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Solution Continuity in Variational Conditions

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Abstract We present some results about Lipschitzian behavior of solutions to variational conditions when the sets over which the conditions are posed, as well as the functions appearing in them, may vary. These results rely on calmness and inner semicontinuity, and we describe some conditions under which those conditions hold, especially when the sets involved in the variational conditions are convex and polyhedral. We then apply the results to find error bounds for solutions of a strongly monotone variational inequality in which both the constraining polyhedral multifunction and the monotone operator are perturbed.

Keywords variational inequalities · sensitivity · polyhedrality

Mathematics Subject Classification (2000) 90C33 · 49J53 · 47H04

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1 Introduction

Variational inequalities, or the more general *variational conditions,* appear in many important problems from application areas including engineering, logistics and transportation, and economics. In many such applications it is important to understand what will happen to a solution of such a problem if the problem's data vary. Another way of thinking about that question is to envision the problem solution as a function or multifunction of the data, and to ask what properties that operator has. Common questions involve existence and local uniqueness of solutions, as well as whether they display any of several types of continuity when regarded as multifunctions of the data.

To specify a finite-dimensional variational inequality we start with a closed convex subset S of $\mathbb{R}^n$ and a single-valued function $f : S \to \mathbb{R}^n$. Then we ask for a point $x \in S$, if any exists, such that $f(x)$ is an inward normal to S at x. Another way to say this is to require that for each $s \in S$,

$$\langle f(x), s-x \rangle \geq 0. \tag{1}$$

With such a problem we can associate a *generalized equation* by writing

$$0 \in f(x) + N_S(x), \tag{2}$$

where $N_S(x)$ is the *normal cone* of S at x, defined by:

$$N_S(x) = \begin{cases} \{x^* \mid \langle x^*, s-x \rangle \leq 0 \text{ for each } s \in S,\} & \text{if } x \in S, \\ \emptyset, & \text{if } x \notin S. \end{cases}$$

As is well known, (1) and (2) are equivalent. They include many special cases that often appear in applications, including systems of nonlinear equations (for which $S = \mathbb{R}^n$), and linear or nonlinear complementarity problems (for which $S = \mathbb{R}^n_+$), among others.

Variational conditions are generalizations of variational inequalities in which we drop the requirement that S be closed and convex and redefine the normal cone according to the more general specifications of variational geometry, for which see [16, Section 6.B]. We do little in this paper with variational conditions, but in Section 4 we briefly discuss some results that apply to them as well as to variational inequalities.

Many investigators have helped to advance our knowledge about how changes in the data of variational conditions and inequalities affect their solutions. A survey of such problems with constraints expressible by fairly smooth functions, containing 79 references, is in [13].

Section 2, just below, describes some recent results about Lipschitz continuity of general multifunctions, and applies these to analyze uniform bounds for changes in the solutions of variational inequalities posed over polyhedral convex sets when the right-hand sides of the sets change, as do the functions appearing in the variational inequalities. Next, Section 3 examines more closely the role of polyhedrality in providing tractability of solutions. We will see there that one aspect of polyhedrality has very strong consequences for problems in which only the right-hand sides of the constraints vary, but then we will also see in Section 4 that if we enlarge the class of permitted perturbations, the situation becomes much worse. Even in that case something can be said, but unfortunately the stronger properties that one would like to have, such as Lipschitz continuity, are currently unavailable. Throughout the paper we use the Euclidean norm unless otherwise stated.

2 Lipschitz continuity and global solution bounds

This section reviews some recent results about Lipschitz continuity for multifunctions in a finite-dimensional space, and then applies these to construct a global bound for differences of solutions to strongly monotone variational inequalities in which both the function involved in the variational inequality and the set over which that inequality is posed may change. We begin in Section 2.1 with some definitions needed to understand the rest of the section, and then present several results relating Lipschitz continuity of multifunctions to properties of calmness, inner semicontinuity, and single-valuedness. We also try to show how these results relate to some others previously known.

