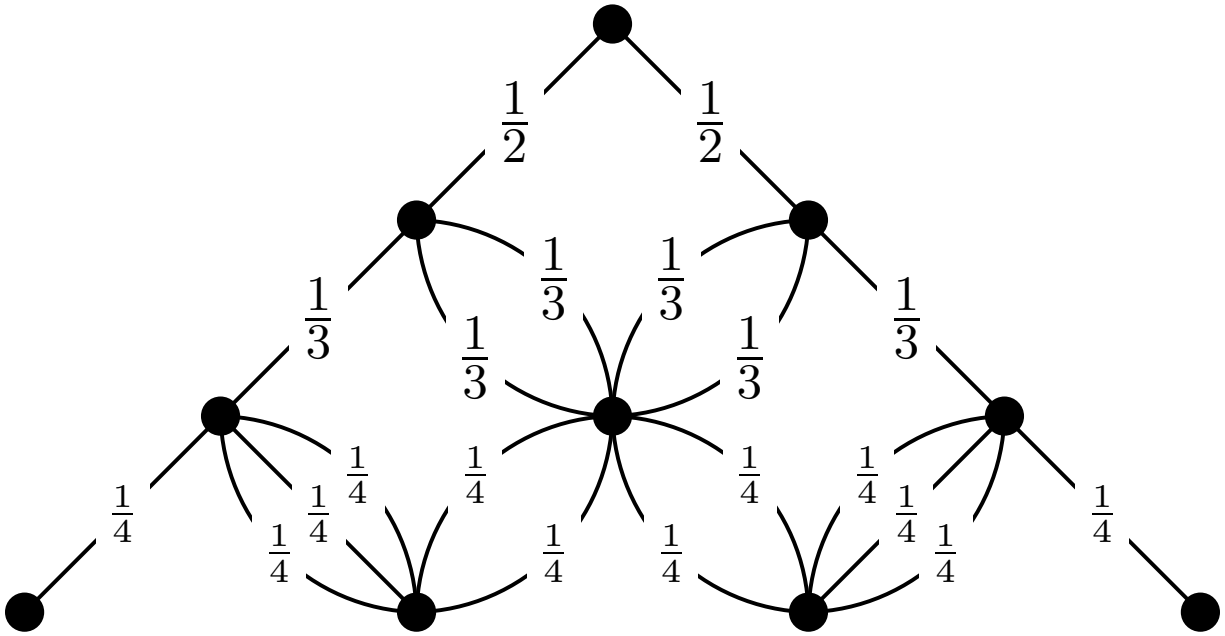
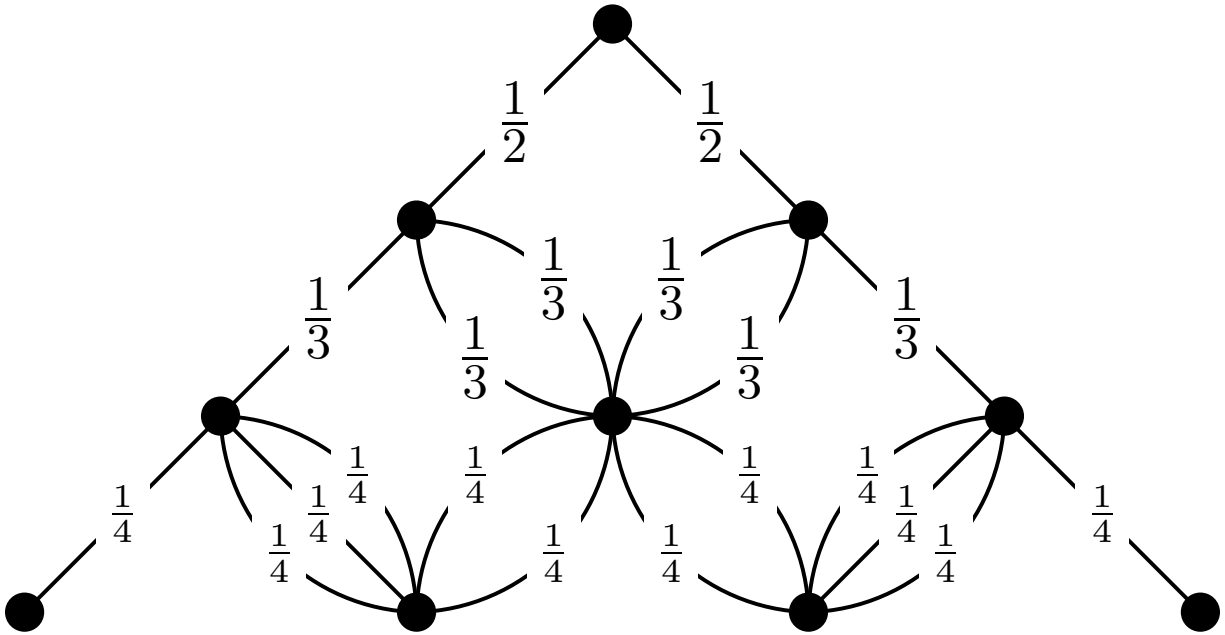
- For the Euler adic, T, it is necessary and sufficient that every cylinder terminating in the same vertex (n, k) has the same measure.
- A T-invariant measure μ is said to be *ergodic* if for any T-invariant set, A, $\mu(A)$ is 0 or 1.

