PLCY 788 Microeconomic Theory I
Problem Set 1
A brief mathematics review of optimization and other fun theorems

One of Economists’ favorite matrices--the Hessian
1. The second order conditions for assessing whether a relative extremum is a minimum or maximum can be written as a set of conditions on the principal minors of the matrix of second derivatives of the objective function (the Hessian or in the case of constrained optimization, the bordered Hessian matrix).

In the unconstrained optimization case, write down the general first and second order conditions for a relative maximum and minimum for a function like \( z = f(x_1, x_2, x_3, \ldots, x_n) \). Write down the determinantal test for the sign of the Hessian.

In the constrained optimization case, write down the general first and second order conditions for a relative minimum and maximum for a problem like the one below. Write down the determinantal test for the sign of the bordered Hessian.

\[ z = f(x_1, x_2, x_3, \ldots, x_n) \quad s.t. \quad g(x_1, x_2, x_3, \ldots, x_4) = 0 \]

Checking the sign of a quadratic form (will come in handy later)
2. For the following quadratic forms, write down the coefficient matrix, and then report the sign of the quadratic form.

\[ q = 5u^2 + 3uv + 2v^2 \quad q = 8uv - u^2 - 31v^2 \]

3. For the following functions, use the first order conditions to find the critical point and the second order conditions to check for a maximum or minimum.

\[ z = 3x^2 + 15y^2 - x - 3y \quad z = -5x^2 - 10y^2 + 5x + 4y \]

4. For the following problems, use the first order conditions to find the critical point(s) and the second order conditions to check for a maximum or minimum. Remember to use your friend the bordered Hessian.

\[ z = (x + 2)(y + 1) \quad s.t. \quad 4x + 6y = 130 \]
\[ z = x^2 + y^2 \quad s.t. \quad x + 4y = 2 \]

Homogeneity and Euler’s Theorem—very handy in public places
5. Determine the degree of homogeneity for each of the following functions. In each case, what is the degree of homogeneity of the derivative function?

\[ f(x, y) = \sqrt{xy} \]
\[ f(x, y, w) = \frac{xy^2}{w} + 2xw \]

Envelope theorem, useful for mailing stuff
6. This handy-dandy theorem allows you to see how a function \( z \) changes with a small change in a parameter such as \( a \), as opposed to the variable \( x \). Essentially, if \( x \) is evaluated at its optimal value \( (x^*) \), then one can observe how \( z \) changes with a change in the parameter \( a \) by simply holding \( x \) constant and evaluating the derivative of \( z \) with respect to (wrt) \( a \). Got it? Here is an example.

Say we have \( f(x;a) = -x^2 + 2ax + 4a^2 \). This is actually a family of functions with each particular member of the family depending on the value of the parameter \( a \). First find the optimal value of \( x \) \( (x^*) \) by taking the derivative of \( f \) wrt \( x \) and setting to 0. You will get \( x^* = a \). Plug \( x^* = a \) into the
original equation to get \( f(x^*;a)=5a^2 \). The derivative of the function \( f \) wrt \( a \) when \( x \) is evaluated at \( x^* \) is 10a—this is how the function \( f \) changes with a small change in \( a \).

Now use the Envelope Theorem instead. Take the derivative of \( f(x;a) \) wrt \( a \) to get 2x+8a. Evaluate \( f \) at \( x^* \), which we found earlier to be equal to \( a \). This gives \( (2a+8a) =10a \). Hooray—it works!

Try this on your own using the function \( f(x;a)=ax-x^2 \) or any of your own favorite functions.