As an application, we show in Section 2.2 how to obtain a global solution bound for the difference of solutions of variational inequalities involving strongly monotone operators and polyhedral convex sets, each of which may change.

2.1 Multifunctions and Lipschitz continuity

In general, solutions of inclusions such as variational inequalities may be set-valued, so we usually treat them as *multifunctions:* that is, operators associating with each point of a space X a subset $F(x)$ (perhaps empty) of a space Y. It is often helpful to think about the *graph* of the multifunction F, which is the subset $\operatorname{gph} F$ of $X \times Y$ consisting of pairs (x,y) such that $y \in F(x)$. The projections of the graph into the component spaces X and Y are, respectively, the *effective domain* $\operatorname{dom} F$ and the *image* $\operatorname{im} F$, of F. Thus, $x \in \operatorname{dom} F$ if and only if $F(x)$ is nonempty.

If Y is a metric space we can consider ways of measuring the amount by which the images under F of two points, say x and x', differ. These images, $F(x)$ and $F(y)$, are subsets of Y, so this means that we need to measure distances between sets. We will do this for the case $Y = \mathbb{R}^m$, but the method can easily be extended to more general spaces. For our purposes the most useful distance will be the *Pompeiu-Hausdorff distance,* defined as follows. We use B to denote the unit ball of whatever Euclidean space we are currently working in, here $\mathbb{R}^m$.

Definition 1 The *Pompeiu-Hausdorff distance* between subsets Y and Y' of $\mathbb{R}^m$ is

$$\rho[Y,Y'] = \inf\{\eta \geq 0 \mid Y \subset Y' + \eta B,\ Y' \subset Y + \eta B\}.$$

This distance may be a nonnegative real number, or may be $+\infty$.

Having this distance, we can define Lipschitz continuity for multifunctions as follows.

Definition 2 Let F be a multifunction from $\mathbb{R}^k$ to $\mathbb{R}^m$, such that for each x in some subset S of $\mathbb{R}^k$ the set $F(x)$ is closed. Let λ be a nonnegative real number. We say F is *Lipschitz continuous* relative to S with modulus λ if for each s and s' in S, $\rho[F(s),F(s')] \leq \lambda \|s - s'\|$.

We will sometimes use the term *Lipschitzian* in place of "Lipschitz continuous."

Lipschitz continuity in the Pompeiu-Hausdorff metric is sometimes inconvenient to use, because some commonly used multifunctions do not satisfy it, and in that case a local condition called the *Aubin property* or, sometimes, *Aubin continuity,* may be used instead. For more on this question see [16, Sections 9.E, 9.F]. However, Lipschitz continuity is well suited to the purposes we have in mind here, so we use it rather than the Aubin property.

There is an important link between Lipschitz continuity and two other important properties that multifunctions may have: namely, calmness and inner semicontinuity. We will describe that link after defining those properties in tailored forms that we need here. The following two definitions are adapted from [16, p. 399] and [9, Definition 1.2] respectively.

Definition 3 Let X be a subset of $\mathbb{R}^n$, x be a point of X, and $S : \mathbb{R}^n \to \mathbb{R}^m$ be a multifunction. S is *calm at x relative to X with modulus λ* if there is some neighborhood V of x relative to X such that for each $x' \in V$ one has $S(x') \subset S(x) + \lambda \|x' - x\| B$, where B is the unit ball in $\mathbb{R}^m$.

Here X may be the underlying space $\mathbb{R}^n$, as we shall assume if it is not explicitly stated to be otherwise.

Definition 4 Let S be a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$, X a subset of $\mathbb{R}^n$, and x a point of X. We say S is *inner semicontinuous at x relative to X* if for each open set Q that meets $S(x)$ there is a neighborhood V of x relative to X such that for each $x' \in V$, Q meets $S(x')$.