Symmetric Measure



- Gives each edge connecting level n to level $n+1$ weight $\frac{1}{n+2}$.
- Gives each cylinder of length n measure $\frac{1}{n+1!}$

Symmetric Measure



- Clearly T-invariant.
- In joint work with Keane, Petersen, and Salama the symmetric measure has been shown to be ergodic.

Cylinders and Permutations

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- Hence there is a bijective correspondence between cylinders of length n with k right turns and the set of permutations of $\{1,2,\ldots,n+1\}$ with k rises.

Adding Rises

For a permutation, $a_1 a_2 \ldots a_{n+1}$ with k rises, there are $n - k + 1$ ways to insert $n + 2$ so that the new permutation has $k + 1$ rises.

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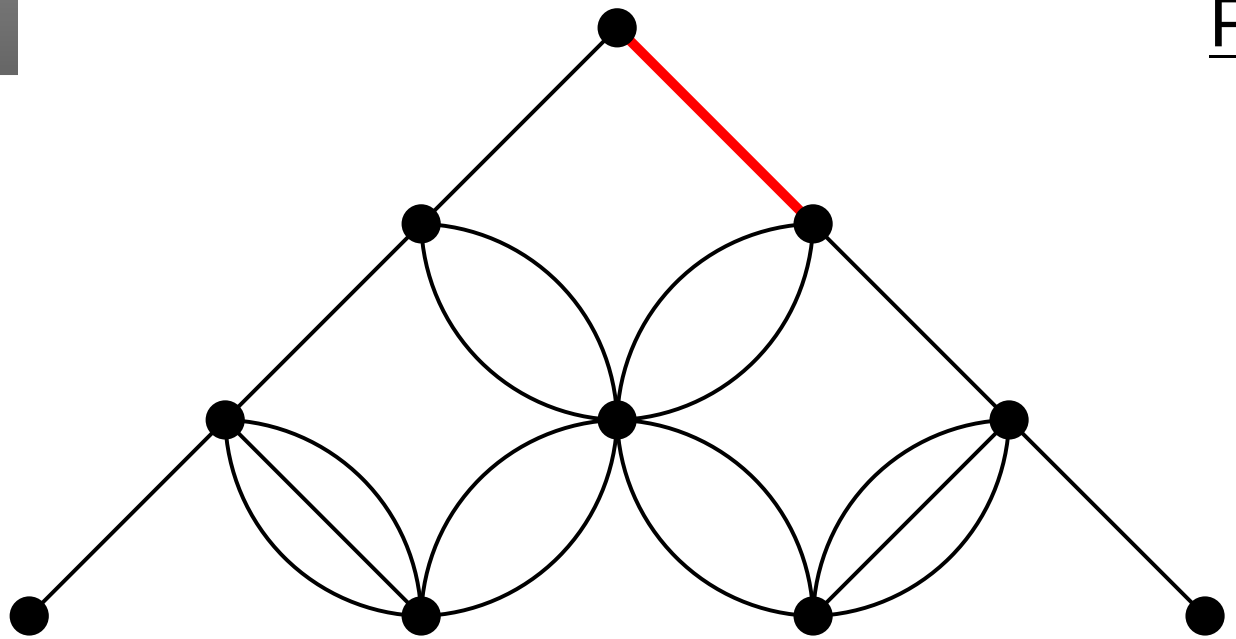
$$a_i^R (n+2)^F a_{i+1}$$

