Notes from class on 3/25 – Finding the mixed strategy equilibrium

There are 2 pure strategy equilibria \{S,S\} and \{O,O\}. Say the probability that player 2 plays ‘sport’ is \(p\), and the probability she plays opera is \(1-p\), while for player 1 (the row player) the respective probabilities are \{q, 1-q\}. Player 2’s expected utility is

\[
U_2(p,q) = p^*q^*U(S,S) + p^*U(O,S) + (1-p)^*qU(S,O) \\
\quad + (1-p)^*(1-q)U(O,O) \\
= pq + (1-p)(1-q)(4) \\
= pq + (1-p)[4(1-q)] \quad (1)
\]

U_2 is increasing in \(p\) when \(q>4(1-q)\) or when \(q>(4/5)\)
U_2 is decreasing in \(p\) when \(q<4(1-q)\) or when \(q<(4/5)\)
U_2 is constant in \(p\) when \(q=4(1-q)\) or when \(q=(4/5)\)

**Best Response of player 2**

If \(q=(4/5)\), player 2’s best response for \(p\) is any number in \([0,1]\).
If \(q>(4/5)\) her best response is \(p=1\).
If \(q<(4/5)\) her best response is \(p=0\).

Note you can also maximize (1) with respect to \(p\) by setting the derivative to 0 and get \(p=(4/5)\).

The expected utility for player 1 is

\[
U_1(p,q) = q^*p^*U(S,S) + (1-q)^*pU(O,S) + q^*(1-p)U(S,O) + (1-q)^*(1-p)U(O,O) \\
= qp + (1-q)p(0) + q(1-p)(0) + (1-q)(1-p)(4) \\
= q[p4] + (1-q)[(1-p)] \quad (2)
\]

U_1 is increasing in \(q\) if \(4p>(1-p)\) or when \(p>(1/5)\).
U_1 is decreasing in \(q\) if \(4p<(1-p)\) or when \(p<(1/5)\).
U_1 is constant when \(4p=(1-p)\) or when \(p=(1/5)\).

**Best response for player 1**

If \(p=(1/5)\), best response is any number in \([0,1]\).
If \(p>(1/5)\) his best response is \(q=1\).
If \(p<(1/5)\) his best response is \(q=0\).

Graph these best response functions in \((p,q)\) space and find the intersections points. There is an intersection at \(\{q=0, p=0\}\) corresponding to the pure strategy \(\{O,O\}\); there is an equilibrium at \(\{q=1, p=1\}\) corresponding to the pure strategy \(\{S,S\}\); and there is an equilibrium at \(\{q=4/5, p=1/5\}\) which is the mixed strategy equilibrium.

Aside: Write down player i’s expected utility from each of the pure strategies s/he can play. Set these two expected utilities equal to each other and solve for the probability that the other person plays S. You should get \(q=4/5\) or \(p=1/5\) depending on which player you do this for. This method is called the ‘pay-off’ equating method. It illustrates the fact that the player must be indifferent between either of the pure strategies in equilibrium.