The link mentioned above appears in the following theorem, in which some terminology is changed from [9, Theorem 1.5].

Theorem 1 *Let S be a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$ having closed values, X be a convex subset of* $\operatorname{dom} S$*, and λ be a nonnegative real number. The following are then equivalent:*

a. At each point of X, S is calm relative to X with modulus λ and is inner semicontinuous relative to X.
b. S is Lipschitz continuous relative to X with modulus λ.

This theorem extends a result of Wu Li [5], which showed that if a multifunction is (in our terminology) calm relative to $\mathbb{R}^n$ at each point of $\mathbb{R}^n$, and in addition is Hausdorff lower semicontinuous on $\mathbb{R}^n$, then it is Lipschitzian on $\mathbb{R}^n$. The term *Hausdorff lower semicontinuous* was defined to mean, for a multifunction T defined on $\mathbb{R}^n$ and at a point $x \in \mathbb{R}^n$, that $\lim_{z \to x} d[T(x), T(z)] = 0$, where $d[T(x), T(z)] = \sup_{u \in T(x)} \inf_{v \in T(z)} \|u - v\|$. By contrast, in order to obtain Lipschitz continuity Theorem 1 requires only that one demonstrate inner semicontinuity, rather than Hausdorff lower semicontinuity.

Our main use here of Theorem 1 will be to demonstrate Lipschitz continuity by establishing calmness and inner semicontinuity, then appealing to this theorem. However, these two properties are rarely given *a priori* in an application. Thus, in order to carry out that program we need first to review some other properties that imply one or the other of calmness and inner semicontinuity.

One result connecting observable properties with calmness is [10, Proposition 1], which says that multifunctions of a certain class, called *polyhedral,* are guaranteed to be everywhere calm with a modulus that depends only on the multifunction. Thus, the modulus of calmness does not depend on the particular point at which we are working, though in general the size of the neighborhood on which the inclusion defining calmness holds does depend on that point.

The defining property of a polyhedral multifunction is that its graph is the union of a finite collection of polyhedral convex sets. In the special case in which the graph is just a single polyhedral convex set, we speak of a *graph-convex* polyhedral multifunction. The class of polyhedral multifunctions is closed under several useful operations, including those of addition, composition, and inversion. This fact is very useful for inferring polyhedrality of a multifunction built up from other multifunctions by means of the functional operations just cited.

As we are going to be concerned in the next section with a variational inequality posed over a polyhedral convex set, we will need to use polyhedrality properties of the normal cone of such a set. A useful result, noted as an observation in [9], says that if S is a graph-convex polyhedral multifunction from $\mathbb{R}^m$ to $\mathbb{R}^n$ then the multifunction $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(u,x) = N_{S(u)}(x)$ is polyhedral. In connection with the characterization of Lipschitz continuity in Theorem 1, this has some immediate consequences.

First, the polyhedrality of $N_{S(u)}(x)$ as a function of (u,x) together with the properties of functional operations mentioned above implies the polyhedrality of the multifunction

$$H(u,y) = \Pi_{S(u)}(y),$$

where $\Pi_{S(u)}$ is the Euclidean projector on $S(u)$, because

$$\Pi_{S(u)}(y) = (I + N_{S(u)})^{-1}(y).$$

We use this fact in the proof of Theorem 2 below. However, the polyhedrality also has an immediate consequence in view of Theorem 1, because it says that H is everywhere calm with a constant modulus. But H is single-valued on the convex set $\operatorname{dom} H = (\operatorname{dom} S) \times \mathbb{R}^n$, so on that set it is also inner semicontinuous relative to $\operatorname{dom} H$. Applying Theorem 1, we conclude that H is Lipschitzian on its domain.

This result (with only u as a variable) was given by Yen [18, Theorem 2.1]. The extension to (u,y) is immediate because a Euclidean projector is Lipschitzian with modulus 1. Yen gave a very different proof; the point of introducing this example here, in addition to the fact that we will use the result below, is to illustrate a particular application of Theorem 1 to give a very short and simple proof of this result.

