SMOOTH, EX-POST IMPLEMENTATION WITH MULTI-DIMENSIONAL INFORMATION.

PRELIMINARY AND INCOMPLETE, DO NOT CITE

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ABSTRACT. A social-choice function maps reports of the agents’ private information into a set of social alternatives. The (weak) ex-post implementation problem is to find transfers such that truthful reporting by the agents is ex-post incentive compatible. Jehiel et al. (2006) prove that in environments with a discrete set of social alternatives, at least two agents, multi-dimensional information and generic preferences, only constant social-choice functions are ex-post implementable. In this paper, we study mechanisms where the outcome space of the social-choice function is continuous (e.g., random allocations) rather than discrete as in JMMZ. In addition, we restrict attention to smooth social-choice functions, where the differential approach can be used. First, we provide a simple proof that extends their impossibility result to our environment when the dimension of the outcome space (i.e., the number of instruments) is one (e.g., random allocations between two alternatives) and the dimensional of an agent signal space is 2 or larger. The proof shows that the only smooth social choice functions that satisfy the local, first-order incentive compatibility constraints for ex-post implementation are constant functions. In the case when the outcome space has dimension equal or larger than \(2N\) and there are \(N\) agents with bidimensional signals, we show that it is possible to satisfy the local, first order incentive compatibility constraints.

1. INTRODUCTION

"Turning now to the case where the number \(n\) of targets surpasses the number \(n'\) of instruments it is clear that we are placed before insoluble tasks. [...] Targets are not free then; had they been chosen otherwise, they would
have been *incompatible* among themselves and their set would be *inconsistent.*” (Tinbergen, 1952, pp. 39-40)

1.1. **The problem and a preview of results.** A social-choice function maps agents’ reports of their private information into outcomes. The planner’s (weak) ex-post implementation problem is to find incentive compatible transfer functions with the property that truthful reporting by the agents is an ex-post equilibrium in the mechanism defined by the social-choice and the transfer functions. For social-choice functions with a discrete outcome space, Jehiel et al. (2006) prove that in environments with at least two agents and generic preferences, only constant social-choice functions are ex-post implementable.¹ We study smooth mechanisms where the outcome space of the social-choice function is continuous and not discrete as in Jehiel et. al. (2006). First, we provide a simple proof of the impossibility result of Jehiel et. al. (2006) for smooth environments where the dimension of the outcome space (i.e., the number of instruments) is one. The proof shows that the only smooth social choice functions that satisfy the local, first-order incentive compatibility constraints for ex-post implementation are the constants. In the case when the outcome space has dimension equal or larger than $2N$ and there are $N$ agents with bidimensional signals, we show that it is possible to satisfy the local, first order incentive compatibility constraints.

1.2. **The Environment.** In this paper the outcome space of the social-choice function is continuous. One possible example of a continuous outcome spaces is the set of feasible allocations of bundles of divisible goods used in the multi-dimensional, monopolistic screening literature; see Laffont et al. (1987), McAfee and McMillan (1988), Wilson (1993), Armstrong (1996), Rochet and Chone (1998), and Basov (2001).

Another example of a continuous outcome space is the set of lotteries over social alternatives used in the virtual implementation literature; see Abreu and Sen (1991), Abreu and Matsushima (1992),

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¹ An important part of the implementation literature is concerned with the so-called strong implementation problem, that is, constructing a mechanism where in all equilibria the outcomes coincide with the ones picked by the social-choice function. Here we only require the existence of one equilibrium inducing the outcome specified by the social-choice function.
Moreover, we restrict attention to smooth environments: preferences, the social-choice function and transfers are analytic, which allows us to use the differential approach of the seminal works of Laffont and Maskin (1979) and Laffont and Maskin (1980).

For a discrete set of outcomes of the social–choice rule, Bikhchandani et al. (2006), Jehiel et al. (2005) and Jehiel et al. (2006) study ex-post incentive compatible (EPIC) rules. Although, smooth social–choice rules can approximate (in any reasonable metric) implementable, discrete social–choice rules, the two classes of social-choice rules are disjoint.

1.3. Remarks. We are interested in ex-post implementation because ex-post incentive compatible mechanisms are robust in the sense that they are detail free with respect to the statistical structure of agents’ information, see Bergemann and Morris (2005) for an in-depth discussion.

For one-agent mechanisms, interim-incentive compatibility (also known as Bayesian incentive compatibility) is equivalent to ex-post incentive compatibility. In this paper, the strategy used to find transfers that make the social-choice function EPIC reduces the problem to a single-agent problem: for a given agent, we fix the (truthful) report of the others and try to obtain restrictions under which there are transfers that make truthful reporting of a given agent incentive compatible. For one-agent mechanisms, there are non-trivial social choice functions where EPIC is possible for an open set of preferences. For many agents, the problem arises when we combine all the individual restrictions needed for truth telling. With multi-dimensional signals, additional restrictions may emerge when we combine the individual restrictions and that makes the implementation problem harder to solve.