Cylinders and Permutations

- The permutation 12 has one rise, and the corresponding cylinder has one right.

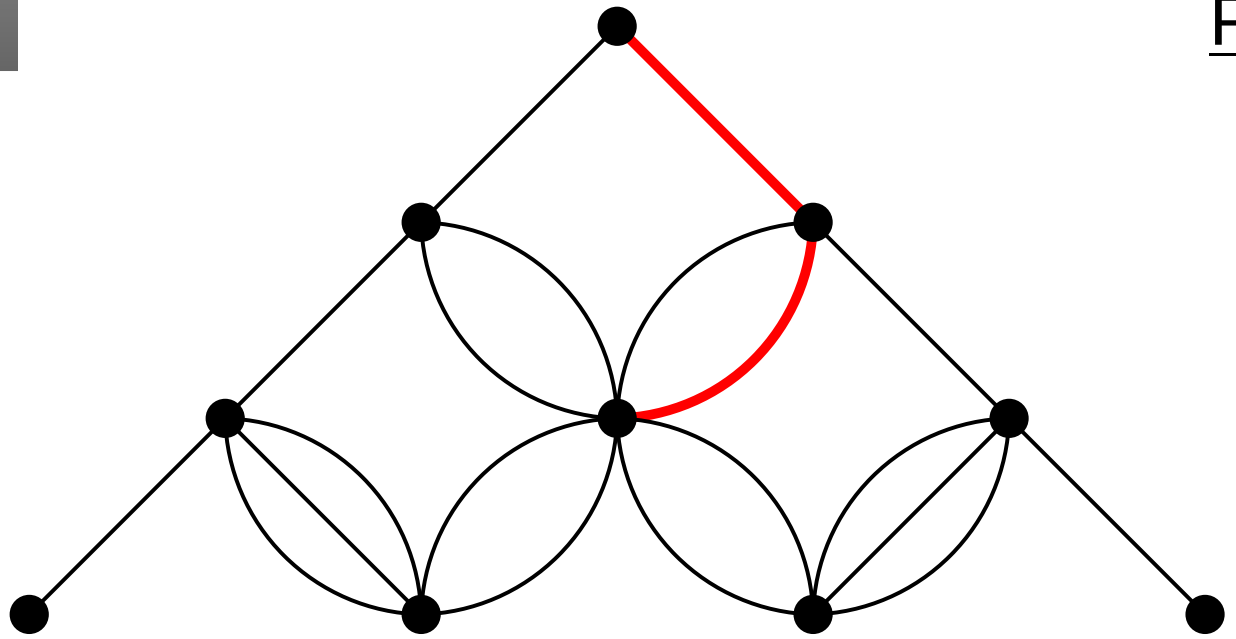


Permutation

12

Cylinders and Permutations

- In order to extend the cylinder 12 by a left turn, insert a 3 in such a way to add a fall.



