Non-Simple Non-Stationary Bratteli-Vershik Systems

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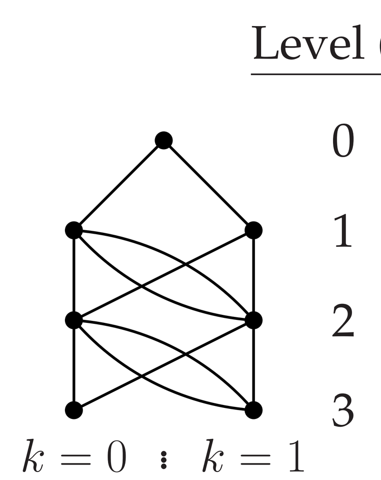
Bratteli Diagram

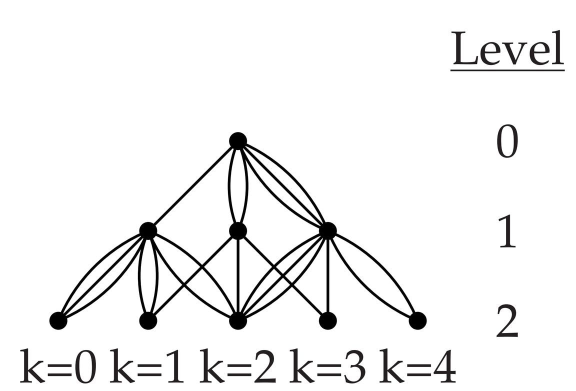
- Infinite downward directed graphs
- Vertices, denoted by (n,k), are partitioned into finite sets, V_n which are referred to as levels.
- V_0 is always a one point set
- Edges partitioned into finite sets E_n, connect vertices in consecutive levels V_{n-1} and V_n
- Incidence matrices describe the number of edges connecting levels n and $n+1$



$$A_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

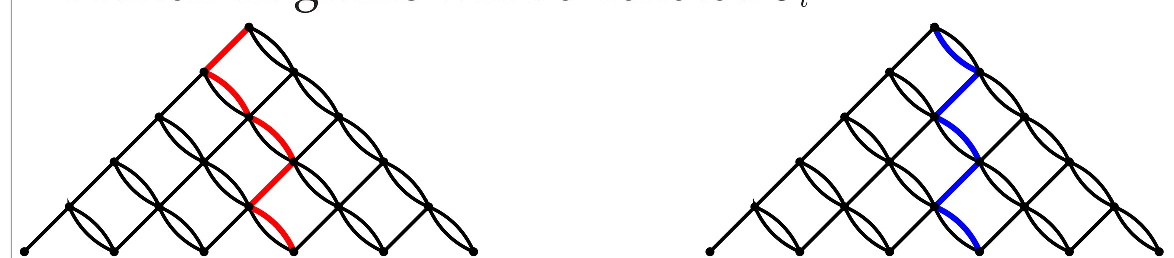
A Special Family

- At each level increase the number of vertices by d.
- There are only edges connecting (n,k) to vertices $(n+1,k+i)$ for $i = 0,1,\dots,d$.
- The specific number of edges connecting (n,k) to $(n+1,k+i)$ is arbitrary for $j \in \{0,1,\dots,d\}$ and 0 elsewhere.
- X is the space of infinite edge paths.
- For $x = x_0x_1x_2\cdots \in X$ denote by x_i the i'th edge of x which connects a vertex in level i to a vertex in level $i+1$.
- X is a compact metric space with metric given by: For $x,y \in X$, $d(x,y) = 2^{-i}$ where $i = \inf\{j | x_j \neq y_j\}$.
- A cylinder set $C = [c_0c_1c_2\dots c_n]$ is a clopen set such that $x \in C$ if and only if $x = c_0c_1\dots c_nx_{n+1}\dots$.