2. The Model

2.1. Notation. All vectors we consider, \( x \in \mathbb{R}^n \), are column vectors. We use the symbol ‘\(^t\) to denote the transpose of a vector or matrix. For a function, \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n, f(x) = (f_1(x), \cdots, f_n(x))' \) we let \( \nabla f(x) \)
denote its gradient at \( x \), that is,

\[
\nabla f(x) = \begin{pmatrix}
\frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_1} f_n(x) \\
\vdots & \cdots & \vdots \\
\frac{\partial}{\partial x_m} f_1(x) & \cdots & \frac{\partial}{\partial x_m} f_n(x)
\end{pmatrix}
\]

and we write \( Df(x) = \nabla f(x)' \) for the derivative of \( f \).

2.2. Assumptions and Definitions. There are \( N \) agents and each agent \( i \) receives a private signal \( s_i \in S_i \); we write \( S = \prod_{k=1}^N S_i \) for the space of signals.

**Assumption 1.** The set of signals of agent \( i \), \( S_i \subset \mathbb{R}^{d_i} \) is a compact, convex, set with non-empty interior.

The planner has at his disposal \( K \) instruments, which we interpret either as social alternatives or allocations. A social-choice rule, \( \psi : S \rightarrow \Delta^K \), maps the agents’ reported signals into outcomes, which are interpreted either as probability distributions over alternatives or allocations, \( \Delta^K = \{ q \in \mathbb{R}^K : q^k \geq 0, \sum_{k=1}^K q^k \leq 1 \} \).

Agents preferences, \( U^i(q, \tau, s) \), over outcomes \( q \in \Delta^K \), and transfers \( \tau \in \mathbb{R} \), may depend on the entire profile of signals, \( s = (s_i, s_{-i}) \). We assume quasi-linear preferences.

**Assumption 2.** Quasi-linear preferences: \( U^i : \Delta^K \times \mathbb{R} \times S \rightarrow \mathbb{R} \) is given by \( U^i(q, \tau, s) = u^i(q, s) - \tau \).

We write the utility of agent \( i \) reporting signals \( \hat{s}_i \), given that \( i \)'s true signals are \( s_i \), and given that the other agents report truthfully, as \( U^i(\hat{s}_i | s_i, s_{-i}) \).

**Definition 1.** The choice rule \( \psi \) is ex-post incentive compatible, EPIC, if for any agent \( i \) there is a transfer function \( \tau^i : S^i \times \prod_{j \neq i} S^j \rightarrow \mathbb{R} \) such that for all \( s_i \) and \( \hat{s}_i \in S_i \):

\[
U^i(s_i | s_i, s_{-i}) = u^i(\psi(s_i, s_{-i}), s_i, s_{-i}) - \tau^i(s_i, s_{-i}) \geq u^i(\psi(\hat{s}_i, s_{-i}), s_i, s_{-i}) - \tau^i(\hat{s}_i, s_{-i}) = U^i(\hat{s}_i | s_i, s_{-i})
\]
Even though the general case where no assumptions are placed upon $u^i$ is important, throughout the paper we shall focus attention in the following random allocation specification.

**Case 1. Random Allocations**

There are $K + 1$ alternatives, denoted as $0, 1, ..., K$. The agent preferences over lottery allocations are given by the expected utility:

$$u^i(q, s) = q'V^i(s) = \sum_{k=0}^{K} q^k V_k^i(s).$$

Thus, $V_k^i(s)$ is interpreted as $i$'s utility for alternative $k$ and $q^k$ is the probability of alternative $k$. Under this view we have the constraints that $q^k \geq 0$ and $\sum q^k = 1$. Let $\mathbf{1}$ be the $K + 1$ unit vector and define $W_i(s) = V^i(s) - \mathbf{1}V_0^i(s)$. From the definition of EPIC above, it follows that $\psi$ is EPIC for $W_i$, if and only if, it is EPIC for $V^i$. Thus, without loss of generality, we will adopt the normalization $V_0^i(s) = 0$ for all $i$ and all $s \in \prod_{i=1}^{N} S_i$. Alternative zero, which we may think of as the status-quo, has its utility normalized to zero for all agents. Note that this implies that, although there are $K + 1$ alternatives, there are only $K$ instruments.

In this paper we are interested in the differential approach. Hence, we restrict attention to smooth mechanisms and analytic preferences.