Permutation

312

Cylinders and Permutations

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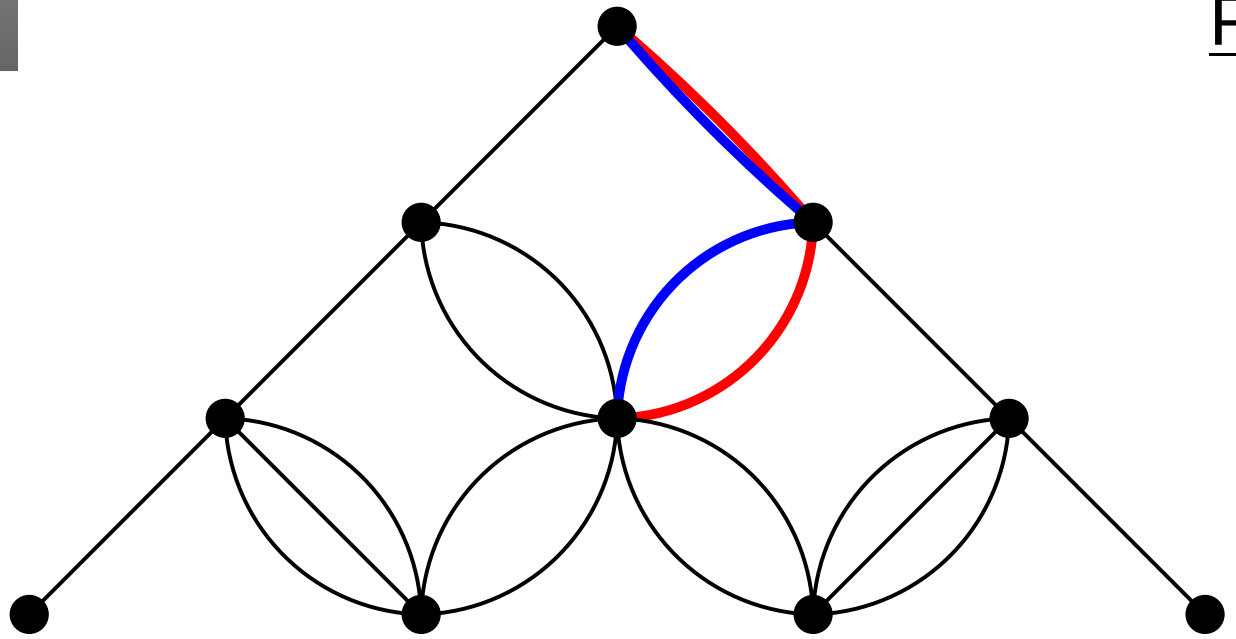

Permutation

312

or 132

Cylinders and Permutations

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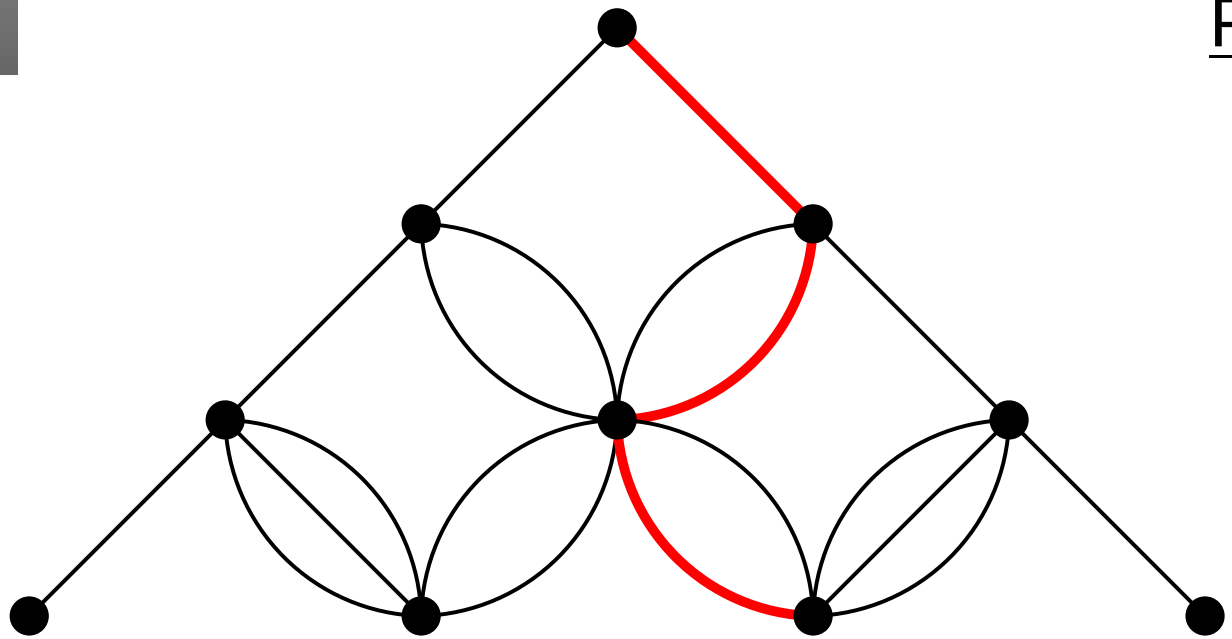