However, in using this line of argument we are not restricted to projections. An immediate extension yields the following more general result [9, Corollary 2.2].

Proposition 1 *Let F be a polyhedral multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$, let λ be its modulus of calmness, and let C be a convex subset of $\operatorname{dom} F$ on which F is single-valued. Then F is Lipschitzian on C with modulus λ.*

This section has reviewed a number of concepts and some recent results that facilitate establishing Lipschitz continuity of some possibly rather complicated multifunctions. Section 2.2 below illustrates how one can apply these to obtain solution bounds for certain variational inequalities.

2.2 Application: global solution bounds

For an example application of the material just covered, we consider solutions of a generalized equation constructed from a single-valued, strongly monotone function and a polyhedral convex set. We make both the function and the set subject to perturbations of a structured kind, and ask under what conditions one can guarantee that these solutions are Lipschitzian in the perturbations. The analysis leading to this bound resulted from a question posed to the authors by Jong-Shi Pang [8].

Theorem 2 *Let S be a graph-convex polyhedral multifunction from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathscr{F}$ be a collection of single-valued functions $f : \operatorname{im} S \to \mathbb{R}^n$. Suppose there are positive real numbers m and M such that each $f \in \mathscr{F}$ is strongly monotone with modulus m and is Lipschitzian with modulus M. Define $\mu = M^{-2}m$ and $\nu = (1 - [1 - (m/M)^2]^{1/2})^{-1}$. Then the following hold:*

a. For each $(f,u) \in \mathscr{F} \times \operatorname{dom} S$, the operator $f + N_{S(u)}$ is maximal monotone and there is a unique point $x = x(f,u)$ satisfying the generalized equation

$$0 \in f(x) + N_{S(u)}(x). \tag{3}$$

b. There is a nonnegative real number τ such that for any two elements (f,u) and (f',u') of $\mathscr{F}\times \operatorname{dom} S$ one has

$$\|x(f',u')-x(f,u)\| \le \nu^{-1}\tau\|u'-u\| + \nu^{-1}\mu\|f'[x(f,u)]-f[x(f,u)]\|. \tag{4}$$

Proof We first adapt the argument of [1, Lemma 2.4] to show that the operator in (3) is maximal monotone. Fix any $f\in\mathscr{F}$ and $u\in\operatorname{dom} S$. Choose a positive number α small enough so that $\alpha M<1$. As the values of the normal-cone operator are cones we have

$$\alpha[f+N_{S(u)}] = \alpha f + N_{S(u)}.$$

To show that $f+N_{S(u)}$ is maximal monotone it therefore suffices to show that $\alpha f+N_{S(u)}$ is maximal monotone.

Fixing any $y\in\mathbb{R}^n$ and rewriting the relation

$$y \in x+\alpha f(x)+N_{S(u)}(x)$$

in the equivalent form

$$x = H_y(x) := (I+N_{S(u)})^{-1}[y-\alpha f(x)],$$

we see that if H_y has a fixed point then y is in the image of $I+\alpha f+N_{S(u)}$. However, the operator $Q(u):=(I+N_{S(u)})^{-1}$ is the Euclidean projector on $S(u)$ and therefore is Lipschitzian with modulus 1. By hypothesis f is Lipschitzian on $S(u)$ with modulus M, so our choice of α ensures that the operator H_y is a strong contraction from $S(u)$ to itself. But $S(u)$, being polyhedral, is closed, so the contraction mapping theorem says that H_y has a unique fixed point in $S(u)$. It follows that $I+\alpha f+N_{S(u)}$ is surjective and therefore, by Minty's theorem, that $\alpha f+N_{S(u)}$ and hence also the operator

$$T(x) := f(x)+N_{S(u)}(x)$$

are maximal monotone with effective domain $S(u)$.