Transformation

- There is a partial order on the edges, the ordering increases from left to right on edges with the same range
- This partial order extends to paths in X, comparable paths have the same tails
- The transformation, T, maps paths into the next largest path, (red path into the blue path), and maps maximal paths into minimal paths
- X and T form the dynamical system (X,T)
- The Bratteli-Vershik systems on the special family of Bratteli diagrams will be denoted $\mathcal{S}_l$



Dimension Group & Co-boundaries

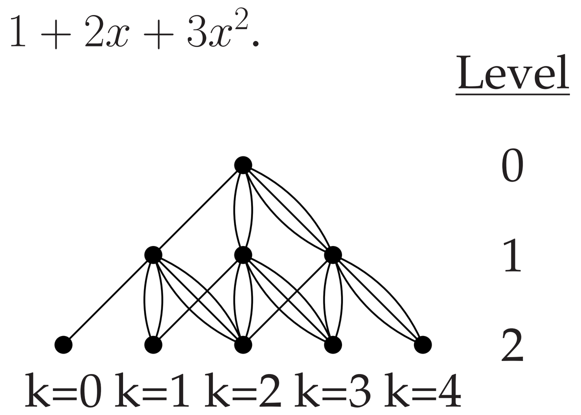
- The dimension group of a B-D is the direct limit of the sequence: $\mathbb{Z} \xrightarrow{A_1} \mathbb{Z}^{|V_1|} \xrightarrow{A_2} \mathbb{Z}^{|V_2|} \xrightarrow{A_3} \mathbb{Z}^{|V_3|} \xrightarrow{A_4} \dots$ where each A_j is the incidence matrix of the B.D.
- $\partial_T C(X,\mathbb{Z})$ are the functions, f, on X such that $f = g \circ T - g$ and $g \in C(X,\mathbb{Z})$
- T is not continuous so $\partial_T C(X,\mathbb{Z}) \not\subseteq C(X,\mathbb{Z})$

Theorem. *For a Bratteli Vershik system, (X,T), in the family $\mathcal{S}_l$ the dimension group, $K_0(\mathcal{V},\mathcal{E})$, is order isomorphic to*

$$C(X,\mathbb{Z})/(\partial_T C(X,\mathbb{Z}) \cap C(X,\mathbb{Z}))$$

The Bratteli Diagram associated to $p(x) = a_0 + a_1x + \cdots + a_dx^d$.

- At each level increase the number of vertices by d.
- The number of edges connecting the root vertex (0,0) to vertex $(1,k)$ is a_k.
- The number of edges connecting (n,k) to $(n+1,k+j)$ is a_j for $j \in \{0,1,\dots,d\}$ and 0 elsewhere.
- Here $p(x) = 1 + 2x + 3x^2$.

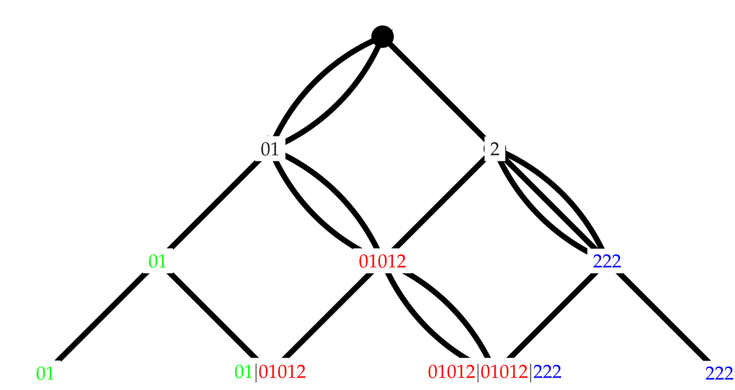


Proposition

- The dimension group of the B.D. associated to $p(x)$ is order isomorphic to the ordered group of rational functions, R, of the form:
$$\frac{r(x)}{p(x)^n} \text{ where } \deg(r(x)) \leq \deg(p(x)^n) \text{ and } r(x) \in \mathbb{Z}[x].$$
- Two elements are equivalent if they are equal in the usual sense.
$$\frac{1}{1+2x} = \frac{1+2x}{(1+2x)^2}$$
- The group operation is addition, in the usual sense.
$$\frac{1}{1+2x} + \frac{1}{(1+2x)^2} = \frac{1+2x}{(1+2x)^2} + \frac{1}{(1+2x)^2} = \frac{2+2x}{(1+2x)^2}$$
- The positive set consists of rational functions, $r(x)/p(x)^n$ for which there is an i such that $r(x)(p(x))^i$ has all nonnegative coefficients.
$$\frac{1-x+3x^2}{(1+2x)^3} = \frac{1+x+x^2+6x^3}{(1+2x)^4}$$