**Assumption 3.** The social–choice rule, transfers and utilities are infinitely differentiable: $\psi(\cdot, s_{-i}) \in C^\infty(S_i)$, $\tau^i(\cdot, s_{-i}) \in C^\infty(S_i)$, and $u^i(\psi, s) \in C^\infty(\Delta^K \times S)$ for all $i$ and $s_{-i}$. When needed we will further assume utilities are analytic, $u^i(\psi, s) \in C^{\omega}(\Delta^K \times S)$; that is, for each $(\psi, s)$ in the domain of $u^i$, the power series representation\(^2\) of $u^i$ is convergent in a neighborhood of $(\psi, s)$.

Any piecewise constant function, for instance the efficient rule in the case of linear preferences, can be approximated by smooth functions.

Let $D_{\hat{s}_i}U^i(\hat{s}_i|s_i, s_{-i})$ be the derivative of $U^i(\hat{s}_i|s_i, s_{-i})$ viewed as a function of $\hat{s}_i$, and $\mathbf{0}'$ be the $d_i$ dimensional zero row vector. The first-order condition for truth-telling, $D_{\hat{s}_i}U^i(\hat{s}_i|s_i, s_{-i}) = \mathbf{0}'$, of an interior

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\(^2\)The power series representation of $f : \mathbb{R}^n \to \mathbb{R}$ at $x$ is:

$$f(\hat{x}) = f(x) + \sum_{a_1=1}^{\infty} \cdots \sum_{a_n=1}^{\infty} C(a)(\hat{x}_1 - x_1)^{a_1} \cdots (\hat{x}_n - x_n)^{a_n}.$$
type \( s_i \in S_i \) can be written as,

\[
(2.1) \quad D_{\psi} u^i(\psi(s), s) D_{s_i} \psi(s) - D_{s_i} \tau^i(s_i, s_{-i}) = 0',
\]
or equivalently, as the partial differential equation system:

\[
(2.2) \quad D_{s_i} \tau^i(s_i, s_{-i}) = \sum_k \frac{\partial u^i}{\partial \psi_k}(\psi(s), s) D_{s_i} \psi_k(s).
\]

Since the necessary first-order conditions must hold for all interior \( s_i \), it follows that:

\[
D_{s_i}^2 \tau^i(s_i, s_{-i}) = \nabla_{s_i} \psi(s) D_{s_i}^2 u^i(\psi(s), s) D_{s_i} \psi(s) + \sum_k \frac{\partial u^i}{\partial \psi_k}(\psi(s), s) D_{s_i}^2 \psi_k(s) + \sum_k \left( \nabla_{s_i} \frac{\partial u^i}{\partial \psi_k} \right) D_{s_i} \psi_k(s),
\]

If there are smooth transfers that implement the social-choice function, we must have that the matrix \( D_{s_i}^2 \tau^i(s_i, s_{-i}) \) is symmetric. Since the first two terms of the Hessian in (2.3) are always symmetric, the last term must be also symmetric. Conversely, by Frobenius’ Theorem (Frobenius, 1877), we have the following proposition.

**Proposition 1.** If \( \sum_k \left( \nabla_{s_i} \frac{\partial u^i}{\partial \psi_k} \right) D_{s_i} \psi_k(s) \) is a symmetric matrix then the partial-differential equation system (2.2) has a local solution for any given initial condition.

As a result, a necessary and sufficient condition for the existence of transfers \( \tau^i \) such that the first-order conditions for EPIC are satisfied is that the social-choice function \( \psi \) satisfy the system of partial differential equations,

\[
(2.4) \quad \frac{1}{2} \sum_{k=0}^{K} \left[ \frac{\partial^2 u^i}{\partial s_i^a \partial \psi_k} \frac{\partial \psi_k}{\partial s_i^b}(s) - \frac{\partial^2 u^i}{\partial s_i^b \partial \psi_k} \frac{\partial \psi_k}{\partial s_i^a}(s) \right] = 0,
\]

with \( a, b = 1, \ldots, d_i \) and \( a \neq b \).

The \( \sum_{i=1}^{N} \frac{d_i^2 - d_i}{2} \) equations in (2.4) form a linear, homogenous, first-order, PDE system, which simplifies considerably in the case of random allocations:

\[
(2.5) \quad \frac{1}{2} \sum_{k=0}^{K} \left[ \frac{\partial V_k^i}{\partial s_i^a}(s) \frac{\partial \psi_k}{\partial s_i^b}(s) - \frac{\partial V_k^i}{\partial s_i^b}(s) \frac{\partial \psi_k}{\partial s_i^a}(s) \right] = 0,
\]

for \( a, b = 1, \ldots, d_i \) and \( a \neq b \).
One may have the false impression we have moved in circles. Transfers, are solution to a PDE system whose integrability conditions are equivalent to yet another PDE system (2.4 or 2.5) where the social choice function is the indeterminate. As we shall see later the new system may carry non-obvious integrability conditions that place additional constraints on the social choice function.