A variant of this argument will also show that the generalized equation $0\in T(x)$ in (3) has a unique solution. As T is maximal monotone, Minty's theorem says that its resolvent $(I+T)^{-1}$ is a contraction defined on all of $\mathbb{R}^n$. However, T inherits the strong monotonicity of f, so this resolvent operator is actually a strong contraction, and therefore the contraction mapping theorem shows that it has a unique fixed point. As the set of fixed points of the resolvent is exactly the set of zeros of T, we see that T has a unique zero $x(f,u)$, which establishes the claim in part (a.) of the theorem. In the remainder of the proof we write x and x' for $x(f,u)$ and $x(f',u')$ respectively.

In this situation we have two desirable properties, strong monotonicity of the (restricted) operators and polyhedrality of the multifunction S. However, in (3) both f and S appear. It is convenient first to separate these by using a standard splitting reformulation to convert (3) to an equivalent fixed-point problem.

If we let μ be the positive number $M^{-2}m$, then the point x is also the unique solution of the equation $0=\mu f(x)+N_{S(u)}(x)$. If we add x to both sides and rearrange the resulting equation, we obtain

$$x = (I+N_{S(u)})^{-1}[I-\mu f](x) = Q(u)[I-\mu f](x). \tag{5}$$

This is one (simple) version of the forward-backward splitting method of Chen and Rockafellar [2]. The form given in (5) separates the part of the expression involving $S(u)$ from that involving f, which will be convenient in the subsequent analysis.

Consider two pairs (u,f) and (u',f') in $(\operatorname{dom} S)\times\mathscr{F}$. We then have

$$\begin{aligned} x'-x &= Q(u')[I-\mu f'](x') - Q(u)[I-\mu f](x) \\ &= \{Q(u')[I-\mu f'](x') - Q(u)[I-\mu f'](x')\} \\ &+ \{Q(u)[I-\mu f'](x') - Q(u)[I-\mu f](x)\}. \end{aligned} \tag{6}$$

We shall bound each of the two quantities enclosed in curly brackets in (6), the first by using polyhedrality and the second by using strong monotonicity. For the first quantity, write $H(u,y)$ for $Q(u)(y)$, and recall that we already noted, just before Proposition 1, that H is polyhedral and single-valued, hence Lipschitzian on $(\operatorname{dom} S)\times\mathbb{R}^n$ with some modulus τ. If we set $y=[I-\mu f'](x')$ we then obtain

$$\begin{aligned} \|Q(u')[I-\mu f'](x') - Q(u)[I-\mu f'](x')\| &= \|H(u',y)-H(u,y)\| \\ &\le \tau\|u'-u\|. \end{aligned} \tag{7}$$

For the second quantity, recall that $Q(u)$, as a projector, is Lipschitzian with modulus 1. Accordingly, we have

$$\begin{aligned} &\|Q(u)[I-\mu f'](x') - Q(u)[I-\mu f](x)\| \\ &\quad \le \|[I-\mu f'](x') - [I-\mu f](x)\| \\ &\quad \le \|(x'-x)-\mu[f'(x')-f'(x)]\| + \mu\|f'(x)-f(x)\|. \end{aligned} \tag{8}$$

If we define ν to be $1-[1-(m/M)^2]^{1/2}>0$, which lies in $(0,1]$ because $0<m\le M$, we have

$$\begin{aligned} &\|(x'-x)-\mu[f'(x')-f'(x)]\|^2 \\ &\quad = \|x'-x\|^2 - 2\mu\langle x'-x, f'(x')-f'(x)\rangle \\ &\quad + \mu^2\|f'(x')-f'(x)\|^2 \\ &\quad \le (1-2\mu m+\mu^2M^2)\|x'-x\|^2 = [(1-\nu)\|x'-x\|]^2. \end{aligned} \tag{9}$$

By combining (6), (7), (8), and (9) we find that

$$\|x'-x\| \le \tau\|u'-u\| + \mu\|f'(x)-f(x)\| + (1-\nu)\|x'-x\|,$$

and by rearrangement we finally have

$$\|x(f',u')-x(f,u)\| \le \nu^{-1}\tau\|u'-u\| + \nu^{-1}\mu\|f'[x(f,u)]-f[x(f,u)]\|,$$

which establishes the claim in part (b.).