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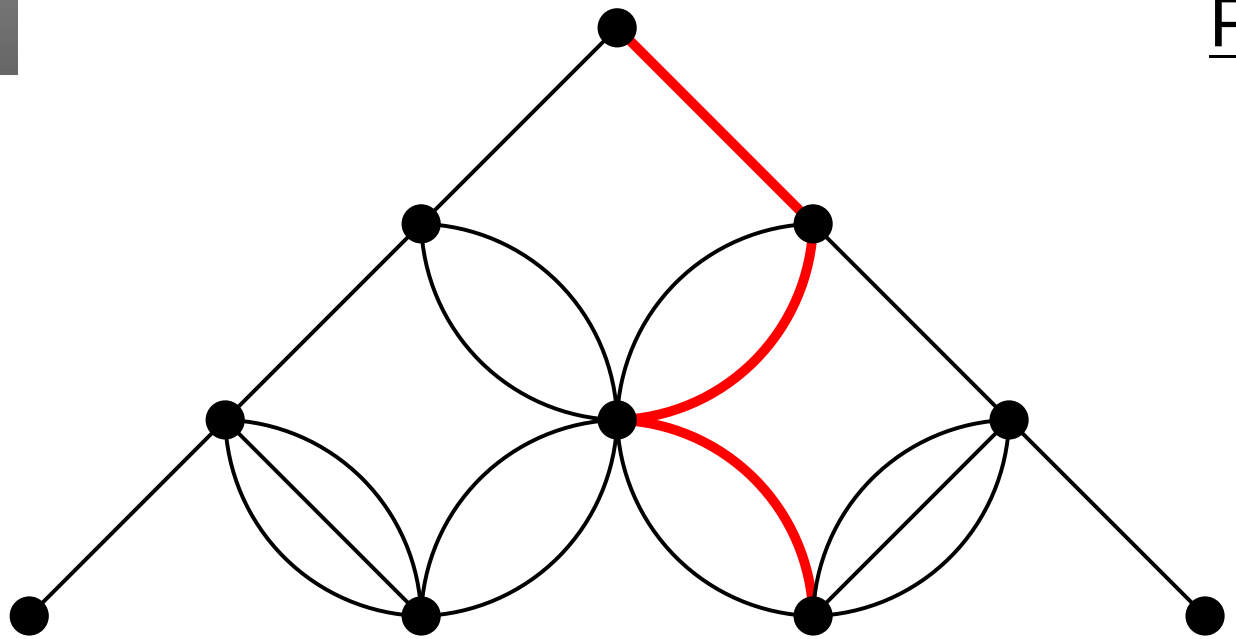