2.3. Second Order Conditions. The analysis above has only looked at the necessary local, first-order, incentive compatibility constraints. The local second order condition for truthfully reporting to be optimal by agent \( i \) is that the matrix

\[
M(s) \equiv \nabla_{s_i} \psi(s) D_{s_i}^2 u^i(\psi(s), s) D_{s_i} \psi(s) + \\
+ \sum_k \frac{\partial u^i}{\partial \psi_k}(\psi(s), s) D_{s_i}^2 \psi_k(s) - D_{s_i}^2 \tau^i(s)
\]

be negative semi-definite at all \( s \). In the case of random allocations, case 1, utility is linear in the decision function and the matrix becomes

\[
M(s) \equiv \sum_k V^i_k(s) D_{s_i}^2 \psi_k(s) - D_{s_i}^2 \tau^i(s).
\]

Using (2.3), this can be rewritten as

\[
M(s) \equiv -\sum_k \nabla_{s_i} V^i_k(s) D_{s_i} \psi_k(s) = -\nabla_{s_i} V^i(s) D_{s_i} \psi(s)
\]

Thus, for the case of random allocations, the second order conditions are that the matrices \( \nabla_{s_i} V^i(s) D_{s_i} \psi(s) \) are positive semi-definite.

3. Results

3.1. The Geometry of EPIC Social-Choice Functions. As in Jehiel et al. (2006), let’s consider the case of two social alternatives, that is \( K = 1 \). Recalling the normalization \( V^i_0 = 0 \), we drop subscripts and write \( \psi \), instead of \( \psi_1 \), for the probability that alternative 1, and \( V^i \) for \( V^i_1 - V^i_0 \). The necessary condition (2.5) for local implementation can now be expressed as:

\[
\frac{\partial V^i}{\partial s^a_i}(s) \frac{\partial \psi}{\partial s^b_i}(s) = \frac{\partial V^i}{\partial s^b_i}(s) \frac{\partial \psi}{\partial s^a_i}(s) \text{ for } a, b = 1, \ldots, d_i \text{ and } a \neq b.
\]

When the agent \( i \)'s utility is constant with respect to her signal \( s^a_i \), the social choice function must also be constant with respect to the signal \( s^b_i \), unless \( i \)'s utility is trivial (i.e. constant with respect to any of her signals). Conversely, if \( \psi \) is constant with respect to \( s^a_i \), the
utility of \( i \) must be constant with respect to \( s_i^a \), unless agent \( i \)'s is a dummy (i.e. \( \psi \) is constant with respect to any \( i \)'s signals).

In the complementary case where the utility function is not trivial and the agent is not a dummy, that is \( \frac{\partial V^i}{\partial s_i^1}(s) \cdot \frac{\partial \psi}{\partial s_i^1}(s) \neq 0 \), we again re-write the necessary condition (2.5) but now in a familiar geometric form:

\[
\text{(MRS)} \quad MRS^i_{a,1} \equiv \frac{\partial V^i}{\partial s_i^a} \cdot \frac{\partial \psi}{\partial s_i^1} = \frac{\partial \psi}{\partial s_i^a} \cdot \frac{\partial V^i}{\partial s_i^1}, \quad \text{for } a = 2, \ldots, d_i.
\]

The MRS condition says that as we change agent \( i \)'s signals keeping his utility constant, then social-choice function remains constant. Geometrically, it means that as we move along \( i \)'s indifference surface, by changing \( i \)'s signal, we also move along the social-choice function indifference surface.

The geometric view naturally leads us to consider the tangent spaces of the social choice function and of the utility function. At a points \( s \), we denote the respective tangent spaces by \( T_{\psi(s)} S \) and \( T_{V^i(s)} S \). These spaces can be viewed as the linear spaces that best approximate the level sets of \( \psi \) and \( V^i \) at the point \( s \).

Remember that, elements in a tangent space of a function may either be interpreted as the direction vectors where the directional derivative of the function vanishes, or, equivalently, as the differential operator that produces the respective directional derivative. The indifference surface of \( V(x, y) = xy \) has a tangent vector that is expressed either as \((y, -x)\), or as \( y\partial_x - x\partial_y \). To verify that \((y, -x)\) is indeed tangent to \( V \) notice that directional derivative vanishes in the direction of \((y, -x)\), that is \( \nabla V \cdot (y, -x) = 0 \). To establish both definitions are equivalent, noticed that for any function \( f(x, y) \) the identity \( \nabla f \cdot (y, -x) \equiv (y\partial_x - x\partial_y) f \) holds true. We adopt the later interpretation, as thinking of tangent vectors as differential operators shall be more fruitful to our goals.