Partition

- Let $(X,T) \in \mathcal{S}_l$ with $d=1$.
- Label the cylinders of length one, 0 through $|\mathcal{E}_1| - 1$, and use to define a partition of X.
- Let $\mathcal{P}(x) = j$ if x is in the cylinder labeled j.
- For each vertex (n,k) choose x in the minimal cylinder into (n,k) and define
$B(n,k) = \mathcal{P}(x)\mathcal{P}(Tx)\mathcal{P}(T^2x)\dots\mathcal{P}(T^{h(n,k)-1}x)$



Subshift

- Let (Σ,σ) be the dynamical system where Σ denotes the space of bi-infinite sequences of $\{0,1,\dots,|\mathcal{E}_1|-1\}$ for which every finite subsequence appears as a subblock in some $B(n,k)$, and σ denotes the shift map.

Theorem. *For $(X,T) \in \mathcal{S}_l$ with $d=1$, and a fully-supported invariant ergodic probability measure, μ, there is a set of measure 0, $X' \subset X$ and a one-to-one Borel measurable map $\phi : X \setminus X' \to \Sigma$ such that $\phi \circ T = \sigma \circ \phi$ on $X \setminus X'$.*

Ergodic Probability Measures for Polynomial System

- An ergodic probability measure is one that gives T-invariant sets measure 1 or 0.

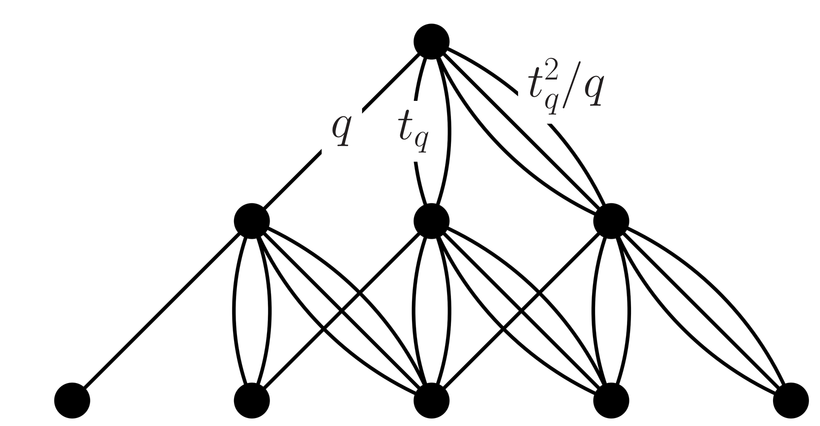
Theorem. *The invariant, fully-supported, ergodic probability measures for the adic T associated to the polynomial $p(x) = a_dx^d + \dots + a_0$, are the one-parameter family of Bernoulli measures*

$$\mathcal{B}\left(\underbrace{q,\dots,q}_{a_0 \text{ times}}, \underbrace{t_q,\dots,t_q}_{a_1 \text{ times}}, \underbrace{\frac{t_q^2}{q},\dots,\frac{t_q^2}{q}}_{a_2 \text{ times}}, \dots, \underbrace{\frac{t_q^d}{q^{d-1}},\dots,\frac{t_q^d}{q^{d-1}}}_{a_d \text{ times}}\right),$$

where $q \in (0,\frac{1}{a_0})$, and t_q is the unique solution in $[0,1]$ to the equation:

$$a_0q^d + a_1q^{d-1}t + \dots + a_dt^d - q^{d-1} = 0.$$

- Bernoulli in the sense of a random walk.
- All weights determined by choice of $q \in (0, 1/a_0)$.
- Say $q = 1/6$, then t_q is the unique solution in [0,1] to: $(1/6)^2 + (1/3)t + 3t^2 - 1/3 = 0$.
- $t_q = 1/6$ and $t_q^2/q = 1/6$



Eigenvalues

- A complex number λ is said to be an eigenvalue of a measure-preserving transformation T (measure denoted by μ) if there exists a function, $f \neq 0$, in $L^2(\mu)$ such that $f(Tx) = \lambda f(x)$ almost everywhere.

Theorem. *The eigenvalues of the transformation T of the Bratteli-Vershik system associated to the polynomial $a_1x + a_0$ with a fully-supported, T-invariant, ergodic probability measure, μ, contain the a_1^n and a_0^n roots of unity for all $n > 0$.*