The bound (4) differs from many bounds obtainable for solutions of variational inequalities in that it is a global bound: there is no requirement that u' be near u, nor that f' and f be close in any sense. On the other hand, this bound involves one of the solutions because of the presence of $\|f'[x(f,u)]-f[x(f,u)]\|$. Although we have not introduced any measure of closeness for f and f', it is natural to ask whether the difference of function values at $x(f,u)$ could be replaced by a difference of function values at some point, say x_0, that did not depend on u and f.

To answer this question, consider the problem in $\mathbb{R}^2$ in which the set $S(u)$ is the halfspace consisting of points whose second coordinate is at least u, and in which the functions $f(x)$

are of the form Px, where P is a symmetric positive definite matrix whose minimum eigenvalue is at least m and whose maximum eigenvalue is not more than M. The solution of the generalized equation $0 \in Px + N_{S(u)}(x)$ is then the point $x(P,u)$ that minimizes $(1/2)\langle x, Px\rangle$ on $\mathbb{R}^2$ subject to the constraint $\langle (0,1), x\rangle \geq u$.

If we start with some positive u and with $P = I$, then we have $x(I,u) = [0,u]$. For matrices

$$P' = \begin{bmatrix} p'_{11} & p'_{12} \\ p'_{21} & p'_{22} \end{bmatrix}$$

that are close to the identity, the solution becomes

$$x(P',u) = u[-p'_{21}/p'_{11},\ 1].$$

Thus, if we take $p'_{11} = 1 = p'_{22}$ and $p'_{12} = \varepsilon = p'_{21}$, then

$$\|x(P',u) - x(I,u)\| = \|u[-\varepsilon,\ 1] - u[0,1]\| = \varepsilon u = \varepsilon \|x(I,u)\|.$$

For these choices our theorem gives the bound

$$\|x(P',u) - x(I,u)\| \leq \nu^{-1}\mu \|P'x(I,u) - Ix(I,u)\|.$$

As $\|P'x(I,u) - Ix(I,u)\| = \varepsilon u$ we see that the bound and the distance between the actual solutions are of the same order, differing only by a constant multiplier. But in this problem u did not change at all, and P changed only by an amount of order ε, which can be arbitrarily small. If the $x(f,u)$ in (4) were replaced by a fixed quantity, then by making u sufficiently large we could obtain a difference of solutions larger than the bound, a contradiction. Therefore no such fixed quantity would work in this bound.

We can, however, remove the point $x(f,u)$ from the bound if we are willing to accept a somewhat cruder bound, replacing $\|f'[x(f,u)] - f[x(f,u)]\|$ by an upper bound on the differences of values of f' and f on all of $\operatorname{im} S$. We then obtain the following corollary.

Corollary 1 *Assume the notation and hypotheses of Theorem 2. For two elements f and f' of $\mathscr{F}$ define*

$$\|f' - f\|_\infty := \sup_{x \in \operatorname{im} S} \|f'(x) - f(x)\|.$$

Then for the τ appearing in Theorem 2 and for any two elements (u,f) and (u',f') of $(\operatorname{dom} S) \times \mathscr{F}$, one has

$$\|x(f',u') - x(f,u)\| \leq \nu^{-1}\tau\|u' - u\| + \nu^{-1}\mu\|f' - f\|_\infty. \tag{10}$$

For some purposes the bound (10) might be more convenient to apply than is (4), but that probably will not be so if one wants to use special properties of the solution $x(f,u)$.