Without loss of generality assume that \( \frac{\partial V^i}{\partial s_i^1}(s) \neq 0 \), for every agent \( i \), and define the vectors (differential operators),

\[
X_i^a \equiv \frac{\partial V^i}{\partial s_i^a} \cdot \partial s_i^1 - \frac{\partial V^i}{\partial s_i^1} \cdot \partial s_i^a \quad \text{for } a = 2, \ldots, d_i.
\]
The system in 2.5 or MRS is equivalent to $X^a \psi = 0$ for all $a$ and $i$.

A tangent vector is defined at a given point $s$, when we vary $s$, we refer to it as vector field.

**Definition 2.** For any two vector fields $X$ and $Y$, their **Lie-bracket** is another vector-field given by $[X, Y] \equiv XY - YX$.

An important property of the Lie-Bracket is the following:

**Fact 1.** For any $X, Y \in T_f S$, also $[X, Y] \in T_f S$.

**Proof.** Since $f$ is constant in the direction of $X$ or $Y$, we have $X f = 0$ and $Y f = 0$. As a result $(XY - YX)f = X(Yf) - Y(Xf) = X0 - Y0 = 0$ and thus $f$ is also constant in the direction of $[X, Y]$, that is $[X, Y]$ belongs to the tangent space of $f$. ■

**Definition 3.** Define $\mathcal{V}(s)$ as the minimal subspace that satisfies:

1. **Tangency to $V^i$:** $X^a_i \in \mathcal{V}$ for all $i$ and $a$.
2. **Involution:** if $X, Y \in \mathcal{V}$ then $[X, Y] \in \mathcal{V}$.

**Lemma 1.** The space $\mathcal{V}$ can be constructed in finite number of steps by the following algorithm:

1. **Input:** $X^a_i \in \mathcal{V}$ for all $i$ and $a$.
2. **Output:** $\mathcal{V}(s)$ of Definition 3.
3. **Initialization:** set $\mathcal{B} = \{X^a_i \in \mathcal{V} : \forall i, a\}$.
4. **Initialization:** order the elements of $\mathcal{B}$ so $\mathcal{B} = \{B_1, \ldots, B_n\}$.
5. **For** $i$ from 1 to $n$ **do:** for $j$ from 1 to $n$ **do**:
6. **If** $j = i$ **then** $j = i + 1$ **else** $Z = [B_i, B_j]$.
7. **If** $Z \notin \text{span}(\mathcal{B})$ **then**:
   7.1 Add $Z$ to $\mathcal{B}$ so that $B_{k+1} = B_k$ for all $k > 1$ and $B_1 = Z$.
   7.2 Re-start the main loop, set $i = 1$.
8. **Else** $j = j + 1$, **end do.** $i = i + 1$, **end do.**
9. Set $\mathcal{V} = \text{span} (\mathcal{B})$.

**Proof.** Whenever $Z \notin \text{span} (\mathcal{B})$, we add a linear-independent element to $\mathcal{B}$ but the maximum number of times that this can occur is less or equal than $\sum_i d_i$, which is the dimension of the space of signals. Thus, the algorithm clearly terminates. The initialization of the algorithm guarantees that tangency is satisfied. We only need to verify that $\text{span} (\mathcal{B})$ is involutive. Pick $X, Y \in \text{span} (\mathcal{B})$ and remember that the coefficients of $X = \sum_k a_k B_k$ and $Y = \sum_j b_j B_j$ are functions:

$$
[X, Y] = \left[ \sum_k a_k B_k, \sum_j b_j B_j \right] = \left( \sum_k a_k B_k \right) \left( \sum_j b_j B_j \right) - \left( \sum_k b_k B_k \right) \left( \sum_j b_j B_j \right)
$$

$$
= \sum_k a_k \left( \sum_j (B_k b_j) B_j + b_j B_k B_j \right) - \sum_j b_j \left( \sum_k (B_j a_k) B_k + a_k B_j B_k \right)
$$

$$
= \sum_{j,k} (a_k (B_k b_j) B_j + b_j (B_j a_k) B_k) + a_k b_j (B_k B_j - B_j B_k)
$$

$$
= \sum_{j,k} (a_k (B_k b_j) B_j + b_j (B_j a_k) B_k) + a_k b_j [B_k, B_j]
$$

Clearly the first two terms belong to $\text{span} (\mathcal{B})$ and by construction of $\mathcal{B}$, for any $k$ and $j$, $[B_k, B_j] \in \text{span} (\mathcal{B})$ and so $[X, Y] \in \text{span} (\mathcal{B})$. ■

**Proposition 2.** Assume two alternatives and the random allocations case. Any $\psi$ that satisfies first-order conditions for EPIC has $T_{\psi(s)} \supset \mathcal{V}(s)$. Moreover, there exists $\psi$ satisfying first-order conditions for EPIC such that $T_{\psi(s)} = \mathcal{V}(s)$.