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Dimension Definitions

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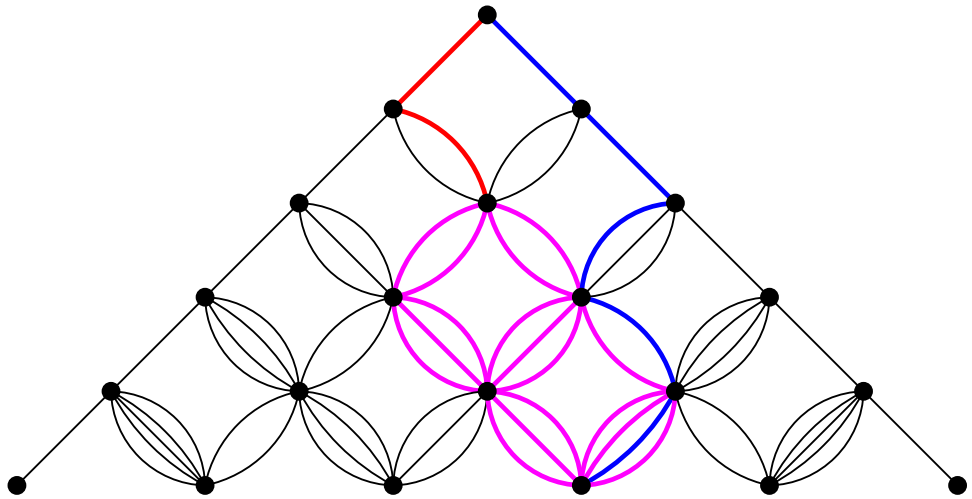

Measures of Cylinder Sets

Let μ be an invariant fully-supported ergodic probability measure for the Euler adic. Then for every cylinder set C and μ-almost every $x \in X$

$$\mu[C] = \lim_{n \to \infty} \frac{\dim(C, k_n(x))}{\dim(k_n(x))}$$

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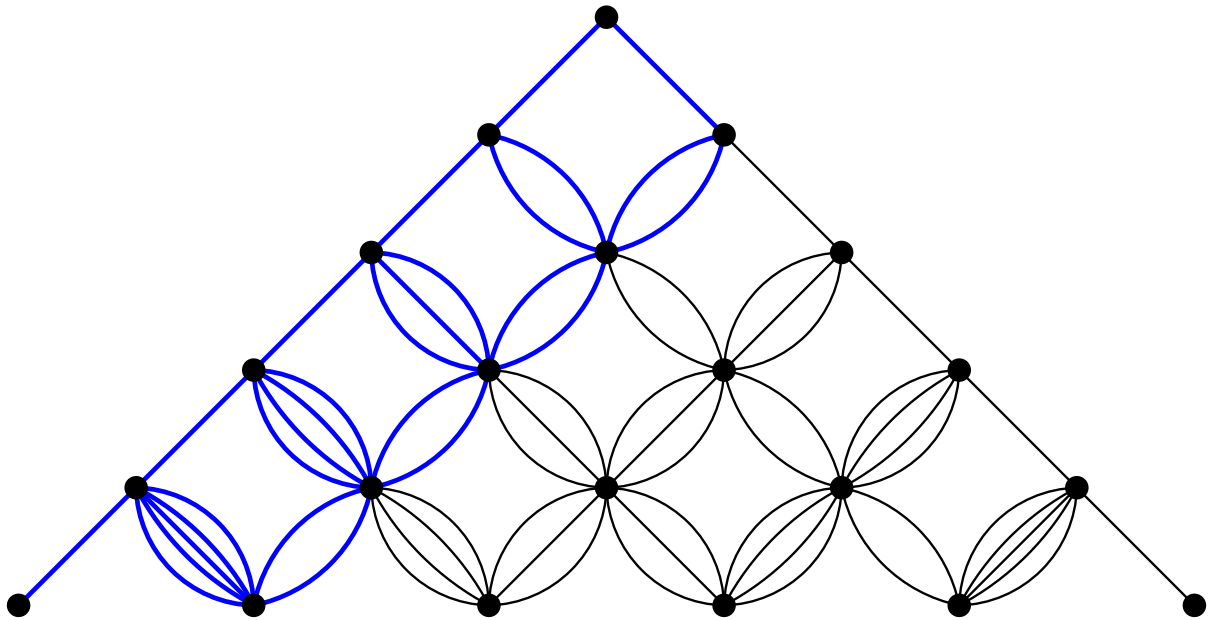
So, for two cylinders C_1, C_2 and μ-almost every x