3 Lipschitzian results without strong monotonicity

In Section 2 we developed a global bound for variation in the solution of a variational inequality in which both the function and the underlying polyhedral convex set may change. We did this by applying some recent results about Lipschitz continuity of multifunctions, but the analysis worked because we also made the assumption that the function in the variational inequality was strongly monotone, which guaranteed a single-valued solution. It was this single-valuedness that allowed us to infer inner semicontinuity of the solution map and thus to apply the general characterization of Lipschitz continuity given in Theorem 1.

Strong monotonicity is very nice when it holds, but in many problems of interest we cannot expect to have it. If we then continue to assume that the underlying set is a graph-convex polyhedral multifunction of some parameter, we would like to find an appropriate assumption, short of strong monotonicity, that will allow us to conclude that the solution of the variational inequality is single-valued and Lipschitzian. It turns out that this is possible, but that the price we have to pay is to give up global bounds.

It might seem unnecessary to develop a new condition, because we could introduce multipliers on the constraints defining the underlying set, and thereby reduce the problem to one posed over a fixed polyhedral convex set, for which there are results that are by now fairly standard; see, e.g., [3]. This, however, will not accomplish what we want, because the multipliers then become part of the solution. As in many cases they will not be unique, the introduction of multipliers will have destroyed the possibility of proving the kinds of results that we want. The only way to avoid this would be to introduce a regularity condition such as constraint nondegeneracy [12] or the well known linear-independence criterion, and that would seriously limit the applicability of the analysis. Therefore we do not want to use multipliers.

This question is attacked in [7]. Theorem 4.2 of that paper characterizes the existence of a locally unique, Lipschitzian solution of the linear generalized equation

$$0 \in Mx + w + N_{S(u)}(x), \tag{11}$$

where M is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$, $w \in \mathbb{R}^n$ and $N_{S(u)}(x)$ is the normal cone at x to the polyhedral convex set

$$S(u) = \{x \in \mathbb{R}^n \mid Ax \leq u\}, \tag{12}$$

with $u \in \mathbb{R}^m$ and A being a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. The theorem shows that the existence of such a solution is equivalent to satisfaction of a certain coherent orientation condition. The condition allows perturbation of either or both of w and u, so that the underlying set, as well as the constant term in the variational inequality, may change. As a consequence of this theorem for linear generalized equations, [7, Theorem 5.1] then presents a sufficient condition for local existence, uniqueness, and Lipschitz continuity properties of solutions of *nonlinear* variational inequalities posed over perturbed polyhedral convex sets. The work of [7] is applied in [6] to analyze the stability of static traffic equilibria.

4 General variations

The results discussed in Section 3 dealt with perturbations of a special type in the constraining set: we assumed that the set was of the form $S(u)$, where S was a graph-convex polyhedral multifunction. A common example of such a perturbation is for the right-hand sides of a system of linear equations and inequalities to be changed. We might next ask for results usable for more general perturbations, such as changes in the matrices defining the linear equations and inequalities, as well as in their right-hand sides.

Unfortunately, the situation is much less favorable here. A counterexample first published in [11] and later applied in simpler forms in [17, p. 642], [4, Example 4.7.4], and [15] shows that even for the problem of projecting a fixed point onto a polyhedral convex subset of $\mathbb{R}^2$ a locally unique solution may not be Lipschitzian, no matter how small the perturbations are. This sets a clear limit to what one can do even with polyhedral sets if one allows perturbation of the matrices as well as the right-hand sides. To avoid this difficulty we could apply stronger conditions, such as the linear-independence condition or the slightly weaker conditions developed in [12], but these imply uniqueness of the multipliers.

Another possibility is to use the very general results developed in [14] for persistence and continuity of solutions to variational conditions. However, although those results require only very weak assumptions they guarantee neither local uniqueness nor Lipschitz continuity.

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