**Proof.** The first part follows immediately since from $X^a_i \psi = 0$ and the definition of $\mathcal{V}$, we have that $Z \psi = 0$ for any $Z$ in $\mathcal{V}$. As $\mathcal{V}$ is involutive, the second part follows from Frobenius’ Theorem (Seiler, 2010, Appendix C, p. 601). ■
Corollary 1. Assume two agents, each with two or more signals and generic preferences. If $\psi$ satisfies the first-order conditions for EPIC then $\nabla \psi = 0$.

Proof. For generic preferences, $\dim(\mathcal{V}(s)) = d_1 + d_2$ and thus $\mathcal{V}(s)$ has maximal dimension: only 0 is orthogonal to it. The computation of $\dim(\mathcal{V})$ is at the appendix. \[\blacksquare\]

Noticed that Lemma 1 and Proposition 2 provide a characterization of social-choice functions that satisfy the FOC for EPIC in the case of two alternatives and random allocations.

Example 1. Assume linear-preferences as in Roberts (1979). In this case $X_i^a = \alpha_i^1 \cdot s_i^a - \alpha_i^a \cdot s_i^1$ where $\alpha_i^a \equiv \frac{\partial V^i}{\partial s_i^a}$. Because the $\alpha$s coefficients are constants, we have that $[X_i^a, X_j^b] = 0$ whenever $i \neq j$. Put simply, there are no additional integrability conditions to consider. Thus, $\mathcal{V} = \text{span}\{X_i^a\}$ and $\dim(\mathcal{V}) \leq \sum_{i=1}^{N} d_i - N$, with equality for generic linear preferences. Any FOC-EPIC social choice function is of the form:

$$\psi(s) = \Psi \left( \sum_{k=1}^{d_1} \alpha_1^k \cdot s_1^k, \ldots, \sum_{k=1}^{d_i} \alpha_i^k \cdot s_i^k, \ldots, \sum_{k=1}^{d_N} \alpha_N^k \cdot s_N^k \right).$$

Moreover, since $\nabla_s V^i(s) D_{s_i} \psi(s) = D_i \Psi \cdot \sum_{k=1}^{d_i} (\alpha_i^k)^2$, second-order conditions are satisfied only if $D_i \Psi \geq 0$ for all $i$.

Example 2. An object must be allocated to agent 1 (alternative 1) or to agent 2 (alternative 2). Agents’ signal profiles are respectively $s_1 = (x, y)$ and $s_2 = (w, z)$. Agent 1’s preferences are $V^1(x, y, w, z) = (x + yz, 0)$ while agent 2’s are $V^2(x, y, w, z) = (0, w + yz + \alpha \cdot xw)$, where $\alpha$ is a parameter.

Here, in Example 2, there are no allocative externalities since an agent utility is zero if the agent does not receive the object.
Claim 1. If $\alpha \neq 0$ then only constant $\psi$ satisfy the first-order conditions for EPIC.

Proof. Normalized utilities are: $V^1(x, y, w, z) = -x - yz$ and $V^2(x, y, w, z) = w + yz + \alpha \cdot xw$. The tangent vectors to the agents’ respective (normalized) indifference surfaces are $X_1 = \partial_y - z\partial_x$ and $X_2 = (1 + \alpha x)\partial_z - y\partial_w$. We complete $\mathcal{V}$ to involution by adding the additional generators: $X_3 \equiv [X_1, X_2] = (1 + \alpha x)\partial_x - \partial_w - z\partial_z$ and $X_4 \equiv [X_1, X_3] = -2\alpha x\partial_x$. For $\alpha \neq 0$, $y \neq 0$ and $z \neq 0$, the vectors $X_1, X_2, X_3$ and $X_4$ are linearly independent since

$$\det\begin{pmatrix} -z & 1 & 0 & 0 \\ 0 & 0 & -y & 1 + \alpha x \\ 1 + \alpha x & -1 & 0 & \alpha z \\ -2\alpha z & 0 & 0 & 0 \end{pmatrix} = -2\alpha^2 y z^2,$$

moreover, as the set has maximal dimension, it spans the whole space, $\mathcal{V}(s) = \mathbb{R}^4$ and thus $\nabla \psi(s) \cdot \mathbb{R}^4 = \{0\}$ implies $\nabla \psi = 0$. ■

Claim 2. If $\alpha = 0$ then only $\psi$ such that $\psi(x, y, w, z) = \Psi(w + x + yz)$, where $\Psi$ is an arbitrary function, satisfy the FOC for EPIC.