$$\frac{\mu[C_1]}{\mu[C_2]} = \lim_{n\to\infty} \frac{\dim(C_1, k_n(x))}{\dim(C_2, k_n(x))}$$

μ-almost every x

- The set of x with $k < \infty$ rights is an invariant set.



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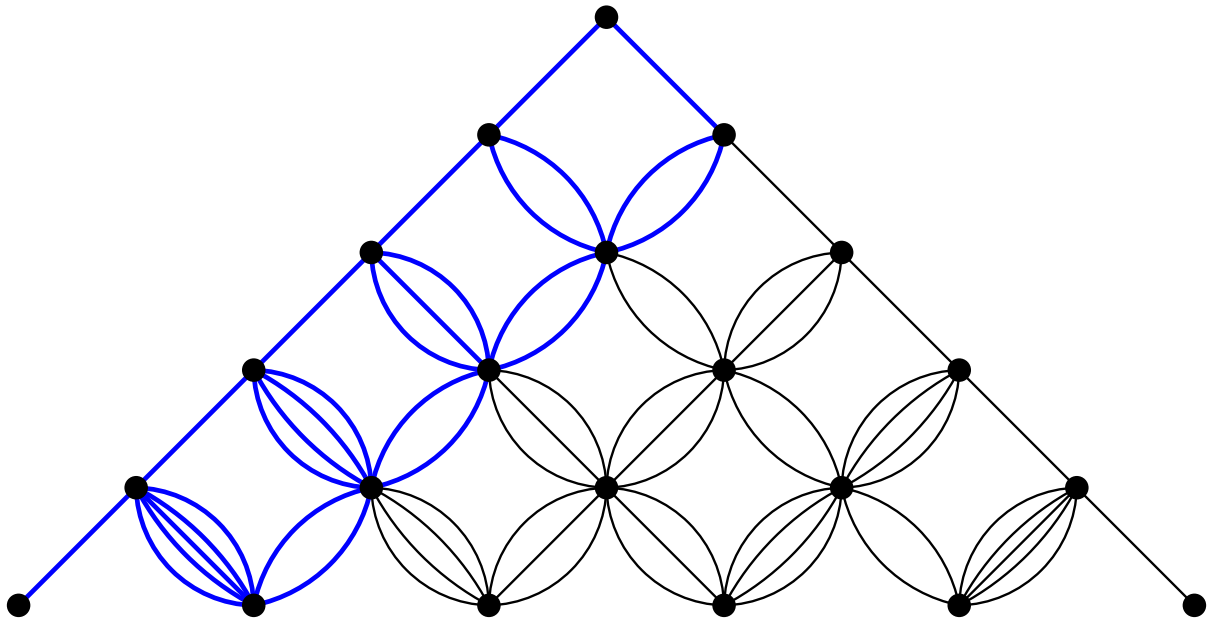