Proof. If $\alpha = 0$, it is sufficient to show the tangent space of of any $\psi$, as defined in the claim, contains the space spanned by the $X_1$, $X_2$ and $X_3$ (notice that $X_4 \equiv 0$ for $\alpha = 0$) given in the above proof. Moreover $X_1\Psi = (\partial_y - z\partial_x)\Psi = 0$, $X_2\Psi = (-y\partial_w)\Psi = 0$, and $X_3\Psi = (-\partial_w)\Psi = 0$ establishes $X_1, X_2, X_3 \in T_{\Psi(s)}$. ■

Claim 3. If $\alpha = 0$ and $\psi(x, y, w, z) \equiv \Psi(w + x + yz)$ satisfies the SOC for EPIC then $\nabla \psi = 0$. 
Proof. The SOC requires that the matrices
\[
\nabla_{x,y} V^1(s) D_{x,y} \psi(s) = \begin{pmatrix}
\partial_x V^1 \cdot \partial_x \psi & \partial_x V^1 \cdot \partial_y \psi \\
\partial_y V^1 \cdot \partial_x \psi & \partial_y V^1 \cdot \partial_y \psi
\end{pmatrix}
\]
and \(\nabla_{w,z} V^2(s) D_{w,z} \psi(s)\) be positive semi-definite. Both matrices have a 0 eigenvalue so the other eigenvalue is given by their trace: \(\partial_x V^1 \cdot \partial_x \psi + \partial_y V^1 \cdot \partial_y \psi = \Psi' \cdot [-1 - z^2]\) and \(\partial_w V^2 \cdot \partial_w \psi + \partial_z V^2 \cdot \partial_z \psi = \Psi' \cdot [1 + y^2]\). Both eigenvalues are non-negative if and only if \(\Psi' = 0\).

3.2. Discussion. For a discrete outcome space, by fixing the signals of the other agents \(s_{-i}\), we can always partition the signal space of agent \(i\) into regions where each region corresponds to the selected social alternative. Moreover, for well-behaved social choice function, typically⁴, any connected boundary segment separates two distinct regions; thus to solve for local, first-order, incentive compatibility constraints, it suffices to restrict attention to the boundary region where the social choice function changes from one alternative to another. In sum, in the discrete case, there is no loss of generality in considering only two alternatives. As we shall see next, this is not the case when we allow for a continuous outcome space.

3.3. The role of the number of instruments. Let’s reconsider a variant of the above example 2 but now we allow for the possibility that the object is not allocated to either of the buyers (call it alternative zero), let \(\psi_1\) be the prob. of alternative 1 (object allocated to agent 1) and \(\psi_2\) the prob. of alternative 2 (object allocated to agent 2).

Example 3. Normalized utilities are: \(V^1(x, y, w, z) = (x + yz, 0)\) and \(V^2(x, y, w, z) = (0, w + yz + \alpha \cdot xw)\). The necessary conditions 2.5 are \((\partial_y - z \partial_x) \psi_1 = 0\) and \(((1 + ax) \partial_z - y \partial_w) \psi_2 = 0\). Any \(\psi = (\Psi_1(w, z, \frac{x + yz}{z}), \Psi_2(x, y, \frac{w + \alpha \cdot xw + yz}{y}))\) solves 2.5.

The result is not surprising since with the additional alternative, the normalized utilities in Example 3, unlike in Example 2, fail to be

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⁴A boundary point may lie in the intersection of three or more regions but these points are non-generic for well-behaved social choice functions.
generic. Let’s consider now an example, with the same space of signals and alternative outcomes as in Example 3 but with preferences given by random generated polynomials of order 2 with rational coefficients.