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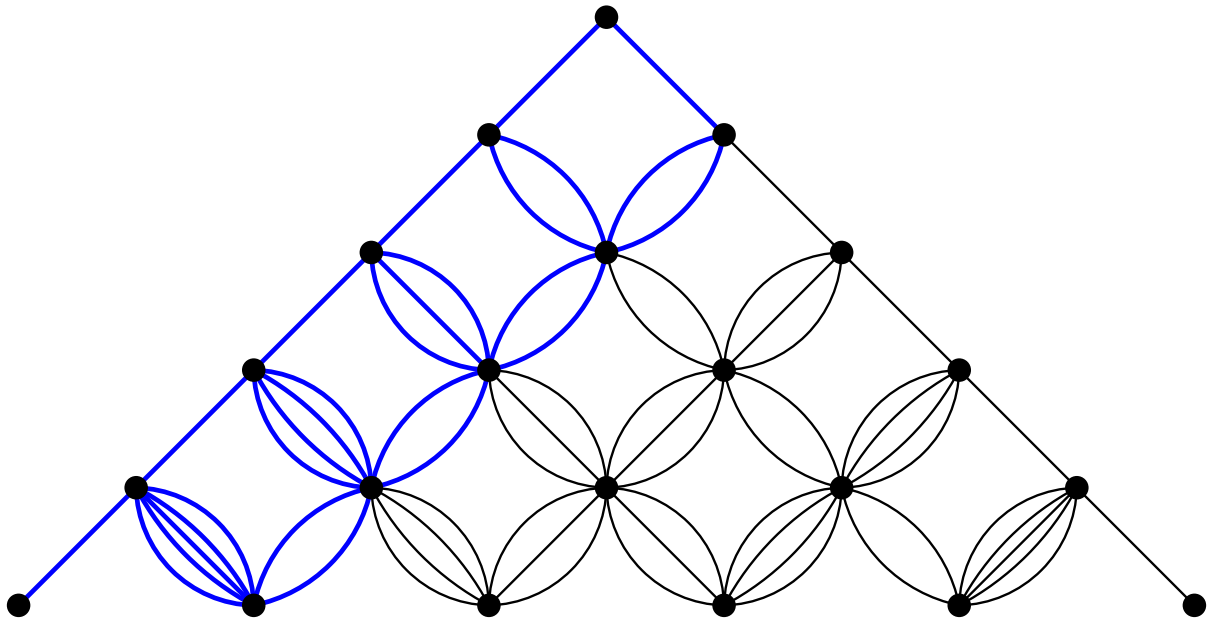
- Hence μ-almost every x has k and $n - k$ unbounded.



Quick Review

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  $$\frac{\mu[C_1]}{\mu[C_2]} = \lim_{n\to\infty} \frac{\dim[C_1, k_n(x)]}{\dim[C_2, k_n(x)]}.$$
- For μ-a.e. $x \in X$, k and $n-k$ are unbounded as n goes to ∞.

Theorem

Theorem 1 *If μ is an invariant fully-supported ergodic probability measure for the Euler adic, then μ is the symmetric measure.*

- Let C_1 and C_2 be different cylinders of length n_0, and $\pi(C_1)$ and $\pi(C_2)$ be their corresponding permutations.

Computation of $\dim(C_1, k_n(x))$

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- In counting $\dim(C, x_n)$ we see Eulerian numbers $A(n-(n_0+1), j)$, with coefficients of various degrees in k and $n-k$.

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- Inserting 1 or 2 into a rise adds a fall, but no rises.
- Hence, $3 \ldots n+1$ is a permutation with k rises
- There are $\frac{k(k-1)}{2} A(n-2, k)$ suitable permutations.

Dominating Terms

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- If any of $1, \ldots, n_0 + 1$ are placed adjacently, we obtain a lower degree product.
- Hence in sum of $\dim(C_1, k_n(x))$, the permutations for which $1, \ldots, n_0 + 1$ are not adjacent dominate the sum.

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- And μ must be the symmetric measure.

Corollary

Theorem 2 *The symmetric measure is ergodic.*

Proof

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- If no ergodic measure has k and $n-k$ unbounded a.e., then the set of ergodic measures is

$$\mathcal{E} = \bigcup_K \mathcal{E}_K.$$

Continuation of Proof

- If η is the symmetric measure, by the Ergodic Decomposition Theorem:

$$\eta = \int_{\mathcal{E}} e \, dP_\eta(e) = \sum_K \int_{\mathcal{E}_K} e \, dP_\eta(e).$$

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- Hence there must be an ergodic measure for which k and $n-k$ unbounded is a set of full measure, i.e. the symmetric measure.