Example 4. Normalized utilities are:

\[ V_1^1 = -77 + 50x^2 - 60xy - 42xw + 7xz - 89x - 70y^2 + 34yw - 68yz - 60y + 16w^2 + 
+ 52wz - 20w - 4z^2 - 89z; \]
\[ V_2^1 = -46 + 69x^2 + 80xy + 28xw - 42xz - 33x + 21y^2 - 35yw + 97yz + 30y - 64w^2 + 
+ 89wz - 16w + 59z^2 - 69z; \]
\[ V_1^2 = 91 - 33x^2 + 87xy - 34xw + 40xz + 77x + y^2 - 10yw - 65yz - 85y + 54w^2 + 18w + 
+ 52z^2 + 36z; \]
\[ V_2^2 = 88 - 22x^2 + 51xy - 27xw + 50xz + 60x - 91y^2 - 47yw - 97yz - 2y - 31w^2 + 
+ 25wz + 31w - 27z^2 + 65z. \]

Here it is also possible to find \( \psi \) that satisfies (2.5). In particular we can construct a local solution for \( \psi \) using its’ Taylor expansion at \((x_0, y_0, w_0, z_0)\) with the following initial conditions:

\[ \psi_1(x_0, y, w_0, z) = \Psi_1(y, z), \psi_2(x, y, w_0, z) = \Psi_2(x, y, z), \text{ and} \]
\[ \frac{\partial \psi_2}{\partial w}(x_0, y, w, z) = \Psi_3(y, w, z), \text{ where the } \Psi_1, \Psi_2 \text{ and } \Psi_3 \text{ are arbitrary functions of their respective arguments.} \]

REFERENCES


\[^{4}\text{Sérgio: Same result applies for polynomials of order 3 or 4, it must be general...} \]
APPENDIX A. SECOND ORDER CONDITIONS

Claudio: I made several changes to this subsection. I did not understand why you imposed SOC for foc in which \( \hat{s}_i \neq s_i \). Check. I also didn’t understand the lemma and the remark. I have left your version of the subsection, just in case.

A.1. Second Order Conditions (Sergio’s version). The analysis above has only looked at the necessary local, first-order, incentive compatibility constraints. To satisfy the second-order conditions, the matrix

\[
M(\hat{s}|s_i) \equiv \nabla_{s_i} \psi(\hat{s}) D^2_{\psi} u^i(\psi(\hat{s}), s) D_{s_i} \psi(\hat{s}) + \sum_{k} \frac{\partial u^i}{\partial \psi_k}(\psi(\hat{s}), s) D^2_{s_i} \psi_k(\hat{s}) - D^2_{s_i} \tau^i(\hat{s}), \text{ where } \hat{s} = (s_i, s_{-i}),
\]

must be negative semi-definite for reports \( \hat{s}_i \) where: the first-order condition holds when true signal is \( s_i \) and others’ signal is \( s_{-i} \); that is, \( D_{s_i} \tau^i(\hat{s}) = D_{\psi} u^i(\psi(\hat{s}), s) D_{s_i} \psi(\hat{s}) \). And since \( \hat{s}_i \) must also satisfies the first-order condition when \( \hat{s}_i \) itself is the true signal, we have similarly \( D_{s_i} \tau^i(\hat{s}) = D_{\psi} u^i(\psi(\hat{s}), \hat{s}) D_{s_i} \psi(\hat{s}) \). By these two first-order conditions give us: \( (D_{\psi} u^i(\psi(\hat{s}), s) - D_{\psi} u^i(\psi(\hat{s}), \hat{s})) D_{s_i} \psi(\hat{s}) = 0' \).

Thus, for the case of random allocations, we can state the sufficient second-order conditions as

**Lemma 2.** The social choice function \( \psi \) satisfies the second-order conditions if and only if the condition \( (V_i(s) - V_i(\hat{s}))) D_{s_i} \psi(\hat{s}) = 0' \) implies that the matrix \( \sum_k (V^i_k(s) - V^i_k(\hat{s})) D^2_{s_i} \psi(\hat{s}) - D_{s_i} V(\hat{s}) \nabla_{s_i} \psi(\hat{s}) \) is negative semi-definite for every i, all \( s_i, \hat{s}_i \), and \( s_{-i} \).

**Proof.** In the case of random allocations, the first term of \( M(\hat{s}|s_i) \) is zero because the preferences are linear in the probabilities so that \( D^2_{\psi} u^i = 0_{\mathbb{K} \times \mathbb{K}} \). The last term of \( M(\hat{s}|s_i) \) (the derivative of transfers with respect to \( s_i \) at \( \hat{s} \)) is computed by differentiating again the first-order condition at \( \hat{s} \): \( D^2_{s_i} \tau^i(s) = \sum_k V^i_k(\hat{s}) D^2_{s_i} \psi(\hat{s}) + D_{s_i} V(\hat{s}) \nabla_{s_i} \psi(\hat{s}) \).

**Remark 1.** If rank \( (\nabla_{s_i} \psi) = K \) then the above condition simplifies to: \( \nabla_{s_i} \psi(s) D_{s_i} V(s) \) is positive semi-definite.